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WATCHING SYSTEMS OF TRIANGULAR GRAPHS

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Abstract. A watching system in a graph $G = (V, E)$ is a set $W = \{\omega_1, \omega_2, \dots, \omega_k\}$, where $\omega_i = (v_i, Z_i), v_i \in V$ and Z_i is a subset of closed neighborhood of v_i such that the sets $L_W(v) = \{\omega_i : v \in Z_i\}$ are non-empty and distinct, for any $v \in V$. In this paper, we study the watching systems of line graph K_n which is called triangular graph and denoted by $T(n)$. The minimum size of a watching system of G is denoted by $\omega(G)$. We show that $\omega(T(n)) = \lceil \frac{2n}{3} \rceil$.

1. Introduction

Throughout this paper we will assume that all graphs are finite, simple, and undirected. We use [5] for terminology and notations not defined here.

For a graph G, let $V = V(G)$ and $E(G)$ denote the set of vertices, edges of G, respectively. For a vertex $x \in V(G)$, the *degree* of x, denoted $\deg(x)$, is the number of edges of G incident with x. We use $\Delta(G) = \Delta$ to denote the maximum degree of vertices of G. For any vertex $v \in V(G)$, the open neighborhood of v is the set $N_G(v) = \{u : uv \in E\}$, while the closed neighborhood of v is the set $N_G[v] = N_G(v) \cup \{v\}$. A *complete graph* is a graph in which each pair of distinct vertices is joined by an edge. We denote the complete graph with n vertices by K_n. A subgraph H of a graph G is called a *spanning subgraph* if $V(H) = V(G)$. For every nonnegative integer r, the graph G is called *r-regular* if the degree of each vertex of G is equal to r. The *triangular graph*, $T(n)$, is the line graph of the complete graph K_n. The vertices of $T(n)$ may be identified with the 2-subsets of $\{v_1, v_2, \dots, v_n\}$ that are adjacent if and only if the 2-subsets have a nonempty intersection.

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Let $C \subseteq V$ be a subset of vertices of G and for all vertex $v \in V$, we define

$$I_C(v) := N_G[v] \bigcap C.$$

If all $I_C(v)$'s are nonempty, then C is a dominating set (or a covering code) of G; if moreover the $I_C(v)$'s are all distinct, then we say that C is an *identifying code* of G. The set $I_C(v)$ is called the identifying set of the vertex v. Note that the graph G has an identifying code if and only if distinct vertices must have distinct closed neighborhoods. A graph with this property is called *twin-free* or *identifiable*. The minimum size of a identifying code in G is denoted by $\iota(G)$. This concept was introduced in 1998 in [6] to model fault diagnosis in multiprocessor systems and have since been studied widely in the communities of both graph theory and coding theory, for instance see [1, 3, 4, 7, 9]. They are also used for the design of indoor detection systems based on wireless sensor network.

Watching systems were introduced in [2], is a generalization of identifying codes. A *watcher* ω of G is a couple $\omega = (v_i, Z_i)$, where v_i is a vertex and $Z_i \subseteq N_G[v_i]$. We will say that ω is located at v_i and that Z_i is its watching area or watching zone. A *watching system* in a graph G is a finite set $W = \{\omega_1, \omega_2, \ldots, \omega_k\}$ where ω_i , $1 \le i \le k$ is a watcher such that $\{Z_1, Z_2, \ldots, Z_k\}$ is an identifying system. So W is a watching system for G if the sets $L_W(v) = \{\omega_i : v \in Z_i\}$ are non-empty and distinct, for any $v \in V$. The *watching system number* of G denote by $\omega(G)$ is the minimum size of watching systems of G.

Note that an identifying code of G, when exists, define a watching system for G, but in a watching system, the selection of neighbor vertices is favorite as watching area from a watcher. This issue is the differentiate of a watching system to an identifying code.

In this paper, we consider the triangular graph, $T(n)$, and compute watching system number of this family of graphs.

2. **Main Theorem**

In this section, we study the watching systems of triangular graph $T(n)$. At first, we mention some earlier results about the watching system of general graphs which are used in the rest of this section.

Theorem 2.1. [2]
a) If G is twin free graph, then $\gamma(G) \le w(G) \le i(G)$,
b)$\lceil log_2 n + 1\rceil \le w(G) \le \gamma(G)\lceil \Delta(G) + 2\rceil$,
c) If G is connected graph, then $w(G) \le \frac{2n}{3}$,
d) If H is a spanning subgraph, then $w(G) \le w(H)$.

Let K_n be the complete graph of order n with vertex set $V = \{v_1, \ldots, v_n\}$. Hence we can assume that $T(n)$ is a graph with vertex set $e_{ij} = v_i v_j \in E(G)$, and two vertices of $T(G)$ is adjacent if and only if two edges of T_n, has non-empty intersection. Suppose that $W = \{\omega_1, \omega_2, \ldots, \omega_m\}$ is a watching system for $T(n)$, with $\omega_l = (e_l, Z_l)$, where $e_l = v_{i_l} v_{j_l}$ is an edge of $K_n, 1 \le i_l, j_l \le n$. In addition $Z_l \subseteq N_{T(n)}[e_l]$. Conceder the following notation:

$E_W = \{e_l : (e_l, Z_l) \in W\}$,

$V_W = \cup_{e_l \in E_W} e_l$,
$R_i =$ The set of elements of V which appear i times in the elements of E_W,
$r_i = |R_i|$.

Lemma 2.2. *With the above notations, we have*
a) $r_0 \leq 1$.
b) $|\{l : e_l \in E_l, e_l \subseteq R_1\}| \leq 1$, *in addition if* $r_0 = 1$, *then* $\{l : e_l \in E_l, e_l \subseteq R_1\} = \emptyset$.

Proof. a) Suppose that $r_0 > 1$. Hence we can choose two distinct elements $a, b \in R_0$. Therefore $e = ab \notin N[e_l]$ for any $e_l \in E_W$ and this fact implies that $L_W(ab) = \emptyset$, which is a contradiction.
b) Suppose that there are $e_t, e_l \in E_W$, such that $e_l = ab, e_t = cd$, and $a, b, c, d \in R_1$. Set $\omega_l = (e_l, Z_l), \omega_t = (e_t, Z_t)$. We have

$$L_W(ab) = \{\omega_l\}, L_W(cd) = \{\omega_t\}.$$

Since $L_W(ac), L_W(ad) \subseteq \{\omega_l, \omega_t\}$, we conclude that

$$L_W(ac) = L_W(ad) = \{\omega_l, \omega_t\}.$$

Hence

$$|\{l : e_l \in E_l, e_l \subseteq R_1\}| \leq 1.$$

Let $r_0 = 1$ and $v_n \notin e_l$, for any $e_l \in E_W$. If $e_t = ab \subseteq R_1$, then $L_W(av_n) = L_W(bv_n) = \{\omega_t\}$, which is a contradiction. □

Lemma 2.3. *With the above notations,*
a) If $r_0 = 0$, *then* $m \geq \lceil \frac{2n-1}{3} \rceil$.
b) If $r_0 = 1$, *then* $m \geq \lceil \frac{2(n-1)}{3} \rceil$.

Proof. a) We know $r_1 + 2(n - r_1) \leq 2m$. On the other hands $r_1 \leq m+1$ by Lemma 2.2. So $2n \leq 3m+1$. Therefore $m \geq \lceil \frac{2n-1}{3} \rceil$.
b) We know $r_1 + 2(n - 1 - r_1) \leq 2m$, by Lemma 2.2, $r_1 \leq m$. So $2n - 2 \leq 3m$ and the result is obtained. □

Theorem 2.4. *Let* $n \geq 3$ *be an integer. Then* $\omega(T(n)) = \lceil \frac{2n}{3} \rceil$.

Proof. Let $V(K_n) = \{v_1, v_2, \ldots, v_n\}$. We consider three cases:

Case 1: $n = 3k$, where $k \geq 3$ is an integer. For $n = 3$ we have $T(3) \cong K_3$ and therefore $\omega(T(3)) = 2$ by Lemma 2.1. For $n = 6$, consider the set

$$W = \{\omega_1, \omega_2, \omega_3, \omega_4\},$$

where

$$\omega_1 = (v_1v_2, N(v_1v_2)), \omega_2 = (v_1v_3, N(v_1v_3)),$$

$$\omega_1 = (v_1v_4, N(v_1v_4)), \omega_2 = (v_1v_5, N(v_1v_5)).$$

It is not difficult to see that W is a watching system for $T(6)$. Hence $\omega(T(6)) \leq 4$ and by applying Lemma 2.3, we conclude that $\omega(T(6)) = 4$.

Now suppose that $n \geq 9$. For $1 \leq i \leq k$, set $e_i = (v_{3i}v_{3i-1}, Z_i), f_i = (v_{3i}v_{3i-2}, T_i)$, where $Z_i = N(v_{3i}v_{3i-1})$, and $T_i = N(v_{3i}v_{3i-2})$. We claim that the set $W = \{e_i, f_i : 1 \leq i \leq k\}$, is a watching system for $T(n)$. We prove this claim by induction on k. Let $k = 3$. It is not difficult to show that the set

$$W = \{e_1, e_2, e_3, f_1, f_2, f_3\}$$

is a watching system for $T(9)$.

Suppose that our claim is true for $k-1$. Hence $W' = \{e'_i, f'_i : 1 \leq i \leq k-1\}$, where $e'_i = (v_{3i}v_{3i-1}, Z'_i), f'_i = (v_{3i}v_{3i-2}, T'_i), Z'_i = N_{T(n-3)}(v_{3i}v_{3i-1})$ and $T'_i = N_{T(n-3)}(v_{3i}v_{3i-2})$ is a watching system for $T(n-3)$. We show that the set $W = \{e_i, f_i : 1 \leq i \leq k\}$, where $e_i = (v_{3i}v_{3i-1}, Z_i), f_i = (v_{3i}v_{3i-2}, T_i)$, $Z_i = N_{T(n)}(v_{3i}v_{3i-1})$, and $T_i = N_{T(n)}(v_{3i}v_{3i-2})$ is a watching system for $T(n)$. For $1 \leq i \leq j \leq n-3$, we have

$$L_W(v_iv_j) = \{f_r, e_s : f'_r, e'_s \in L_{W'}(v_iv_j)\}.$$

For $j \leq k-1$, we have

$$L_W(v_nv_{3j}) = \{f_j, e_j, f_k, e_k\},$$

$$L_W(v_{n-1}v_{3j}) = \{f_j, e_j, e_k\},$$

$$L_W(v_{n-2}v_{3j}) = \{f_j, e_j, f_k\},$$

$$L_W(v_nv_{3j-1}) = \{e_j, f_k, e_k\},$$

$$L_W(v_{n-1}v_{3j-1}) = \{e_j, e_k\},$$

$$L_W(v_{n-2}v_{3j-1}) = \{e_j, f_k\},$$

$$L_W(v_nv_{3j-2}) = \{f_j, f_k, e_k\},$$

$$L_W(v_{n-1}v_{3j-2}) = \{f_j, e_k\},$$

$$L_W(v_{n-2}v_{3j-2}) = \{f_j, f_k\}.$$

In addition,

$$L_W(v_nv_{n-1}) = \{f_k\}, L_W(v_nv_{n-2}) = \{e_k\}, L_W(v_{n-1}v_{n-2}) = \{e_k, f_k\}.$$

Since W' is a watching system of $T(n-3)$, we conclude that W is a watching system of $T(n)$. Hence $w(T(n)) \leq \frac{2n}{3}$. On the other hand $\omega(T(n)) \geq \frac{2n}{3}$ by Lemma 2.3. Hence $\omega(T(n)) = \frac{2n}{3}$.

Case 2: $n = 3k+1$, $k \geq 1$.

Set $w_1 = (v_1v_3, A_1), w_2 = (v_2v_3, A_2)$ and $w_3 = (v_3v_4, A_3)$, where

$$A_1 = N[v_1v_3] \setminus \{v_2v_3, v_3v_4\}, A_2 = N[v_2v_3] \setminus \{v_1v_3, v_3v_4\}, A_3 = N[v_3v_4] \setminus \{v_1v_3, v_2v_3\}.$$

For $2 \leq i \leq k$, set $e_i = (v_{3i}v_{3i-1}, Z_i)$, $f_i = (v_{3i}v_{3i+1}, T_i)$, where $Z_i = N(v_{3i}v_{3i-1})$, and $T_i = N(v_{3i}v_{3i+1})$. We claim that the set $W = \{w_1, w_2, w_3, e_i, f_i : 2 \leq i \leq k\}$, is a watching system for $T(n)$. We prove this claim by induction on k. For $k = 1$ we have

$$W = \{A_1, A_2, A_3\}$$

and this implies that

$$L_W(v_1v_2) = \{A_1, A_2\}, L_W(v_1v_3) = \{A_1\}, L_W(v_1v_4 = \{A_1, A_3\}$$

$$L_W(v_2v_3) = \{A_2\}, L_W(v_2v_4) = \{A_2, A_3\}, L_W(v_3v_4) = \{A_1, A_2, A_3\}.$$

Hence W is a watching system for $T(4)$. The argument of the rest of proof is similar with the Case 1. Hence $\omega(T(n)) \leq 2k + 1 = \lceil \frac{2n}{3} \rceil$

Also $\omega(T(n)) \geq 2k$ by Lemma 2.3. If $\omega(T(n)) = 2k$, then $r_0 = 1$, by applying Lemma 2.3. Hence $|R_1| = 2k, |R_2| = k$ and hence there are two watcher $\omega_1 = (v_1v_3, Z_1), \omega_2 = (v_2v_3, Z_2)$ with $v_1, v_2 \in R_1$ and $v_3 \in R_2$. Suppose that $v_4 \notin V_W$. Hence the labeling sets

$$L_W(v_1v_2), L_W(v_1v_4), L_W(v_2v_3), L_W(v_2v_4),$$

are non-empty distinct subsets $\{\omega_1, \omega_2\}$ and this is a contradiction. Therefore $2k + 1 \leq \omega(T(n))$. Hence

$$\omega(T(n)) = 2k + 1 = \lceil \frac{2n}{3} \rceil.$$

Hence $\omega(T(n)) = \lceil \frac{2n}{3} \rceil$.

Case 3: $n = 3k + 2, k \geq 1$.

In this case consider the set

$$W = \{w_1, w_2, w_3, w_4, e_i, f_i : 2 \leq i \leq k\},$$

where

$$w_1 = (v_1v_3, N[v_1v_3] \setminus \{v_2v_3, v_3v_4, v_3v_5\}), w_2 = (v_2v_3, N[v_2v_3] \setminus \{v_1v_3, v_3v_4, v_3v_5\})$$

$$w_3 = (v_3v_4, N[v_3v_4] \setminus \{v_1v_3, v_2v_3, v_3v_5\}), w_4 = (v_3v_5, N[v_3v_5] \setminus \{v_1v_3, v_2v_3, v_3v_4\})$$

and for $2 \leq i \leq k$

$$e_i = (v_{3i}v_{3i+1}, Z_i), f_i = (v_{3i}v_{3i+2}, T_i),$$

with

$$Z_i = N(v_{3i}v_{3i+1}), T_i = N(v_{3i}v_{3i+2}).$$

For $k = 1$, we have $W = \{w_1, w_2, w_3, w_4\}$ and therefore

$$L_W(v_1v_2) = \{w_1, w_2\}, L_W(v_1v_3) = \{w_1\}, L_W(v_1v_4) = \{w_1, w_3\}, L_W(v_1v_5) = \{w_1, w_4\}$$

$$L_W(v_2v_3) = \{w_2\}, L_W(v_2v_4) = \{w_2, w_3\}, L_W(v_2v_5) = \{w_2, w_4\},$$

$$L_W(v_3v_4) = \{w_3\}, L_W(v_3v_5) = \{w_4\}, L_W(v_4v_5) = \{w_3, w_4\}.$$

Hence W ia a watching system for $T(5)$. Now by induction and the same argument as Case 1, we prove that W is a watching system for $T(n)$. Hence $\omega(T(n)) \leq 2k + 2$. By applying Lemma 2.3. we

have $w(T(n)) \geq 2k+1$.
For $k=1$, the graph $T(5)$ has 10 vertices and hence $\omega(T(5)) \geq 4$ and hence $\omega(T(5)) = 4 = 2k+2$. Suppose that $k \geq 2$ and $w(T(n)) = 2k+1$. Recall that R_i be the set of $V = \{v_1, v_2, \dots, v_n\}$ which appear i times in the element of E_W and $|R_i| = r_i$. By the Lemma 2.3, $r_0 \leq 1$. If $r_0 = 1$, then $|R_1| = 2k, |R_2| = k+1$. Since $k \geq 2$, then there exist $v_1, v_2 \in R_1, v_3 \in R_2$ such that $v_1v_3, v_2v_3 \in E_W$. Hence there are two watchers $w_1 = (v_1v_3, Z_1), w_2 = (v_2v_3, Z_2) \in W$ with $Z_1 \subseteq N[v_1v_3]$ and $Z_2 \subseteq N[v_2v_3]$. Suppose that $v_n \in R_0$. Hence the sets

$$L_W(v_1v_2), L_W(v_1v_3), L_W(v_1v_n), L_W(v_2v_n)$$

are distinct non-empty subsets of $\{w_1, w_2\}$, which is a contradiction. If $r_0 = 0$, then $|R_1| = 2k+2, |R_2| = k$. By applying Lemma 2.3, there are watchers

$$w_1 = (v_1v_4, Z_1), w_2 = (v_2v_5, Z_2), W_3 = (v_3v_5, Z_3)$$

belong to W such that $v_1, v_2, v_3, v_4 \in R_1$ and $v_5 \in R_2$. Hence the sets

$$L_W(v_1v_2), L_W(v_1v_3), L_W(v_1v_4), L_W(v_1v_5),$$

$$L_W(v_2v_3), L_W(v_2v_4), L_W(v_2v_5), L_W(v_3v_4),$$

are distinct non-empty subsets of $\{w_1, w_2, w_3\}$, which is a contradiction. Hence

$$2k+1 < \omega(T(n)) \leq 2k+2,$$

and we conclude that $\omega(T(n)) = 2k+2 = \lceil \frac{2n}{3} \rceil$. □

Remark 2.5. *In* [8], *it is proved that $\iota(T(n)) = n-1$ for $n > 5$ and for $n = 4, 5, \iota(T(n)) = 5$. In this paper we show that $\omega(T(n)) = \lceil \frac{2n}{3} \rceil$. This means that in this family of graphs the watching system is more efficient than identifying code.*

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