

Transactions on CombinatoricsISSN (print): 2251-8657, ISSN (on-line): 2251-8665Vol. 3 No. 1 (2014), pp. 51-57.© 2014 University of Isfahan



WATCHING SYSTEMS OF TRIANGULAR GRAPHS

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Communicated by Ali Reza Ashrafi

ABSTRACT. A watching system in a graph G = (V, E) is a set $W = \{\omega_1, \omega_2, \dots, \omega_k\}$, where $\omega_i = (v_i, Z_i), v_i \in V$ and Z_i is a subset of closed neighborhood of v_i such that the sets $L_W(v) = \{\omega_i : v \in Z_i\}$ are non-empty and distinct, for any $v \in V$. In this paper, we study the watching systems of line graph K_n which is called triangular graph and denoted by T(n). The minimum size of a watching system of G is denoted by $\omega(G)$. We show that $\omega(T(n)) = \lceil \frac{2n}{3} \rceil$.

1. Introduction

Throughout this paper we will assume that all graphs are finite, simple, and undirected. We use [5] for terminology and notations not defined here.

For a graph G, let V = V(G) and E(G) denote the set of vertices, edges of G, respectively. For a vertex $x \in V(G)$, the degree of x, denoted deg(x), is the number of edges of G incident with x. We use $\Delta(G) = \Delta$ to denote the maximum degree of vertices of G. For any vertex $v \in V(G)$, the open neighborhood of v is the set $N_G(v) = \{u : uv \in E\}$, while the closed neighborhood of v is the set $N_G[v] = N_G(v) \cup \{v\}$. A complete graph is a graph in which each pair of distinct vertices is joined by an edge. We denote the complete graph with n vertices by K_n . A subgraph H of a graph G is called a spanning subgraph if V(H) = V(G). For every nonnegative integer r, the graph G is called r-regular if the degree of each vertex of G is equal to r. The triangular graph, T(n), is the line graph of the complete graph K_n . The vertices of T(n) may be identified with the 2-subsets of $\{v_1, v_2, \ldots, v_n\}$ that are adjacent if and only if the 2-subsets have a nonempty intersection.

MSC(2010): Primary: 05C05; Secondary: 05C12.

Keywords: Identifying code, Watching system, Triangular graph.

Received: 26 December 2013, Accepted: 9 January 2014.

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Let $C \subseteq V$ be a subset of vertices of G and for all vertex $v \in V$, we define

$$I_C(v) := N_G[v] \bigcap C$$

If all $I_C(v)$'s are nonempty, then C is a dominating set (or a covering code) of G; if moreover the $I_C(v)$'s are all distinct, then we say that C is an *identifying code* of G. The set $I_C(v)$ is called the identifying set of the vertex v. Note that the graph G has an identifying code if and only if distinct vertices must have distinct closed neighborhoods. A graph with this property is called *twin-free* or *identifiable*. The minimum size of a identifying code in G is denoted by $\iota(G)$. This concept was introduced in 1998 in [6] to model fault diagnosis in multiprocessor systems and have since been studied widely in the communities of both graph theory and coding theory, for instance see [1, 3, 4, 7, 9]. They are also used for the design of indoor detection systems based on wireless sensor network.

Watching systems were introduced in [2], is a generalization of identifying codes. A watcher ω of Gis a couple $\omega = (v_i, Z_i)$, where v_i is a vertex and $Z_i \subseteq N_G[v_i]$. We will say that ω is located at v_i and that Z_i is its watching area or watching zone. A watching system in a graph G is a finite set $W = \{\omega_1, \omega_2, \ldots, \omega_k\}$ where ω_i , $1 \leq i \leq k$ is a watcher such that $\{Z_1, Z_2, \ldots, Z_k\}$ is an identifying system. So W is a watching system for G if the sets $L_W(v) = \{\omega_i : v \in Z_i\}$ are non-empty and distinct, for any $v \in V$. The watching system number of G denote by $\omega(G)$ is the minimum size of watching systems of G.

Note that an identifying code of G, when exists, define a watching system for G, but in a watching system, the selection of neighbor vertices is favorite as watching area from a watcher. This issue is the differentiate of a watching system to an identifying code.

In this paper, we consider the triangular graph, T(n), and compute watching system number of this family of graphs.

2. Main Theorem

In this section, we study the watching systems of triangular graph T(n). At first, we mention some earlier results about the watching system of general graphs which are used in the rest of this section.

Theorem 2.1. [2]

a) If G is twin free graph, then $\gamma(G) \leq w(G) \leq i(G)$,

- $b)\lceil log_2n+1\rceil \leq w(G) \leq \gamma(G)\lceil \Delta(G)+2\rceil,$
- c) If G is connected graph, then $w(G) \leq \frac{2n}{3}$,
- d) If H is a spanning subgraph, then $w(G) \leq w(H)$.

Let K_n be the complete graph of order n with vertex set $V = \{v_1, \ldots, v_n\}$. Hence we can assume that T(n) is a graph with vertex set $e_{ij} = v_i v_j \in E(G)$, and two vertices of T(G) is adjacent if and only if two edges of T_n , has non-empty intersection. Suppose that $W = \{\omega_1, \omega_2, \ldots, \omega_m\}$ is a watching system for T(n), with $\omega_l = (e_l, Z_l)$, where $e_l = v_{il}v_{jl}$ is an edge of $K_n, 1 \leq i_l, j_l \leq n$. In addition $Z_l \subseteq N_{T(n)}[e_l]$. Conceder the following notation:

$$E_W = \{e_l : (e_l, Z_l) \in W\},\$$

 $V_W = \cup_{e_l \in E_W} e_l,$

 R_i = The set of elements of V which appear i times in the elements of E_W , $r_i = |R_i|$.

Lemma 2.2. With the above notations, we have

a) $r_0 \leq 1$. b) $|\{l : e_l \in E_l, e_l \subseteq R_1\}| \leq 1$, in addition if $r_0 = 1$, then $\{l : e_l \in E_l, e_l \subseteq R_1\} = \emptyset$.

Proof. a) Suppose that $r_0 > 1$. Hence we can choose two distinct elements $a, b \in R_0$. Therefore $e = ab \notin N[e_l]$ for any $e_l \in E_W$ and this fact implies that $L_W(ab) = \emptyset$, which is a contradiction.

b) Suppose that there are $e_t, e_l \in E_W$, such that $e_l = ab, e_t = cd$, and $a, b, c, d \in R_1$. Set $\omega_l = (e_l, Z_l), \omega_t = (e_t, Z_t)$. We have

$$L_W(ab) = \{\omega_l\}, L_W(cd) = \{\omega_t\}$$

Since $L_W(ac), L_W(ad) \subseteq \{\omega_l, \omega_t\}$, we conclude that

$$L_W(ac) = L_W(ad) = \{\omega_l, \omega_t\}.$$

Hence

$$|\{l: e_l \in E_l, e_l \subseteq R_1\}| \le 1.$$

Let $r_0 = 1$ and $v_n \notin e_l$, for any $e_l \in E_W$. If $e_t = ab \subseteq R_1$, then $L_W(av_n) = L_W(bv_n) = \{\omega_t\}$, which is a contradiction.

Lemma 2.3. With the above notations, a) If $r_0 = 0$, then $m \ge \lceil \frac{2n-1}{3} \rceil$. b) If $r_0 = 1$, then $m \ge \lceil \frac{2(n-1)}{3} \rceil$.

Proof. a) We know $r_1 + 2(n-r_1) \le 2m$. On the other hands $r_1 \le m+1$ by Lemma 2.2. So $2n \le 3m+1$. Therefore $m \ge \lfloor \frac{2n-1}{3} \rfloor$.

b) We know $r_1 + 2(n - 1 - r_1) \le 2m$, by Lemma 2.2, $r_1 \le m$. So $2n - 2 \le 3m$ and the result is obtained.

Theorem 2.4. Let $n \ge 3$ be an integer. Then $\omega(T(n)) = \lceil \frac{2n}{3} \rceil$.

Proof. Let $V(K_n) = \{v_1, v_2, \ldots, v_n\}$. We consider three cases:

Case 1: n = 3k, where $k \ge 3$ is an integer. For n = 3 we have $T(3) \cong K_3$ and therefore $\omega(T(3)) = 2$ by Lemma 2.1. For n = 6, consider the set

$$W = \{\omega_1, \omega_2, \omega_3, \omega_4\},\$$

where

$$\omega_1 = (v_1 v_2, N(v_1 v_2)), \omega_2 = (v_1 v_3, N(v_1 v_3)),$$

$$\omega_1 = (v_1 v_4, N(v_1 v_4)), \omega_2 = (v_1 v_5, N(v_1 v_5)).$$

It is not difficult to see that W is a watching system for T(6). Hence $\omega(T(6)) \leq 4$ and by applying Lemma 2.3, we conclude that $\omega(T(6)) = 4$.

Now suppose that $n \ge 9$. For $1 \le i \le k$, set $e_i = (v_{3i}v_{3i-1}, Z_i)$, $f_i = (v_{3i}v_{3i-2}, T_i)$, where $Z_i = N(v_{3i}v_{3i-1})$, and $T_i = N(v_{3i}v_{3i-2})$. We claim that the set $W = \{e_i, f_i : 1 \le i \le k\}$, is a watching system for T(n). We prove this claim by induction on k. Let k = 3. It is not difficult to show that the set

$$W = \{e_1, e_2, e_3, f_1, f_2, f_3\}$$

is a watching system for T(9).

Suppose that our claim is true for k - 1. Hence $W' = \{e'_i, f'_i : 1 \le i \le k - 1\}$, where $e'_i = (v_{3i}v_{3i-1}, Z'_i), f'_i = (v_{3i}v_{3i-2}, T'_i), Z'_i = N_{T(n-3)}(v_{3i}v_{3i-1})$ and $T'_i = N_{T(n-3)}(v_{3i}v_{3i-2})$ is a watching system for T(n-3). We show that the set $W = \{e_i, f_i : 1 \le i \le k\}$, where $e_i = (v_{3i}v_{3i-1}, Z_i), f_i = (v_{3i}v_{3i-2}, T_i), Z_i = N_{T(n)}(v_{3i}v_{3i-1}),$ and $T_i = N_{T(n)}(v_{3i}v_{3i-2})$ is a watching system for T(n). For $1 \le i \le j \le n-3$, we have

$$L_W(v_i v_j) = \{f_r, e_s : f'_r, e'_s \in L_{W'}(v_i v_j)\}$$

For $j \leq k - 1$, we have

$$L_W(v_n v_{3j}) = \{f_j, e_j, f_k, e_k\},\$$

$$L_W(v_{n-1}v_{3j}) = \{f_j, e_j, e_k\},\$$

$$L_W(v_{n-2}v_{3j}) = \{f_j, e_j, f_k\},\$$

$$L_W(v_n v_{3j-1}) = \{e_j, f_k, e_k\},\$$

$$L_W(v_{n-1}v_{3j-1}) = \{e_j, f_k\},\$$

$$L_W(v_n v_{3j-2}) = \{f_j, f_k, e_k\},\$$

$$L_W(v_{n-1}v_{3j-2}) = \{f_j, e_k\},\$$

$$L_W(v_{n-2}v_{3j-2}) = \{f_j, e_k\},\$$

In addition,

$$L_W(v_n v_{n-1}) = \{f_k\}, L_W(v_n v_{n-2}) = \{e_k\}, L_W(v_{n-1} v_{n-2}) = \{e_k, f_k\}.$$

Since W' is a watching system of T(n-3), we conclude that W is a watching system of T(n). Hence $w(T(n)) \leq \frac{2n}{3}$. On the other hand $\omega(T(n)) \geq \frac{2n}{3}$ by Lemma 2.3. Hence $\omega(T(n)) = \frac{2n}{3}$.

Case 2: $n = 3k + 1, k \ge 1$. Set $w_1 = (v_1v_3, A_1), w_2 = (v_2v_3, A_2)$ and $w_3 = (v_3v_4, A_3)$, where

 $A_1 = N[v_1v_3] \setminus \{v_2v_3, v_3v_4\}, A_2 = N[v_2v_3] \setminus \{v_1v_3, v_3v_4\}, A_3 = N[v_3v_4] \setminus \{v_1v_3, v_2v_3\}.$

For $2 \leq i \leq k$, set $e_i = (v_{3i}v_{3i-1}, Z_i)$, $f_i = (v_{3i}v_{3i+1}, T_i)$, where $Z_i = N(v_{3i}v_{3i-1})$, and $T_i = N(v_{3i}v_{3i+1})$. We claim that the set $W = \{w_1, w_2, w_3, e_i, f_i : 2 \leq i \leq k\}$, is a watching system for T(n). We prove this claim by induction on k. For k = 1 we have

$$W = \{A_1, A_2, A_3\}$$

and this implies that

$$L_W(v_1v_2) = \{A_1, A_2\}, L_W(v_1v_3) = \{A_1\}, L_W(v_1v_4 = \{A_1, A_3\})$$
$$L_W(v_2v_3) = \{A_2\}, L_W(v_2v_4) = \{A_2, A_3\}, L_W(v_3v_4) = \{A_1, A_2, A_3\}.$$

Hence W is a watching system for T(4). The argument of the rest of proof is similar with the Case 1. Hence $\omega(T(n)) \leq 2k + 1 = \lceil \frac{2n}{3} \rceil$

Also $\omega(T(n)) \ge 2k$ by Lemma 2.3. If $\omega(T(n)) = 2k$, then $r_0 = 1$, by applying Lemma 2.3. Hence $|R_1| = 2k, |R_2| = k$ and hence there are two watcher $\omega_1 = (v_1v_3, Z_1), \omega_2 = (v_2v_3, Z_2)$ with $v_1, v_2 \in R_1$ and $v_3 \in R_2$. Suppose that $v_4 \notin V_W$. Hence the labeling sets

$$L_W(v_1v_2), L_W(v_1v_4), L_W(v_2v_3), L_W(v_2v_4),$$

are non-empty distinct subsets $\{\omega_1, \omega_2\}$ and this is a contradiction. Therefore $2k + 1 \leq \omega(T(n))$. Hence

$$\omega(T(n)) = 2k + 1 = \lceil \frac{2n}{3} \rceil.$$

Hence $\omega(T(n)) = \lceil \frac{2n}{3} \rceil$.

Case 3: $n = 3k + 2, k \ge 1$.

In this case consider the set

$$W = \{w_1, w_2, w_3, w_4, e_i, f_i : 2 \le i \le k\},\$$

where

$$w_1 = (v_1v_3, N[v_1v_3] \setminus \{v_2v_3, v_3v_4, v_3v_5\}), w_2 = (v_2v_3, N[v_2v_3] \setminus \{v_1v_3, v_3v_4, v_3v_5\})$$

$$w_3 = (v_3v_4, N[v_3v_4] \setminus \{v_1v_3, v_2v_3, v_3v_5\}), w_4 = (v_3v_5, N[v_3v_5] \setminus \{v_1v_3, v_2v_3, v_3v_4\})$$

and for $2 \leq i \leq k$

$$e_i = (v_{3i}v_{3i+1}, Z_i), f_i = (v_{3i}v_{3i+2}, T_i)$$

with

$$Z_i = N(v_{3i}v_{3i+1}), T_i = N(v_{3i}v_{3i+2}).$$

For k = 1, we have $W = \{w_1, w_2, w_3, w_4\}$ and therefore

$$\begin{split} L_W(v_1v_2) &= \{w_1, w_2\}, L_W(v_1v_3) = \{w_1\}, L_W(v_1v_4) = \{w_1, w_3\}, L_W(v_1v_5) = \{w_1, w_4\} \\ \\ L_W(v_2v_3) &= \{w_2\}, L_W(v_2v_4) = \{w_2, w_3\}, L_W(v_2v_5) = \{w_2, w_4\}, \\ \\ L_W(v_3v_4) &= \{w_3\}, L_W(v_3v_5) = \{w_4\}, L_W(v_4v_5) = \{w_3, w_4\}. \end{split}$$

Hence W is a watching system for T(5). Now by induction and the same argument as Case 1, we prove that W is a watching system for T(n). Hence $\omega(T(n)) \leq 2k + 2$. By applying Lemma 2.3. we

have $w(T(n)) \ge 2k + 1$.

For k = 1, the graph T(5) has 10 vertices and hence $\omega(T(5)) \ge 4$ and hence $\omega(T(5)) = 4 = 2k + 2$. Suppose that $k \ge 2$ and w(T(n)) = 2k + 1. Recall that R_i be the set of $V = \{v_1, v_2, \ldots, v_n\}$ which appear *i* times in the element of E_W and $|R_i| = r_i$. By the Lemma 2.3, $r_0 \le 1$. If $r_0 = 1$, then $|R_1| = 2k, |R_2| = k + 1$. Since $k \ge 2$, then there exist $v_1, v_2 \in R_1, v_3 \in R_2$ such that $v_1v_3, v_2v_3 \in E_W$. Hence there are two watchers $w_1 = (v_1v_3, Z_1), w_2 = (v_2v_3, Z_2) \in W$ with $Z_1 \subseteq N[v_1v_3]$ and $Z_2 \subseteq N[v_2v_3]$. Suppose that $v_n \in R_0$. Hence the sets

$$L_W(v_1v_2), L_W(v_1v_3), L_W(v_1v_n), L_W(v_2v_n)$$

are distinct non-empty subsets of $\{w_1, w_2\}$, which is a contradiction. If $r_0 = 0$, then $|R_1| = 2k + 2$, $|R_2| = k$. By applying Lemma 2.3, there are watchers

$$w_1 = (v_1v_4, Z_1), w_2 = (v_2v_5, Z_2), W_3 = (v_3v_5, Z_3)$$

belong to W such that $v_1, v_2, v_3, v_4 \in R_1$ and $v_5 \in R_2$. Hence the sets

$$L_W(v_1v_2), L_W(v_1v_3), L_W(v_1v_4), L_W(v_1v_5),$$

$$L_W(v_2v_3), L_W(v_2v_4), L_W(v_2v_5), L_W(v_3v_4),$$

are distinct non-empty subsets of $\{w_1, w_2, w_3\}$, which is a contradiction. Hence

$$2k+1 < \omega(T(n)) \le 2k+2$$

and we conclude that $\omega(T(n)) = 2k + 2 = \lceil \frac{2n}{3} \rceil$.

Remark 2.5. In [8], it is proved that $\iota(T(n)) = n - 1$ for n > 5 and for $n = 4, 5, \iota(T(n)) = 5$. In this paper we show that $\omega(T(n)) = \lceil \frac{2n}{3} \rceil$. This means that in this family of graphs the watching system is more efficient than identifying code.

Acknowledgments

The authors are deeply grateful to the referee for his/her valuable suggestions.

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