www.combinatorics.ir

Transactions on Combinatorics
ISSN (print): 2251-8657, ISSN (on-line): 2251-8665
Vol. 3 No. 2 (2014), pp. 1-9.

www.ui.ac.ir

GROUP MAGICNESS OF CERTAIN PLANAR GRAPHS

M. J. NIKMEHR* AND S. BAHRAMIAN

Communicated by Hamid Reza Maimani

Abstract. Let A be a non-trivial abelian group and $A^* = A \setminus \{0\}$. A graph G is said to be A-magic graph if there exists a labeling $l : E(G) \to A^*$ such that the induced vertex labeling $l^+ : V(G) \to A$, define by

$$l^+(v) = \sum_{uv \in E(G)} l(uv)$$

is a constant map. The set of all constant integers such that $\sum_{u \in N(v)} l(uv) = c$, for each $v \in N(v)$, where $N(v)$ denotes the set of adjacent vertices to vertex v in G, is called the index set of G and denoted by $\mathrm{In}_A(G)$. In this paper we determine the index set of certain planar graphs for $\mathbb{Z}_h$, where $h \in \mathbb{N}$, such as wheels and fans.

1. **Introduction**

For an abelian group A (written additively) let $A^* = A \setminus \{0\}$. A map $l : E(G) \to A^*$ is called a *labeling* of G. Given a labeling on the edge set of G one can introduce a vertex labeling $l^+ : V(G) \to A$, by $l^+(v) = \sum_{uv \in E(G)} l(uv)$. A graph G is said to be A-*magic labeling* if there is a labeling $l : E(G) \to A^*$ such that for every vertex $v \in V(G)$, the sum of values of all edges incident with v is equal to the same constant; that is, $l^+(v) = c$ for sum fixed $c \in A$. A magic graph introduced by J. Sedlaced [5, 6]. An A-magic graph G is said to be $\mathbb{Z}_h$-*magic graph* if we choose the group A as $\mathbb{Z}_h$ the group of integers mod h. These $\mathbb{Z}_h$-magic graph are referred as h-*magic graphs*. Clearly, if a graph is h-magic, then it is not necessary k-magic ($k \neq h$). An h-magic graph G is said to be h-*zero-sum* if there is a magic labeling of G in $\mathbb{Z}_h$ which induces a vertex labeling with sum zero. The *null set* of a graph G, denoted by $N(G)$, is the set of all natural numbers $h \in \mathbb{N}$, such that G admits an h-zero-sum magic labeling. Let G be a graph and $l : E(G) \to A^*$ be a labeling on the edge set of G and A be an abelian group. The set of all constant integers such that $\sum_{u \in N(v)} l(uv) = c$, for each $u \in N(v)$, where $N(v)$ denotes

MSC(2010): Primary: 05C15; Secondary: 20D60.

Keywords: Index Set, Magic, Zero-Sum, Null Set.

Received: 23 September 2012, Accepted: 19 January 2014.

*Corresponding author.

the set of adjacent vertices to vertex v, is called the *index set* of G and denoted by $\mathrm{In}_A(G)$. The join $G \vee H$ of disjoint graph G and H is the graph obtained from $G+H$ by joining each vertex of G to each vertex of H. Tao-Ming wang and Shi-wei Hu [7] in (2011), determined the index set of regular graphs for an abelian group $\mathbb{Z}$. A detailed study about zero-sum magic graphs and their null sets done by E. Salehi in [1, 2]. It was determined in [3, 4] null set of wheels and fans that is, $0 \in \mathrm{In}_{\mathbb{Z}_h}(G)$, where G is the above graphs. In this paper we determine the index set of wheels and fans.

2. Index Set of Wheels

For $n \geq 3$, wheel on $n+1$ vertices, denoted by $\mathrm{W_n}$ and is defined to be $C_n \vee K_1$, where C_n is the cycle of order n. The null set of wheels determined in [3].

Theorem 2.1. *For any positive integer $n \geq 3$,* $N(\mathrm{W_n}) = \begin{cases} \mathbb{N} \setminus \{2\} & n = 0 \pmod 3; \\ \mathbb{N} \setminus \{2,3\} & \text{otherwise.} \end{cases}$

Let G be a graph. It is obvious that $1 \in \mathrm{In}_{\mathbb{Z}_2}(\mathrm{G})$ if and only if the degree of every vertex is odd. Therefore, since the degree set of the $\mathrm{W_n}$ is $\{3, n\}$, $\mathrm{W_n}$ has 2-magic labeling with index 1, if and only if n is odd. First we need the following.

Remark 2.2. If G is a graph and $c \in \mathrm{In}_{\mathbb{Z}_h}(G)$, then we have

$$2 \sum_{e \in E(G)} l(e) = c \mid V(G) \mid \qquad (\mathrm{mod}\ h).$$

Lemma 2.3. *For any positive integers n and $h \geq 3$,* $\mathrm{In}_{\mathbb{Z}_\mathrm{h}}(\mathrm{C_n}) = \begin{cases} \mathbb{Z}_h \setminus \{0\} & n \text{ is odd;} \\ \mathbb{Z}_h & n \text{ is even.} \end{cases}$

Proof. First suppose that n is odd. Note that in any h-magic labeling of an odd cycle, the edges should have the same value. So, $0 \notin \mathrm{In}_{\mathbb{Z}_\mathrm{h}}(\mathrm{C_n})$. Let $h = 2k+1$ and $x \in \mathbb{Z}_h \setminus \{0\}$. We assign value $(k+1)x$ to all edges of C_n. So, $x \in \mathrm{In}_{\mathbb{Z}_\mathrm{h}}(\mathrm{C_n})$ and thus $\mathrm{In}_{\mathbb{Z}_\mathrm{h}}(\mathrm{C_n}) = \mathbb{Z}_\mathrm{h} \setminus \{0\}$.
Now, assume that n is even and $x \in \mathbb{Z}_h$. If assign value 2 and -1 to all edges of C_n, alternatively, then $1 \in \mathrm{In}_{\mathbb{Z}_\mathrm{h}}(\mathrm{C_n})$. Suppose that $x \neq 1$. We assign the values $x-1$ and 1, alternatively to all edges of C_n. So, $x \in \mathrm{In}_{\mathbb{Z}_\mathrm{h}}(\mathrm{C_n})$, as desired. □

Let $u, u_1, \ldots, u_n$ be the vertices of $\mathrm{W_n}$ and assume that $u_1, \ldots, u_n$ are arranged clockwise around a circle and u is the center vertex of wheel. In some cases, for convenience, we way use u_{n+1} for u_1 and u_{-1}, u_0 for u_{n-1}, u_n, respectively. Now, we have the following theorem.

Theorem 2.4. *If $n \geq 3$ and $h \geq 4$ are positive integers, then*

$$\mathrm{In}_{\mathbb{Z}_\mathrm{h}}(\mathrm{W_n}) = \begin{cases} 2\mathbb{Z}_h & \textit{if } n \textit{ and } h \textit{ are even;} \\ \mathbb{Z}_h & \text{otherwise.} \end{cases}$$

Proof. We would like to define a function $l : E(\mathrm{W_n}) \to \mathbb{Z}_h \setminus \{0\}$ such that l is an edge magic labeling of $\mathrm{W_n}$. By Theorem 2.1, $0 \in \mathrm{In}_{\mathbb{Z}_\mathrm{h}}(\mathrm{W_n})$. We consider two cases:

Case 1. Suppose that $n = 2r + 1$. We consider two subcases:

Subcase (i) Let r be odd. The labeling of the edges is done as follows:
$l(uu_i) = 1$, for $1 \leq i \leq r+1$ and $l(uu_i) = -1$, for $r+2 \leq i \leq n$. Also, define:

$$l(u_iu_{i+1}) = \begin{cases} -1 & 1 \leq i \leq r+1 \text{ and } i \text{ is odd;} \\ 1 & 1 \leq i \leq r+1 \text{ and } i \text{ is even or } r+2 \leq i \leq n. \end{cases}$$

Thus, $l^+(u) = r+1+r(-1) = 1 \pmod h$ and obviously, for every i, $l^+(u_i) = 1$. Therefore, $1 \in \mathrm{In}_{\mathbb{Z}_h}(\mathrm{W_n})$ and $\mathrm{In}_{\mathbb{Z}_h}(\mathrm{W_n}) = \mathbb{Z}_h$, as desired.

Subcase (ii) If r is even, then assign value 2 to uu_1 and uu_2 and define:
$l(uu_i) = -1$, for $3 \leq i \leq r+3$ and $l(uu_i) = 1$, for $r+4 \leq i \leq n$. Also, define:

$$l(u_1u_2) = -2 \text{ and } l(u_iu_{i+1}) = \begin{cases} 1 & 2 \leq i \leq r+3 \text{ or } r+4 \leq i \leq n \text{ and } i \text{ is odd;} \\ -1 & r+4 \leq i \leq n \text{ and } i \text{ is even.} \end{cases}$$

Thus, $l^+(u) = 4 + (r+1)(-1) + r - 2 = 1 \pmod h$ and obviously, for every i, $l^+(u_i) = 1$. Now, if h is odd, then $1 \in \mathrm{In}_{\mathbb{Z}_h}(\mathrm{W_n})$ and $\mathrm{In}_{\mathbb{Z}_h}(\mathrm{W_n}) = \mathbb{Z}_h$. If $h = 2k$, then the above edge labeling of $\mathrm{W_n}$ shows that $\mathrm{In}_{\mathbb{Z}_h}(\mathrm{W_n}) = 2\mathbb{Z}_h \setminus \{\mathrm{k}\}$. Also, if assign value k to all edges of $\mathrm{W_n}$, then $k \in \mathrm{In}_{\mathbb{Z}_h}(\mathrm{W_n})$. Therefore, $\mathrm{In}_{\mathbb{Z}_h}(\mathrm{W_n}) = \mathbb{Z}_h$, as desired.

Case 2. Let $n = 2r$ and $x = 2t$ be a non-zero element of $\mathbb{Z}_h$. We consider two subcases:

Subcase (i) If r is even, then the labeling done as follows:
$l(uu_1) = l(uu_2) = x$, $l(uu_3) = l(uu_4) = -t$.

$$l(uu_5) = l(uu_6) = 1,\ l(uu_7) = l(uu_8) = -1,\ \ldots,\ l(uu_{n-3}) = l(uu_{n-2}) = 1 \text{ and}$$
$$l(uu_{n-1}) = l(uu_n) = -1,$$

and define:
$l(u_1u_2) = -x$, $l(u_3u_4) = t$, $l(u_5u_6) = -1$, $l(u_7u_8) = 1$, $\ldots$, $l(u_{n-3}u_{n-2}) = -1$ and $l(u_{n-1}u_n) = 1$. Also, assign value x to the remaining edges of $\mathrm{W_n}$.

Subcase (ii) If r is odd, then the labeling done as follows:
$l(uu_1) = l(uu_2) = t$ and

$$l(uu_3) = l(uu_4) = 1,\ l(uu_5) = l(uu_6) = -1,\ \ldots,\ l(uu_{n-3}) = l(uu_{n-2}) = 1 \text{ and}$$
$$l(uu_{n-1}) = l(uu_n) = -1.$$

Also, define:
$l(u_1u_2) = -t$, $l(u_3u_4) = -1$, $l(u_5u_6) = 1$, $\ldots$, $l(u_{n-3}u_{n-2}) = -1$ and $l(u_{n-1}u_n) = 1$ and assign value x to the remaining edges of $\mathrm{W_n}$.

First suppose that h is odd and $t = 1$. Then Subcases (i) and (ii) show that $2 \in \mathrm{In}_{\mathbb{Z}_\mathrm{h}}(\mathrm{W_n})$ and so, $\mathrm{In}_{\mathbb{Z}_\mathrm{h}}(\mathrm{W_n}) = \mathbb{Z}_\mathrm{h}$. Now, assume that h is even. If $c \in \mathbb{Z}_h \setminus \{0\}$ is odd, then by Remark 2.2, we have $c \notin \mathrm{In}_{\mathbb{Z}_\mathrm{h}}(\mathrm{W_n})$ and Subcases (i) and (ii) show that $\mathrm{In}_{\mathbb{Z}_\mathrm{h}}(\mathrm{W_n}) = 2\mathbb{Z}_\mathrm{h}$. The proof is complete. □

Remark 2.5. If $h = 3$ and $n \geq 3$ is positive integer, then Theorem 2.1 and the proof of Theorem 2.4, imply that:

$$\mathrm{In}_{\mathbb{Z}_3}(\mathrm{W_n}) = \begin{cases} \mathbb{Z}_3 \setminus \{0\} & n \neq 0 \ (\mathrm{mod}\ 3); \\ \mathbb{Z}_3 & n = 0 \ (\mathrm{mod}\ 3). \end{cases}$$

3. Index Set of Fans

For $n \geq 2$, fan on $n+1$ vertices, denoted by $\mathrm{F_n}$ and is defined to be $P_n \vee K_1$, where P_n is the path of order n. The null set of fans determined in [3].

Theorem 3.1. $N(\mathrm{F}_2) = 2\mathbb{N}$, $N(\mathrm{F}_3) = 2\mathbb{N} \setminus \{2\}$ *and for any* $n \geq 4$,

$$N(\mathrm{F_n}) = \begin{cases} \mathbb{N} \setminus \{2\} & n = 1 \,(\mathrm{mod}\, 3); \\ \mathbb{N} \setminus \{2,3\} & \text{otherwise}. \end{cases}$$

Remark 3.2. Note that the degree set of $\mathrm{F_n}$ is $\{2, 3, n\}$. So, it has no 2-magic labeling with index 1. On the other hand by Theorem 3.1, we have $0 \notin \mathrm{In}_{\mathbb{Z}_2}(\mathrm{F_n})$. Therefore, $\mathrm{In}_{\mathbb{Z}_2}(\mathrm{F_n}) = \emptyset$. Also, $\mathrm{F}_2 = C_3$ and by Lemma 2.3, $\mathrm{In}_{\mathbb{Z}_\mathrm{h}}(\mathrm{F}_2) = \mathbb{Z}_\mathrm{h} \setminus \{0\}$, where h is a positive integer.

For the general case, let $u_1 \sim u_2 \sim \ldots \sim u_n$ be the vertices of the path P_n and u be the central vertex of the fan. We call the edges uu_i $(1 \leq i \leq n)$ blades of the fan.

Lemma 3.3. *Let* $h \geq 3$ *be a positive integer. Then*

$$\mathrm{In}_{\mathbb{Z}_\mathrm{h}}(\mathrm{F}_3) = \begin{cases} \emptyset & h \text{ is odd;} \\ \mathbb{Z}_h & h \text{ is even.} \end{cases}$$

First suppose that h is odd and $l : E(\mathrm{F}_3) \to \mathbb{Z}_h \setminus \{0\}$ is an h-magic labeling with index c of F_3, that illustrated in Figure 1, where $a, b \neq c$ and $a, b, z \in \mathbb{Z}_h \setminus \{0\}$. We should have $a + b + z = c$, $z + 2c - (a + b) = c$.

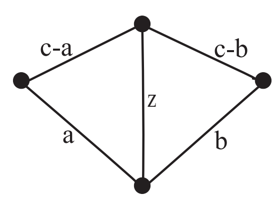

Figure 1

If we add these equations, then we get $2z = 0$ (mod h). Thus $z = 0$ (mod h), a contradiction. Now, assume that $h = 2k$ and x is a non-zero element of $\mathbb{Z}_h$. If $x = 2t$ (for some t), then consider Figure 2 Part (a) and if $x = 2t + 1$ (for some t), then consider Figure 2 Part (b). Thus, $x \in \mathrm{In}_{\mathbb{Z}_\mathrm{h}}(\mathrm{F}_3)$ and so, $\mathrm{In}_{\mathbb{Z}_\mathrm{h}}(\mathrm{F}_3) = \mathbb{Z}_\mathrm{h}$.

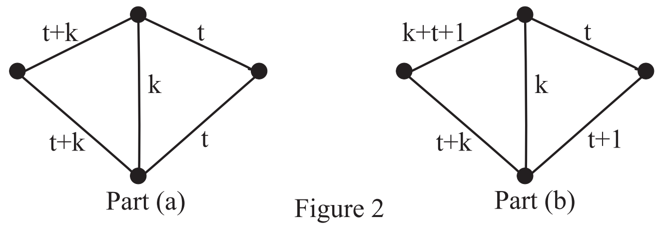

Figure 2

The proof is complete.

Lemma 3.4. *For a positive integer $h \geq 3$,*

$$\mathrm{In}_{\mathbb{Z}_\mathrm{h}}(\mathrm{F}_4) = \begin{cases} \{0\} & h = 3; \\ \mathbb{Z}_h & h \neq 3 \text{ is odd}; \\ 2\mathbb{Z}_h & \text{otherwise.} \end{cases}$$

Proof. By Theorem 3.1, we have $0 \in \mathrm{In}_{\mathbb{Z}_\mathrm{h}}(\mathrm{F}_4)$. We consider three cases:

Case 1. Suppose that $h = 3$ and $1 \in \mathrm{In}_{\mathbb{Z}_3}(\mathrm{F}_4)$. Since $d(u) = 4$, four edges incident with u have value $\{2, 2, 2, 1\}$ or $\{1, 1, 1, 1\}$. Since, $d(u_1) = d(u_4) = 2$, thus, $l(uu_1) \neq 1$ and $l(uu_4) \neq 1$. So, 4 edges incident with u have value $\{2, 2, 2, 1\}$. Therefore, $l(uu_1) = l(uu_4) = 2$. With no loss of generality suppose that $l(uu_2) = 2$. So, we have $l(u_1u_2) = 2$. Therefore, $l(u_2u_3) = 0$, a contradiction. Also, if $2 \in \mathrm{In}_{\mathbb{Z}_3}(\mathrm{F}_4)$ and we multiply the values of all edges of F_4 in 2, then we obtain that $1 \in \mathrm{In}_{\mathbb{Z}_3}(\mathrm{F}_4)$, a contradiction. Therefore, $2 \notin \mathrm{In}_{\mathbb{Z}_3}(\mathrm{F}_4)$ and so, $\mathrm{In}_{\mathbb{Z}_3}(\mathrm{F}_4) = \{0\}$.

Case 2. Let $3 \neq h = 2k + 1$ and x be a non-zero element of $\mathbb{Z}_h$. If $x = 2t$ and $x \neq -2, 2, 4$, then Figure 3 Part (a) shows that $x \in \mathrm{In}_{\mathbb{Z}_\mathrm{h}}(\mathrm{F}_4)$ and if $x = 2t + 1$ and $x \neq -1, 1$, then Figure 3 Part (b) shows that $x \in \mathrm{In}_{\mathbb{Z}_\mathrm{h}}(\mathrm{F}_4)$.

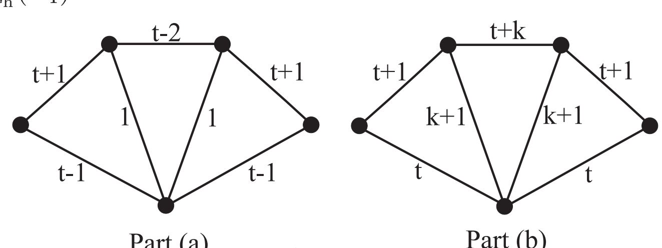

Figure 3

If $x = 1$, then Figure 4, Part (a) shows that $1 \in \mathrm{In}_{\mathbb{Z}_\mathrm{h}}(\mathrm{F}_\mathrm{n})$ and if we multiply all values of edges in value -1, then we obtain that $-1 \in \mathrm{In}_{\mathbb{Z}_\mathrm{h}}(\mathrm{F}_4)$. If $x = 2$, then Figure 4, Part (b) shows that $2 \in \mathrm{In}_{\mathbb{Z}_\mathrm{h}}(\mathrm{F}_4)$ and if we multiply all values of edges in value -1 and 2, respectively, then we obtain that $-2 \in \mathrm{In}_{\mathbb{Z}_\mathrm{h}}(\mathrm{F}_4)$ and $4 \in \mathrm{In}_{\mathbb{Z}_\mathrm{h}}(\mathrm{F}_4)$, respectively. Therefore, $\mathrm{In}_{\mathbb{Z}_\mathrm{h}}(\mathrm{F}_4) = \mathbb{Z}_\mathrm{h}$, as desired.

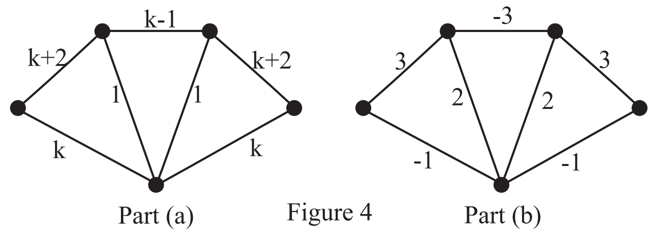

Figure 4

Case 3. Let $h = 2k$. If $c \in \mathbb{Z}_h \setminus \{0\}$ is odd, then by Remark 2.2, $c \notin \mathrm{In}_{\mathbb{Z}_\mathrm{h}}(\mathrm{F}_4)$. Let $x = 2t$ be a non-zero element of $\mathbb{Z}_h$. Figure 5 shows that $x \in \mathrm{In}_{\mathbb{Z}_\mathrm{h}}(\mathrm{F}_4)$. Therefore, $\mathrm{In}_{\mathbb{Z}_\mathrm{h}}(\mathrm{F}_4) = 2\mathbb{Z}_\mathrm{h}$.

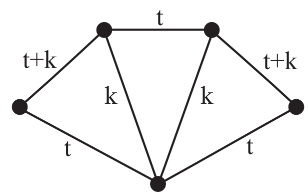

Figure 5

The proof is complete. $\square$

Lemma 3.5. *For a positive integer $h \geq 3$,* $\mathrm{In}_{\mathbb{Z}_\mathrm{h}}(\mathrm{F}_5) = \begin{cases} \emptyset & h = 3; \\ \mathbb{Z}_h & h \neq 3. \end{cases}$

Proof. By Theorem 3.1, we have $0 \notin \mathrm{In}_{\mathbb{Z}_3}(\mathrm{F}_5)$ and if $h \neq 3$, then $0 \in \mathrm{In}_{\mathbb{Z}_\mathrm{h}}(\mathrm{F}_5)$. Let $1 \in \mathrm{In}_{\mathbb{Z}_3}(\mathrm{F}_5)$. Since $d(u_1) = d(u_5) = 2$, then $l(uu_1) \neq 1$ and $l(uu_5) \neq 1$. Therefore, $l(uu_1) = l(uu_5) = 2$. So, for $2 \leq i \leq 4$, uu_i have value 1 or 2. First suppose that for $2 \leq i \leq 4$, $l(uu_i) = 1$. Then we have $l(u_1u_2) = l(u_4u_5) = 2$ and $l(u_iu_{i+1}) = 1$ $(i = 2, 3)$ and so $l^+(u_3) = 0 \pmod 3$, a contradiction. Also, if suppose that for $i = 2, 3, 4$, $l(uu_i) = 2$, then we have $l(u_1u_2) = l(u_4u_5) = 2$, so $l(u_2u_3) = l(u_3u_4) = 0$, a contradiction. Also, if $2 \in \mathrm{In}_{\mathbb{Z}_3}(\mathrm{F}_5)$ and we multiply the value of all edges of F_5 in 2, then we find that $1 \in \mathrm{In}_{\mathbb{Z}_3}(\mathrm{F}_5)$, a contradiction.

Now, let $h \neq 3$ and $x \neq 2$ be a non-zero element of $\mathbb{Z}_h$. Figure 6 Part (a) shows that $x \in \mathrm{In}_{\mathbb{Z}_\mathrm{h}}(\mathrm{F}_5)$ and Figure 6 Part (b) shows that, $2 \in \mathrm{In}_{\mathbb{Z}_\mathrm{h}}(\mathrm{F}_5)$.

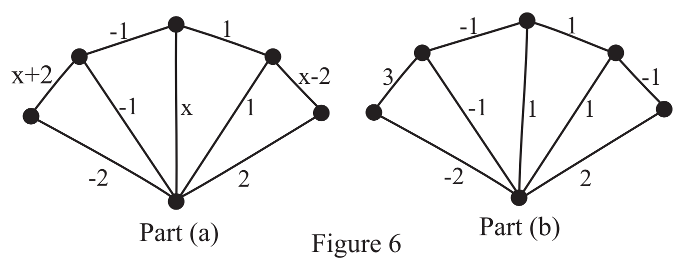

Figure 6

Thus, $\mathrm{In}_{\mathbb{Z}_\mathrm{h}}(\mathrm{F}_5) = \mathbb{Z}_\mathrm{h}$ and the proof is complete. $\square$

Theorem 3.6. *If $n \geq 6$ and $h \geq 4$ are positive integers, then*

$$\mathrm{In}_{\mathbb{Z}_\mathrm{h}}(\mathrm{F}_\mathrm{n}) = \begin{cases} 2\mathbb{Z}_h & n \text{ and } h \text{ are both even;} \\ \mathbb{Z}_h & \text{otherwise.} \end{cases}$$

Proof. By Theorem 3.1, $0 \in \mathrm{In}_{\mathbb{Z}_\mathrm{h}}(\mathrm{F}_\mathrm{n})$. Let x be a non-zero element of $\mathbb{Z}_h$. We would like to define a function $l : E(\mathrm{F}_\mathrm{n}) \to \mathbb{Z}_h \setminus \{0\}$ such that l is an edge magic labeling of F_n. We consider three cases:

Case 1. Let $n = 2r+1$ and r be even. First suppose that $x \neq 2, -2$. We label the edges of $\mathrm{F_n}$ as follows:
$-l(uu_1) = l(uu_n) = 2$, $l(uu_{r+1}) = x$ and $-l(uu_i) = l(uu_{n-i+1}) = 1$, for $2 \leq i \leq r$. Also, define:

$$l(u_iu_{i+1}) = \begin{cases} x+2 & 1 \leq i \leq r \text{ and } i \text{ is odd;} \\ -1 & 1 \leq i \leq r \text{ and } i \text{ is even;} \\ 1 & r+1 \leq i \leq n \text{ and } i \text{ is odd;} \\ x-2 & r+1 \leq i < n \text{ and } i \text{ is even.} \end{cases}$$

So, $l^+(u) = 2-2+x+(r-1)(1)+(r-1)(-1) = x \pmod h$ and obviously, for every i, $l^+(u_i) = x$.
Now, assume that $x = 2$. Label all edges of $\mathrm{F_n}$ as follows:
$l(uu_{n-1}) = l(uu_i) = -1$, for $2 \leq i \leq r-2$, $l(uu_{r+1}) = -2$ and assign value 1 to the remaining edges.
Also, define:

$$l(u_iu_{i+1}) = \begin{cases} 1 & 1 \leq i \leq r-2 \text{ and } i \text{ is odd;} \\ 2 & r+1 \leq i \leq n-2 \text{ and } i \text{ is odd or } 1 \leq i \leq r-2 \text{ and } i \text{ is even;} \\ -1 & r+1 \leq i \leq n-2 \text{ and } i \text{ is even,} \end{cases}$$

and $l(u_{r-1}u_r) = -1$, $l(u_ru_{r+1}) = 1$ and $l(u_{n-1}u_n) = 1$.
So, $l^+(u) = -2-1+(r-3)(-1)+(r+2)(1) = 2 \pmod h$ and obviously, for every i, $l^+(u_i) = x$. Also, if we multiply all values of edges in -1, then we obtain that $-2 \in \mathrm{In}_{\mathbb{Z}_\mathrm{h}}(\mathrm{F_n})$, as desired.

Case 2. Let $n = 2r+1$ and r be odd. First suppose that $x \neq 2$. Label all edges of $\mathrm{F_n}$ as follows:
$l(uu_1) = 2$, $l(uu_n) = x-2$, $l(uu_{n-1}) = -2$, $l(uu_i) = 1$, for $2 \leq i \leq r+1$ and $l(uu_i) = -1$, for $r+2 \leq i \leq n-2$. Also, define:

$$l(u_iu_{i+1}) = \begin{cases} 1 & 1 \leq i \leq n-2 \text{ and } i \text{ is even;} \\ x-2 & 1 \leq i \leq r \text{ and } i \text{ is odd;} \\ x & r+1 \leq i \leq n-1 \text{ and } i \text{ is odd,} \end{cases}$$

and $l(u_{n-1}u_n) = 2$.
So, $l^+(u) = x-2+2-2+r+(r-2)(-1) = x \pmod h$ and obviously, for every i, $l^+(u_i) = x$.
Now, assume that $x = 2$. Label all edges of $\mathrm{F_n}$ as follows:
$l(uu_1) = l(uu_n) = -l(uu_2) = -l(uu_{n-1}) = 1$, $-l(uu_r) = l(uu_{r+1}) = l(uu_{r+2}) = 2$, $l(uu_i) = 2$, for $3 \leq i \leq r-1$ and $l(uu_i) = -2$, for $r+3 \leq i \leq n-2$. Also, define:
$l(u_1u_2) = l(u_{n-1}u_n) = 1$, $l(u_{r+1}u_{r+2}) = -2$ and for $2 \leq i \leq r-1$,

$$l(u_iu_{i+1}) = \begin{cases} 2 & i \text{ is even;} \\ -2 & i \text{ is odd.} \end{cases}$$

Also, assign value 2 to the remaining edges and obtain the result.
So, $l^+(u) = 2-2+4-2+(r-3)(2)+(r-3)(-2) = 2 \pmod h$ and obviously, for every i, $l^+(u_i) = x$.

Case 3. Let $n = 2r$. First suppose that $h = 2k + 1$. We consider two subcases:

Subcase (**i**) Suppose that r is odd. Define:
$l(uu_1) = l(uu_n) = k + 1$, $l(uu_2) = l(uu_{n-1}) = k$ and $l(uu_i) = \begin{cases} k+1 & 3 \leq i \leq r+1; \\ k & r+2 \leq i \leq n-2. \end{cases}$

Also, define:
$l(u_1u_2) = k + 1$ and $l(u_iu_{i+1}) = \begin{cases} k & 2 \leq i \leq r+1 \text{ and } i \text{ is odd}; \\ k+1 & r+2 \leq i \leq n \text{ and } i \text{ is odd}; \\ 1 & 2 \leq i \leq n \text{ and } i \text{ is even}. \end{cases}$

So, we have $l^+(u) = 2(k+1) + 2k + (r-1)(k+1) + (r-3)k = 1 \pmod h$ and obviously, for every i, $l^+(u_i) = 1$. Thus, $1 \in \mathrm{In}_{\mathbb{Z}_h}(\mathrm{F_n})$ and therefore, $\mathrm{In}_{\mathbb{Z}_h}(\mathrm{F_n}) = \mathbb{Z}_h$.

Subcase (**ii**) Suppose that r is even. Assign the value $k + 1$ to uu_1 and uu_n and value 1 to uu_2 and uu_{n-1}. Also define,

$$l(uu_i) = \begin{cases} k & 3 \leq i \leq r+2; \\ k+1 & r+3 \leq i \leq n-2. \end{cases}$$

Also, define:
$l(u_1u_2) = l(u_{n-1}u_n) = k + 1$ and $l(u_iu_{i+1}) = \begin{cases} 2 & 2 \leq i \leq r+2 \text{ and } i \text{ is odd}; \\ 1 & r+3 \leq i \leq n-2 \text{ and } i \text{ is odd}; \\ k & 2 \leq i \leq n-2 \text{ and } i \text{ is even}. \end{cases}$

So, $l^+(u) = 2(k+1) + 2 + rk + (r-4)(k+1) = 1 \pmod h$ and obviously, for every i, $l^+(u_i) = 1$. Thus, $1 \in \mathrm{In}_{\mathbb{Z}_h}(\mathrm{F_n})$ and so, $\mathrm{In}_{\mathbb{Z}_h}(\mathrm{F_n}) = \mathbb{Z}_h$.
Now, assume that $h = 2k$. If $c \in \mathbb{Z}_h \setminus \{0\}$ is odd, then by Remark 2.2, $c \notin \mathrm{In}_{\mathbb{Z}_h}(\mathrm{F_n})$. Let $x = 2t$. Define:
$l(uu_1) = l(uu_n) = t + k$ and $l(uu_i) = k$, for $2 \leq i \leq n - 1$. Also, define:

$$l(u_iu_{i+1}) = \begin{cases} t+k & i \text{ is odd}; \\ t & i \text{ is even}. \end{cases}$$

Therefore, $x \in \mathrm{In}_{\mathbb{Z}_h}(\mathrm{F_n})$ and so, $\mathrm{In}_{\mathbb{Z}_h}(\mathrm{F_n}) = 2\mathbb{Z}_h$. The proof is complete. □

Finally, we have the following remark.

Remark 3.7. If $h = 3$ and $n \geq 6$ are positive integers, then Theorem 3.1 and the proof of Theorem 3.6, imply that:

$$\mathrm{In}_{\mathbb{Z}_3}(\mathrm{F_n}) = \begin{cases} \mathbb{Z}_3 \setminus \{0\} & n \neq 1 \pmod 3; \\ \mathbb{Z}_3 & n = 1 \pmod 3). \end{cases}$$

References

[1] E. Salehi, Zero-sum magic graphs and their null sets, *Ars Combin.*, **82** (2007) 41-53.

[2] E. Salehi, Integer-magic spectra of cycle related graphs, *Iranian J. Math. Sci Inform*, **1** no. 2 (2006) 53-63.

[3] E. Salehi and S. Hansen, Zero sum magic and null sets of planar graphs, *J. Combin. Math. Combin. Comput.*, **72** (2010) 55-64.

[4] E. Salehi, On zero-sum magic graphs and their null sets, *Bull. Inst. Math. Acad. Sin. (N.S.)*, **3** (2008) 255-264.

[5] J. Sedláček, On magic graphs, *Math. Slovaca*, **26** (1976) 329-335.

[6] J. Sedláček, Some properties of magic graphs, in Graphs, Hypergraph, and Bloc Syst, *Proc. Symp. Comb. Anal, Zielona Gora*, (1976) 247-253.

[7] T.-M. Wang and S.-W. Hu, Constant sum flows in regular graphs, *Lecture Notes in Computer Science*, **6681** (2011) 168-175.

Mohammad Javad Nikmehr

Department of Mathematics, Karaj Branch, Islamic Azad University, Karaj, Iran

Samaneh Bahramian

Department of Mathematics, Karaj Branch, Islamic Azad University, Karaj, Iran