GROUP MAGICNESS OF CERTAIN PLANAR GRAPHS

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Abstract. Let \( A \) be a non-trivial abelian group and \( A^* = A \setminus \{0\} \). A graph \( G \) is said to be \( A \)-magic graph if there exists a labeling \( l : E(G) \to A^* \) such that the induced vertex labeling \( l^+ : V(G) \to A \), define by

\[
l^+(v) = \sum_{uv \in E(G)} l(uv)
\]

is a constant map. The set of all constant integers such that \( \sum_{u \in N(v)} l(uv) = c \), for each \( v \in N(v) \), where \( N(v) \) denotes the set of adjacent vertices to vertex \( v \) in \( G \), is called the index set of \( G \) and denoted by \( \text{In}_{\text{A}}(G) \). In this paper we determine the index set of certain planar graphs for \( \mathbb{Z}_h \), where \( h \in \mathbb{N} \), such as wheels and fans.

1. Introduction

For an abelian group \( A \) (written additively) let \( A^* = A \setminus \{0\} \). A map \( l : E(G) \to A^* \) is called a labeling of \( G \). Given a labeling on the edge set of \( G \) one can introduce a vertex labeling \( l^+ : V(G) \to A \), by \( l^+(v) = \sum_{uv \in E(G)} l(uv) \). A graph \( G \) is said to be \( A \)-magic labeling if there is a labeling \( l : E(G) \to A^* \) such that for every vertex \( v \in V(G) \), the sum of values of all edges incident with \( v \) is equal to the same constant; that is, \( l^+(v) = c \) for sum fixed \( c \in A \). A magic graph introduced by J. Sedlacek [5, 6]. An \( h \)-magic graph \( G \) is said to be \( \mathbb{Z}_h \)-magic graph if we choose the group \( A \) as \( \mathbb{Z}_h \) the group of integers mod \( h \). These \( \mathbb{Z}_h \)-magic graph are referred as \( h \)-magic graphs. Clearly, if a graph is \( h \)-magic, then it is not necessary \( k \)-magic (\( k \neq h \)). An \( h \)-magic graph \( G \) is said to be \( h \)-zero-sum if there is a magic labeling of \( G \) in \( \mathbb{Z}_h \) which induces a vertex labeling with sum zero. The null set of a graph \( G \), denoted by \( N(G) \), is the set of all natural numbers \( h \in \mathbb{N} \), such that \( G \) admits an \( h \)-zero-sum magic labeling. Let \( G \) be a graph and \( l : E(G) \to A^* \) be a labeling on the edge set of \( G \) and \( A \) be an abelian group. The set of all constant integers such that \( \sum_{u \in N(v)} l(uv) = c \), for each \( u \in N(v) \), where \( N(v) \) denotes

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the set of adjacent vertices to vertex \( v \), is called the index set of \( G \) and denoted by \( \text{In}_A(G) \). The join \( G \lor H \) of disjoint graph \( G \) and \( H \) is the graph obtained from \( G + H \) by joining each vertex of \( G \) to each vertex of \( H \). Tao-Ming Wang and Shi-wei Hu \cite{7} in (2011), determined the index set of regular graphs for an abelian group \( \mathbb{Z} \). A detailed study about zero-sum magic graphs and their null sets done by E. Salehi in \cite{1, 2}. It was determined in \cite{3, 4} null set of wheels and fans that is, \( 0 \in \text{In}_{\mathbb{Z}_h}(G) \), where \( G \) is the above graphs. In this paper we determine the index set of wheels and fans.

2. Index Set of Wheels

For \( n \geq 3 \), wheel on \( n+1 \) vertices, denoted by \( W_n \) and is defined to be \( C_n \lor K_1 \), where \( C_n \) is the cycle of order \( n \). The null set of wheels determined in \cite{3}.

**Theorem 2.1.** For any positive integer \( n \geq 3 \), \( N(W_n) = \begin{cases} \mathbb{N} \setminus \{2\} & n = 0 \pmod{3}; \\ \mathbb{N} \setminus \{2, 3\} & \text{otherwise}. \end{cases} \)

Let \( G \) be a graph. It is obvious that \( 1 \in \text{In}_{\mathbb{Z}_h}(G) \) if and only if the degree of every vertex is odd. Therefore, since the degree set of the \( W_n \) is \( \{3, n\} \), \( W_n \) has 2-magic labeling with index 1, if and only if \( n \) is odd. First we need the following.

**Remark 2.2.** If \( G \) is a graph and \( c \in \text{In}_{\mathbb{Z}_h}(G) \), then we have

\[
2 \sum_{e \in E(G)} l(e) = c \mid V(G) \mid \pmod{h}.
\]

**Lemma 2.3.** For any positive integers \( n \) and \( h \geq 3 \), \( \text{In}_{\mathbb{Z}_h}(C_n) = \begin{cases} \mathbb{Z}_h \setminus \{0\} & n \text{ is odd}; \\ \mathbb{Z}_h & n \text{ is even}. \end{cases} \)

**Proof.** First suppose that \( n \) is odd. Note that in any \( h \)-magic labeling of an odd cycle, the edges should have the same value. So, \( 0 \notin \text{In}_{\mathbb{Z}_h}(C_n) \). Let \( h = 2k + 1 \) and \( x \in \mathbb{Z}_h \setminus \{0\} \). We assign value \( (k + 1)x \) to all edges of \( C_n \). So, \( x \in \text{In}_{\mathbb{Z}_h}(C_n) \) and thus \( \text{In}_{\mathbb{Z}_h}(C_n) = \mathbb{Z}_h \setminus \{0\} \).

Now, assume that \( n \) is even and \( x \in \mathbb{Z}_h \). If assign value 2 and \(-1\) to all edges of \( C_n \), alternatively, then \( 1 \in \text{In}_{\mathbb{Z}_h}(C_n) \). Suppose that \( x \neq 1 \). We assign the values \( x - 1 \) and 1, alternatively to all edges of \( C_n \). So, \( x \in \text{In}_{\mathbb{Z}_h}(C_n) \), as desired. \( \square \)

Let \( u, u_1, \ldots, u_n \) be the vertices of \( W_n \) and assume that \( u_1, \ldots, u_n \) are arranged clockwise around a circle and \( u \) is the center vertex of wheel. In some cases, for convenience, we way use \( u_{n+1} \) for \( u_1 \) and \( u_{-1}, u_0 \) for \( u_{n-1}, u_n \), respectively. Now, we have the following theorem.

**Theorem 2.4.** If \( n \geq 3 \) and \( h \geq 4 \) are positive integers, then

\[
\text{In}_{\mathbb{Z}_h}(W_n) = \begin{cases} 2\mathbb{Z}_h & \text{if } n \text{ and } h \text{ are even}; \\ \mathbb{Z}_h & \text{otherwise}. \end{cases}
\]

**Proof.** We would like to define a function \( l : E(W_n) \to \mathbb{Z}_h \setminus \{0\} \) such that \( l \) is an edge magic labeling of \( W_n \). By Theorem \(
\begin{align*}
\text{In}_{\mathbb{Z}_h}(W_n) & = 0 \in \text{In}_{\mathbb{Z}_h}(W_n) \text{. We consider two cases:}
\end{align*}
\)
Case 1. Suppose that \( n = 2r + 1 \). We consider two subcases:

Subcase (i) Let \( r \) be odd. The labeling of the edges is done as follows:
\[
l(uu_i) = 1, \text{ for } 1 \leq i \leq r + 1 \text{ and } l(uu_i) = -1, \text{ for } r + 2 \leq i \leq n.
\]
Also, define:
\[
l(u_iu_{i+1}) = \begin{cases} 
-1 & 1 \leq i \leq r + 1 \text{ and } i \text{ is odd;} \\
1 & 1 \leq i \leq r + 1 \text{ and } i \text{ is even or } r + 2 \leq i \leq n.
\end{cases}
\]
Thus, \( l^+(u) = r + 1 + r(-1) = 1 \mod h \) and obviously, for every \( i \), \( l^+(u_i) = 1 \). Therefore, \( 1 \in \text{In}_{Z_h}(W_n) \) and \( \text{In}_{Z_h}(W_n) = Z_h \), as desired.

Subcase (ii) If \( r \) is even, then assign value 2 to \( uu_1 \) and \( uu_2 \) and define:
\[
l(uu_i) = -1, \text{ for } 3 \leq i \leq r + 3 \text{ and } l(uu_i) = 1, \text{ for } r + 4 \leq i \leq n.
\]
Also, define:
\[
l(u_1u_2) = -2 \text{ and } l(u_iu_{i+1}) = \begin{cases} 
1 & 2 \leq i \leq r + 3 \text{ or } r + 4 \leq i \leq n \text{ and } i \text{ is odd;} \\
-1 & r + 4 \leq i \leq n \text{ and } i \text{ is even.}
\end{cases}
\]
Thus, \( l^+(u) = 4 + (r + 1)(-1) + r - 2 = 1 \mod h \) and obviously, for every \( i \), \( l^+(u_i) = 1 \). Now, if \( h \) is odd, then \( 1 \in \text{In}_{Z_h}(W_n) \) and \( \text{In}_{Z_h}(W_n) = Z_h \). If \( h = 2k \), then the above edge labeling of \( W_n \) shows that \( \text{In}_{Z_h}(W_n) = 2Z_h \setminus \{k\} \). Also, if assign value \( k \) to all edges of \( W_n \), then \( k \in \text{In}_{Z_h}(W_n) \). Therefore, \( \text{In}_{Z_h}(W_n) = Z_h \), as desired.

Case 2. Let \( n = 2r \) and \( x = 2t \) be a non-zero element of \( Z_h \). We consider two subcases:

Subcase (i) If \( r \) is even, then the labeling done as follows:
\[
l(uu_1) = l(uu_2) = x, \ l(uu_3) = l(uu_4) = -t.
\]
\[
l(uu_5) = l(uu_6) = 1, \ l(uu_7) = l(uu_8) = -1, \ldots, \ l(uu_{n-3}) = l(uu_{n-2}) = 1 \text{ and } \ l(uu_{n-1}) = l(uu_n) = -1,
\]
and define:
\[
l(u_1u_2) = -x, \ l(u_3u_4) = t, \ l(u_5u_6) = -1, \ l(u_7u_8) = 1, \ldots, \ l(u_{n-3}u_{n-2}) = -1 \text{ and } l(u_{n-1}u_n) = 1.
\]
Also, assign value \( x \) to the remaining edges of \( W_n \).

Subcase (ii) If \( r \) is odd, then the labeling done as follows:
\[
l(uu_1) = l(uu_2) = t \text{ and } \ l(uu_3) = l(uu_4) = 1, \ l(uu_5) = l(uu_6) = -1, \ldots, \ l(uu_{n-3}) = l(uu_{n-2}) = 1 \text{ and } \ l(uu_{n-1}) = l(uu_n) = -1.
\]
Also, define:
\[
l(u_1u_2) = -t, \ l(u_3u_4) = -1, \ l(u_5u_6) = 1, \ldots, \ l(u_{n-3}u_{n-2}) = -1 \text{ and } l(u_{n-1}u_n) = 1 \text{ and assign value } \ x \text{ to the remaining edges of } W_n.
First suppose that \( h \) is odd and \( t = 1 \). Then Subcases (i) and (ii) show that \( 2 \in \text{In}_{\mathbb{Z}_h}(W_n) \) and so, \( \text{In}_{\mathbb{Z}_h}(W_n) = \mathbb{Z}_h \). Now, assume that \( h \) is even. If \( c \in \mathbb{Z}_h \setminus \{0\} \) is odd, then by Remark 2.2 we have \( c \not\in \text{In}_{\mathbb{Z}_h}(W_n) \) and Subcases (i) and (ii) show that \( \text{In}_{\mathbb{Z}_h}(W_n) = 2\mathbb{Z}_h \). The proof is complete. \( \square \)

**Remark 2.5.** If \( h = 3 \) and \( n \geq 3 \) is a positive integer, then Theorem 2.1 and the proof of Theorem 2.4 imply that:

\[
\text{In}_{\mathbb{Z}_3}(W_n) = \begin{cases} 
\mathbb{Z}_3 \setminus \{0\} & n \not\equiv 0 \pmod{3}; \\
\mathbb{Z}_3 & n \equiv 0 \pmod{3}.
\end{cases}
\]

### 3. Index Set of Fans

For \( n \geq 2 \), fan on \( n+1 \) vertices, denoted by \( F_n \), and is defined to be \( P_n \lor K_1 \), where \( P_n \) is the path of order \( n \). The null set of fans determined in [3].

**Theorem 3.1.** \( N(F_2) = 2\mathbb{N} \), \( N(F_3) = 2\mathbb{N} \setminus \{2\} \) and for any \( n \geq 4 \),

\[
N(F_n) = \begin{cases} 
\mathbb{N} \setminus \{2\} & n \equiv 1 \pmod{3}; \\
\mathbb{N} \setminus \{2, 3\} & \text{otherwise}.
\end{cases}
\]

**Remark 3.2.** Note that the degree set of \( F_n \) is \( \{2, 3, n\} \). So, it has no 2-magic labeling with index 1. On the other hand by Theorem 3.1 we have \( \emptyset \not\in \text{In}_{\mathbb{Z}_2}(F_n) \). Therefore, \( \text{In}_{\mathbb{Z}_2}(F_n) = \emptyset \). Also, \( F_2 = C_3 \) and by Lemma 2.3 \( \text{In}_{\mathbb{Z}_h}(F_2) = \mathbb{Z}_h \setminus \{0\} \), where \( h \) is a positive integer.

For the general case, let \( u_1 \sim u_2 \sim \ldots \sim u_n \) be the vertices of the path \( P_n \) and \( u \) be the central vertex of the fan. We call the edges \( uu_i \) (\( 1 \leq i \leq n \)) blades of the fan.

**Lemma 3.3.** Let \( h \geq 3 \) be a positive integer. Then

\[
\text{In}_{\mathbb{Z}_h}(F_3) = \begin{cases} 
\emptyset & h \text{ is odd}; \\
\mathbb{Z}_h & h \text{ is even}.
\end{cases}
\]

First suppose that \( h \) is odd and \( l : E(F_3) \to \mathbb{Z}_h \setminus \{0\} \) is an \( h \)-magic labeling with index \( c \) of \( F_3 \), that illustrated in Figure 1, where \( a, b \neq c \) and \( a, b, z \in \mathbb{Z}_h \setminus \{0\} \). We should have \( a + b + z = c \), \( z + 2c - (a + b) = c \).

![Figure 1](image)

If we add these equations, then we get \( 2z = 0 \pmod{h} \). Thus \( z = 0 \pmod{h} \), a contradiction. Now, assume that \( h = 2k \) and \( x \) is a non-zero element of \( \mathbb{Z}_h \). If \( x = 2t \) (for some \( t \)), then consider Figure 2 Part (a) and if \( x = 2t + 1 \) (for some \( t \)), then consider Figure 2 Part (b). Thus, \( x \in \text{In}_{\mathbb{Z}_h}(F_3) \) and so, \( \text{In}_{\mathbb{Z}_h}(F_3) = \mathbb{Z}_h \).
The proof is complete.

**Lemma 3.4.** For a positive integer \( h \geq 3 \),

\[
\text{In}_{Z_h}(F_4) = \begin{cases} 
\{0\} & h = 3; \\
Z_h & h \neq 3 \text{ is odd}; \\
2Z_h & \text{otherwise}. 
\end{cases}
\]

**Proof.** By Theorem 3.1, we have \( 0 \in \text{In}_{Z_h}(F_4) \). We consider three cases:

**Case 1.** Suppose that \( h = 3 \) and \( 1 \in \text{In}_{Z_3}(F_4) \). Since \( d(u) = 4 \), four edges incident with \( u \) have value \( \{2, 2, 2, 1\} \) or \( \{1, 1, 1, 1\} \). Since, \( d(u_1) = d(u_4) = 2 \), thus, \( l(uu_1) \neq 1 \) and \( l(uu_4) \neq 1 \). So, 4 edges incident with \( u \) have value \( \{2, 2, 2, 1\} \). Therefore, \( l(uu_1) = l(uu_4) = 2 \). With no loss of generality suppose that \( l(uu_2) = 2 \). So, we have \( l(u_1u_2) = 2 \). Therefore, \( l(u_2u_3) = 0 \), a contradiction. Also, if \( 2 \in \text{In}_{Z_3}(F_4) \) and we multiply the values of all edges of \( F_4 \) in 2, then we obtain that \( 1 \in \text{In}_{Z_3}(F_4) \), a contradiction. Therefore, \( 2 \notin \text{In}_{Z_3}(F_4) \) and so, \( \text{In}_{Z_3}(F_4) = \{0\} \).

**Case 2.** Let \( 3 \neq h = 2k + 1 \) and \( x \) be a non-zero element of \( Z_h \). If \( x = 2t \) and \( x \neq -2, 2, 4 \), then Figure 3 Part (a) shows that \( x \in \text{In}_{Z_h}(F_4) \) and if \( x = 2t + 1 \) and \( x \neq -1, 1 \), then Figure 3 Part (b) shows that \( x \in \text{In}_{Z_h}(F_4) \).

If \( x = 1 \), then Figure 4, Part (a) shows that \( 1 \in \text{In}_{Z_h}(F_n) \) and if we multiply all values of edges in value \( -1 \), then we obtain that \( -1 \in \text{In}_{Z_h}(F_4) \). If \( x = 2 \), then Figure 4, Part (b) shows that \( 2 \in \text{In}_{Z_h}(F_4) \) and if we multiply all values of edges in value \( -1 \) and 2, respectively, then we obtain that \( -2 \in \text{In}_{Z_h}(F_4) \) and \( 4 \in \text{In}_{Z_h}(F_4) \), respectively. Therefore, \( \text{In}_{Z_h}(F_4) = Z_h \), as desired.
Case 3. Let $h = 2k$. If $c \in \mathbb{Z}_h \setminus \{0\}$ is odd, then by Remark 2.2 $c \notin \text{In}_{Z_h}(F_4)$. Let $x = 2t$ be a non-zero element of $\mathbb{Z}_h$. Figure 5 shows that $x \in \text{In}_{Z_h}(F_4)$. Therefore, $\text{In}_{Z_h}(F_4) = 2\mathbb{Z}_h$.

The proof is complete. □

Lemma 3.5. For a positive integer $h \geq 3$, $\text{In}_{Z_h}(F_5) = \begin{cases} \emptyset & h = 3; \\ \mathbb{Z}_h & h \neq 3. \end{cases}$

Proof. By Theorem 3.1 we have $0 \notin \text{In}_{Z_3}(F_5)$ and if $h \neq 3$, then $0 \in \text{In}_{Z_h}(F_5)$. Let $x$ be a non-zero element of $\mathbb{Z}_h$. We would like to define a function $l : E(F_5) \to \mathbb{Z}_h \setminus \{0\}$ such that $l$ is an edge magic labeling of $F_5$. We consider three cases:

Thus, $\text{In}_{Z_h}(F_5) = \mathbb{Z}_h$ and the proof is complete. □

Theorem 3.6. If $n \geq 6$ and $h \geq 4$ are positive integers, then

$$
\text{In}_{Z_h}(F_n) = \begin{cases} 2\mathbb{Z}_h & n \text{ and } h \text{ are both even;} \\ \mathbb{Z}_h & \text{otherwise.} \end{cases}
$$

Proof. By Theorem 3.1 $0 \in \text{In}_{Z_3}(F_n)$. Let $x$ be a non-zero element of $\mathbb{Z}_h$. We would like to define a function $l : E(F_n) \to \mathbb{Z}_h \setminus \{0\}$ such that $l$ is an edge magic labeling of $F_n$. We consider three cases:
Case 1. Let $n = 2r + 1$ and $r$ be even. First suppose that $x \neq 2, -2$. We label the edges of $F_n$ as follows:

\[
l(uu_1) = l(uu_n) = 2, \quad l(uu_{r+1}) = x \quad \text{and} \quad -l(uu_i) = l(uu_{n-i+1}) = 1, \quad \text{for} \quad 2 \leq i \leq r.
\]

Also, define:

\[
l(u_iu_{i+1}) = \begin{cases} 
  x + 2 & 1 \leq i \leq r \quad \text{and} \quad i \text{ is odd;} \\
  -1 & 1 \leq i \leq r \quad \text{and} \quad i \text{ is even;} \\
  1 & r + 1 \leq i \leq n \quad \text{and} \quad i \text{ is odd;} \\
  x - 2 & r + 1 < i < n \quad \text{and} \quad i \text{ is even.}
\end{cases}
\]

So, $l^+(u) = 2 - 2 + x + (r - 1)(1) + (r - 1)(-1) = x \pmod{h}$ and obviously, for every $i$, $l^+(u_i) = x$.

Now, assume that $x = 2$. Label all edges of $F_n$ as follows:

\[
l(uu_{n-1}) = l(uu_i) = -1, \quad \text{for} \quad 2 \leq i \leq r - 2, \quad l(uu_{r+1}) = -2 \quad \text{and} \quad \text{assign value 1 to the remaining edges. Also, define:}
\]

\[
l(u_iu_{i+1}) = \begin{cases} 
  1 & 1 \leq i \leq r - 2 \quad \text{and} \quad i \text{ is odd;} \\
  2 & r + 1 \leq i \leq n - 2 \quad \text{and} \quad i \text{ is odd or} \quad 1 \leq i \leq r - 2 \quad \text{and} \quad i \text{ is even;} \\
  -1 & r + 1 \leq i \leq n - 2 \quad \text{and} \quad i \text{ is even,}
\end{cases}
\]

and $l(u_{r-1}u_r) = -1, \quad l(u_ru_{r+1}) = 1 \quad \text{and} \quad l(u_{n-1}u_n) = 1$.

So, $l^+(u) = -2 - 1 + (r - 3)(-1) + (r + 2)(1) = 2 \pmod{h}$ and obviously, for every $i$, $l^+(u_i) = x$. Also, if we multiply all values of edges in $-1$, then we obtain that $-2 \in \text{Int}_n(F_n)$, as desired.

Case 2. Let $n = 2r + 1$ and $r$ be odd. First suppose that $x \neq 2$. Label all edges of $F_n$ as follows:

\[
l(uu_1) = 2, \quad l(uu_n) = x - 2, \quad l(uu_{n-1}) = -2, \quad l(uu_i) = 1, \quad \text{for} \quad 2 \leq i \leq r + 1 \quad \text{and} \quad l(uu_i) = -1, \quad \text{for} \quad r + 2 \leq i \leq n - 2.
\]

Also, define:

\[
l(u_iu_{i+1}) = \begin{cases} 
  1 & 1 \leq i \leq n - 2 \quad \text{and} \quad i \text{ is even;} \\
  x - 2 & 1 \leq i \leq r \quad \text{and} \quad i \text{ is odd;} \\
  x & r + 1 \leq i \leq n - 1 \quad \text{and} \quad i \text{ is odd,}
\end{cases}
\]

and $l(u_{n-1}u_n) = 2$.

So, $l^+(u) = x - 2 + 2 - 2 + r + (r - 2)(-1) = x \pmod{h}$ and obviously, for every $i$, $l^+(u_i) = x$.

Now, assume that $x = 2$. Label all edges of $F_n$ as follows:

\[
l(uu_1) = l(uu_n) = -l(uu_2) = -l(uu_{n-1}) = 1, \quad -l(uu_r) = l(uu_{r+1}) = l(uu_{r+2}) = 2, \quad l(uu_i) = 2, \quad \text{for} \quad 3 \leq i \leq r - 1 \quad \text{and} \quad l(uu_i) = -2, \quad \text{for} \quad r + 3 \leq i \leq n - 2.
\]

Also, define:

\[
l(u_1u_2) = l(u_{n-1}u_n) = 1, \quad l(u_{r+1}u_{r+2}) = -2 \quad \text{and} \quad \text{for} \quad 2 \leq i \leq r - 1,
\]

\[
l(u_iu_{i+1}) = \begin{cases} 
  2 & i \text{ is even;} \\
  -2 & i \text{ is odd.}
\end{cases}
\]

Also, assign value 2 to the remaining edges and obtain the result.

So, $l^+(u) = 2 - 2 + 4 - 2 + (r - 3)(2) + (r - 3)(-2) = 2 \pmod{h}$ and obviously, for every $i$, $l^+(u_i) = x$.\]
Case 3. Let $n = 2r$. First suppose that $h = 2k + 1$. We consider two subcases:

**Subcase (i)** Suppose that $r$ is odd. Define:

\[ l(uu_1) = l(uu_n) = k + 1, \quad l(uu_2) = l(uu_{n-1}) = k \] and

\[ l(uu_i) = \begin{cases}  
  k + 1 & 3 \leq i \leq r + 1; \\
  k & r + 2 \leq i \leq n - 2. 
\end{cases} \]

Also, define:

\[ l(u_1u_2) = k + 1 \] and

\[ l(u_iu_{i+1}) = \begin{cases}  
  k & 2 \leq i \leq r + 1 \text{ and } i \text{ is odd}; \\
  k + 1 & r + 2 \leq i \leq n \text{ and } i \text{ is odd}; \\
  1 & 2 \leq i \leq n \text{ and } i \text{ is even}. 
\end{cases} \]

So, we have $l^+(u) = 2(k + 1) + 2k + (r - 1)(k + 1) + (r - 3)k = 1 \pmod{h}$ and obviously, for every $i$, $l^+(u_i) = 1$. Thus, $1 \in \mathbb{I}_{Z_h}(F_n)$ and therefore, $\mathbb{I}_{Z_h}(F_n) = \mathbb{Z}_h$.

**Subcase (ii)** Suppose that $r$ is even. Assign the value $k + 1$ to $uu_1$ and $uu_n$ and value 1 to $uu_2$ and $uu_{n-1}$. Also define,

\[ l(uu_i) = \begin{cases}  
  k & 3 \leq i \leq r + 2; \\
  k + 1 & r + 3 \leq i \leq n - 2. 
\end{cases} \]

Also, define:

\[ l(u_1u_2) = l(u_{n-1}u_n) = k + 1 \] and

\[ l(u_iu_{i+1}) = \begin{cases}  
  2 & 2 \leq i \leq r + 2 \text{ and } i \text{ is odd}; \\
  1 & r + 3 \leq i \leq n - 2 \text{ and } i \text{ is odd}; \\
  k & 2 \leq i \leq n - 2 \text{ and } i \text{ is even}. 
\end{cases} \]

So, $l^+(u) = 2(k + 1) + 2k + (r - 4)(k + 1) = 1 \pmod{h}$ and obviously, for every $i$, $l^+(u_i) = 1$. Thus, $1 \in \mathbb{I}_{Z_h}(F_n)$ and so, $\mathbb{I}_{Z_h}(F_n) = \mathbb{Z}_h$.

Now, assume that $h = 2k$. If $c \in \mathbb{Z}_h \setminus \{0\}$ is odd, then by Remark 2.2, $c \not\in \mathbb{I}_{Z_h}(F_n)$. Let $x = 2t$. Define:

\[ l(uu_1) = l(uu_n) = t + k \] and

\[ l(uu_i) = k, \text{ for } 2 \leq i \leq n - 1. \] Also, define:

\[ l(u_iu_{i+1}) = \begin{cases}  
  t + k & i \text{ is odd}; \\
  t & i \text{ is even}. 
\end{cases} \]

Therefore, $x \in \mathbb{I}_{Z_h}(F_n)$ and so, $\mathbb{I}_{Z_h}(F_n) = 2\mathbb{Z}_h$. The proof is complete. \(\square\)

Finally, we have the following remark.

**Remark 3.7.** If $h = 3$ and $n \geq 6$ are positive integers, then Theorem 3.1 and the proof of Theorem 3.6 imply that:

\[ \mathbb{I}_{Z_3}(F_n) = \begin{cases}  
  \mathbb{Z}_3 \setminus \{0\} & n \not\equiv 1 \pmod{3}; \\
  \mathbb{Z}_3 & n \equiv 1 \pmod{3}. 
\end{cases} \]
REFERENCES


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