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## GROUP MAGICNESS OF CERTAIN PLANAR GRAPHS

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ABSTRACT. Let  $A$  be a non-trivial abelian group and  $A^* = A \setminus \{0\}$ . A graph  $G$  is said to be  $A$ -magic graph if there exists a labeling  $l : E(G) \rightarrow A^*$  such that the induced vertex labeling  $l^+ : V(G) \rightarrow A$ , define by

$$l^+(v) = \sum_{uv \in E(G)} l(uv)$$

is a constant map. The set of all constant integers such that  $\sum_{u \in N(v)} l(uv) = c$ , for each  $v \in N(v)$ , where  $N(v)$  denotes the set of adjacent vertices to vertex  $v$  in  $G$ , is called the index set of  $G$  and denoted by  $\text{In}_A(G)$ . In this paper we determine the index set of certain planar graphs for  $\mathbb{Z}_h$ , where  $h \in \mathbb{N}$ , such as wheels and fans.

### 1. Introduction

For an abelian group  $A$  (written additively) let  $A^* = A \setminus \{0\}$ . A map  $l : E(G) \rightarrow A^*$  is called a *labeling* of  $G$ . Given a labeling on the edge set of  $G$  one can introduce a vertex labeling  $l^+ : V(G) \rightarrow A$ , by  $l^+(v) = \sum_{uv \in E(G)} l(uv)$ . A graph  $G$  is said to be  *$A$ -magic labeling* if there is a labeling  $l : E(G) \rightarrow A^*$  such that for every vertex  $v \in V(G)$ , the sum of values of all edges incident with  $v$  is equal to the same constant; that is,  $l^+(v) = c$  for sum fixed  $c \in A$ . A magic graph introduced by J. Sedlaced [5, 6]. An  $A$ -magic graph  $G$  is said to be  *$\mathbb{Z}_h$ -magic graph* if we choose the group  $A$  as  $\mathbb{Z}_h$  the group of integers mod  $h$ . These  $\mathbb{Z}_h$ -magic graph are referred as  *$h$ -magic graphs*. Clearly, if a graph is  $h$ -magic, then it is not necessary  $k$ -magic ( $k \neq h$ ). An  $h$ -magic graph  $G$  is said to be  *$h$ -zero-sum* if there is a magic labeling of  $G$  in  $\mathbb{Z}_h$  which induces a vertex labeling with sum zero. The *null set* of a graph  $G$ , denoted by  $N(G)$ , is the set of all natural numbers  $h \in \mathbb{N}$ , such that  $G$  admits an  $h$ -zero-sum magic labeling. Let  $G$  be a graph and  $l : E(G) \rightarrow A^*$  be a labeling on the edge set of  $G$  and  $A$  be an abelian group. The set of all constant integers such that  $\sum_{u \in N(v)} l(uv) = c$ , for each  $u \in N(v)$ , where  $N(v)$  denotes

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the set of adjacent vertices to vertex  $v$ , is called the *index set* of  $G$  and denoted by  $\text{In}_A(G)$ . The join  $G \vee H$  of disjoint graph  $G$  and  $H$  is the graph obtained from  $G + H$  by joining each vertex of  $G$  to each vertex of  $H$ . Tao-Ming wang and Shi-wei Hu [7] in (2011), determined the index set of regular graphs for an abelian group  $\mathbb{Z}$ . A detailed study about zero-sum magic graphs and their null sets done by E. Salehi in [1, 2]. It was determined in [3, 4] null set of wheels and fans that is,  $0 \in \text{In}_{\mathbb{Z}_h}(G)$ , where  $G$  is the above graphs. In this paper we determine the index set of wheels and fans.

### 2. Index Set of Wheels

For  $n \geq 3$ , wheel on  $n+1$  vertices, denoted by  $W_n$  and is defined to be  $C_n \vee K_1$ , where  $C_n$  is the cycle of order  $n$ . The null set of wheels determined in [3].

**Theorem 2.1.** For any positive integer  $n \geq 3$ ,  $N(W_n) = \begin{cases} \mathbb{N} \setminus \{2\} & n = 0 \pmod{3}; \\ \mathbb{N} \setminus \{2, 3\} & \text{otherwise.} \end{cases}$

Let  $G$  be a graph. It is obvious that  $1 \in \text{In}_{\mathbb{Z}_2}(G)$  if and only if the degree of every vertex is odd. Therefore, since the degree set of the  $W_n$  is  $\{3, n\}$ ,  $W_n$  has 2-magic labeling with index 1, if and only if  $n$  is odd. First we need the following.

**Remark 2.2.** If  $G$  is a graph and  $c \in \text{In}_{\mathbb{Z}_h}(G)$ , then we have

$$2 \sum_{e \in E(G)} l(e) = c |V(G)| \pmod{h}.$$

**Lemma 2.3.** For any positive integers  $n$  and  $h \geq 3$ ,  $\text{In}_{\mathbb{Z}_h}(C_n) = \begin{cases} \mathbb{Z}_h \setminus \{0\} & n \text{ is odd;} \\ \mathbb{Z}_h & n \text{ is even.} \end{cases}$

*Proof.* First suppose that  $n$  is odd. Note that in any  $h$ -magic labeling of an odd cycle, the edges should have the same value. So,  $0 \notin \text{In}_{\mathbb{Z}_h}(C_n)$ . Let  $h = 2k + 1$  and  $x \in \mathbb{Z}_h \setminus \{0\}$ . We assign value  $(k + 1)x$  to all edges of  $C_n$ . So,  $x \in \text{In}_{\mathbb{Z}_h}(C_n)$  and thus  $\text{In}_{\mathbb{Z}_h}(C_n) = \mathbb{Z}_h \setminus \{0\}$ .

Now, assume that  $n$  is even and  $x \in \mathbb{Z}_h$ . If assign value 2 and  $-1$  to all edges of  $C_n$ , alternatively, then  $1 \in \text{In}_{\mathbb{Z}_h}(C_n)$ . Suppose that  $x \neq 1$ . We assign the values  $x - 1$  and 1, alternatively to all edges of  $C_n$ . So,  $x \in \text{In}_{\mathbb{Z}_h}(C_n)$ , as desired. □

Let  $u, u_1, \dots, u_n$  be the vertices of  $W_n$  and assume that  $u_1, \dots, u_n$  are arranged clockwise around a circle and  $u$  is the center vertex of wheel. In some cases, for convenience, we may use  $u_{n+1}$  for  $u_1$  and  $u_{-1}, u_0$  for  $u_{n-1}, u_n$ , respectively. Now, we have the following theorem.

**Theorem 2.4.** If  $n \geq 3$  and  $h \geq 4$  are positive integers, then

$$\text{In}_{\mathbb{Z}_h}(W_n) = \begin{cases} 2\mathbb{Z}_h & \text{if } n \text{ and } h \text{ are even;} \\ \mathbb{Z}_h & \text{otherwise.} \end{cases}$$

*Proof.* We would like to define a function  $l : E(W_n) \rightarrow \mathbb{Z}_h \setminus \{0\}$  such that  $l$  is an edge magic labeling of  $W_n$ . By Theorem 2.1,  $0 \in \text{In}_{\mathbb{Z}_h}(W_n)$ . We consider two cases:

**Case 1.** Suppose that  $n = 2r + 1$ . We consider two subcases:

**Subcase (i)** Let  $r$  be odd. The labeling of the edges is done as follows:

$l(uu_i) = 1$ , for  $1 \leq i \leq r + 1$  and  $l(uu_i) = -1$ , for  $r + 2 \leq i \leq n$ . Also, define:

$$l(u_i u_{i+1}) = \begin{cases} -1 & 1 \leq i \leq r + 1 \text{ and } i \text{ is odd;} \\ 1 & 1 \leq i \leq r + 1 \text{ and } i \text{ is even or } r + 2 \leq i \leq n. \end{cases}$$

Thus,  $l^+(u) = r+1+r(-1) = 1 \pmod{h}$  and obviously, for every  $i$ ,  $l^+(u_i) = 1$ . Therefore,  $1 \in \text{In}_{\mathbb{Z}_h}(W_n)$  and  $\text{In}_{\mathbb{Z}_h}(W_n) = \mathbb{Z}_h$ , as desired.

**Subcase (ii)** If  $r$  is even, then assign value 2 to  $uu_1$  and  $uu_2$  and define:

$l(uu_i) = -1$ , for  $3 \leq i \leq r + 3$  and  $l(uu_i) = 1$ , for  $r + 4 \leq i \leq n$ . Also, define:

$$l(u_1 u_2) = -2 \text{ and } l(u_i u_{i+1}) = \begin{cases} 1 & 2 \leq i \leq r + 3 \text{ or } r + 4 \leq i \leq n \text{ and } i \text{ is odd;} \\ -1 & r + 4 \leq i \leq n \text{ and } i \text{ is even.} \end{cases}$$

Thus,  $l^+(u) = 4 + (r + 1)(-1) + r - 2 = 1 \pmod{h}$  and obviously, for every  $i$ ,  $l^+(u_i) = 1$ . Now, if  $h$  is odd, then  $1 \in \text{In}_{\mathbb{Z}_h}(W_n)$  and  $\text{In}_{\mathbb{Z}_h}(W_n) = \mathbb{Z}_h$ . If  $h = 2k$ , then the above edge labeling of  $W_n$  shows that  $\text{In}_{\mathbb{Z}_h}(W_n) = 2\mathbb{Z}_h \setminus \{k\}$ . Also, if assign value  $k$  to all edges of  $W_n$ , then  $k \in \text{In}_{\mathbb{Z}_h}(W_n)$ . Therefore,  $\text{In}_{\mathbb{Z}_h}(W_n) = \mathbb{Z}_h$ , as desired.

**Case 2.** Let  $n = 2r$  and  $x = 2t$  be a non-zero element of  $\mathbb{Z}_h$ . We consider two subcases:

**Subcase (i)** If  $r$  is even, then the labeling done as follows:

$l(uu_1) = l(uu_2) = x$ ,  $l(uu_3) = l(uu_4) = -t$ .

$$l(uu_5) = l(uu_6) = 1, l(uu_7) = l(uu_8) = -1, \dots, l(uu_{n-3}) = l(uu_{n-2}) = 1 \text{ and } l(uu_{n-1}) = l(uu_n) = -1,$$

and define:

$l(u_1 u_2) = -x$ ,  $l(u_3 u_4) = t$ ,  $l(u_5 u_6) = -1$ ,  $l(u_7 u_8) = 1$ ,  $\dots$ ,  $l(u_{n-3} u_{n-2}) = -1$  and  $l(u_{n-1} u_n) = 1$ .

Also, assign value  $x$  to the remaining edges of  $W_n$ .

**Subcase (ii)** If  $r$  is odd, then the labeling done as follows:

$l(uu_1) = l(uu_2) = t$  and

$$l(uu_3) = l(uu_4) = 1, l(uu_5) = l(uu_6) = -1, \dots, l(uu_{n-3}) = l(uu_{n-2}) = 1 \text{ and } l(uu_{n-1}) = l(uu_n) = -1.$$

Also, define:

$l(u_1 u_2) = -t$ ,  $l(u_3 u_4) = -1$ ,  $l(u_5 u_6) = 1$ ,  $\dots$ ,  $l(u_{n-3} u_{n-2}) = -1$  and  $l(u_{n-1} u_n) = 1$  and assign value  $x$  to the remaining edges of  $W_n$ .

First suppose that  $h$  is odd and  $t = 1$ . Then Subcases (i) and (ii) show that  $2 \in \text{In}_{\mathbb{Z}_h}(W_n)$  and so,  $\text{In}_{\mathbb{Z}_h}(W_n) = \mathbb{Z}_h$ . Now, assume that  $h$  is even. If  $c \in \mathbb{Z}_h \setminus \{0\}$  is odd, then by Remark 2.2, we have  $c \notin \text{In}_{\mathbb{Z}_h}(W_n)$  and Subcases (i) and (ii) show that  $\text{In}_{\mathbb{Z}_h}(W_n) = 2\mathbb{Z}_h$ . The proof is complete.  $\square$

**Remark 2.5.** If  $h = 3$  and  $n \geq 3$  is positive integer, then Theorem 2.1 and the proof of Theorem 2.4, imply that:

$$\text{In}_{\mathbb{Z}_3}(W_n) = \begin{cases} \mathbb{Z}_3 \setminus \{0\} & n \not\equiv 0 \pmod{3}; \\ \mathbb{Z}_3 & n \equiv 0 \pmod{3}. \end{cases}$$

### 3. Index Set of Fans

For  $n \geq 2$ , fan on  $n+1$  vertices, denoted by  $F_n$  and is defined to be  $P_n \vee K_1$ , where  $P_n$  is the path of order  $n$ . The null set of fans determined in [3].

**Theorem 3.1.**  $N(F_2) = 2\mathbb{N}$ ,  $N(F_3) = 2\mathbb{N} \setminus \{2\}$  and for any  $n \geq 4$ ,

$$N(F_n) = \begin{cases} \mathbb{N} \setminus \{2\} & n \equiv 1 \pmod{3}; \\ \mathbb{N} \setminus \{2, 3\} & \text{otherwise.} \end{cases}$$

**Remark 3.2.** Note that the degree set of  $F_n$  is  $\{2, 3, n\}$ . So, it has no 2-magic labeling with index 1. On the other hand by Theorem 3.1, we have  $0 \notin \text{In}_{\mathbb{Z}_2}(F_n)$ . Therefore,  $\text{In}_{\mathbb{Z}_2}(F_n) = \emptyset$ . Also,  $F_2 = C_3$  and by Lemma 2.3,  $\text{In}_{\mathbb{Z}_h}(F_2) = \mathbb{Z}_h \setminus \{0\}$ , where  $h$  is a positive integer.

For the general case, let  $u_1 \sim u_2 \sim \dots \sim u_n$  be the vertices of the path  $P_n$  and  $u$  be the central vertex of the fan. We call the edges  $uu_i$  ( $1 \leq i \leq n$ ) blades of the fan.

**Lemma 3.3.** Let  $h \geq 3$  be a positive integer. Then

$$\text{In}_{\mathbb{Z}_h}(F_3) = \begin{cases} \emptyset & h \text{ is odd}; \\ \mathbb{Z}_h & h \text{ is even.} \end{cases}$$

First suppose that  $h$  is odd and  $l : E(F_3) \rightarrow \mathbb{Z}_h \setminus \{0\}$  is an  $h$ -magic labeling with index  $c$  of  $F_3$ , that illustrated in Figure 1, where  $a, b \neq c$  and  $a, b, z \in \mathbb{Z}_h \setminus \{0\}$ . We should have  $a + b + z = c$ ,  $z + 2c - (a + b) = c$ .

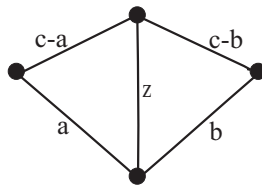


Figure 1

If we add these equations, then we get  $2z = 0 \pmod{h}$ . Thus  $z = 0 \pmod{h}$ , a contradiction. Now, assume that  $h = 2k$  and  $x$  is a non-zero element of  $\mathbb{Z}_h$ . If  $x = 2t$  (for some  $t$ ), then consider Figure 2 Part (a) and if  $x = 2t + 1$  (for some  $t$ ), then consider Figure 2 Part (b). Thus,  $x \in \text{In}_{\mathbb{Z}_h}(F_3)$  and so,  $\text{In}_{\mathbb{Z}_h}(F_3) = \mathbb{Z}_h$ .

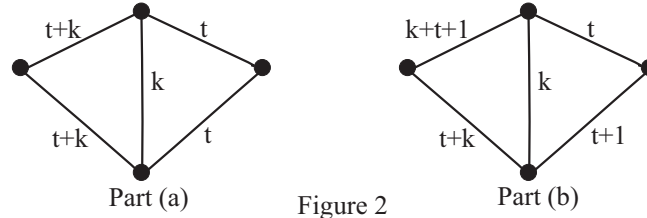


Figure 2

The proof is complete.

**Lemma 3.4.** For a positive integer  $h \geq 3$ ,

$$\text{In}_{\mathbb{Z}_h}(\mathbb{F}_4) = \begin{cases} \{0\} & h = 3; \\ \mathbb{Z}_h & h \neq 3 \text{ is odd}; \\ 2\mathbb{Z}_h & \text{otherwise.} \end{cases}$$

*Proof.* By Theorem 3.1, we have  $0 \in \text{In}_{\mathbb{Z}_h}(\mathbb{F}_4)$ . We consider three cases:

**Case 1.** Suppose that  $h = 3$  and  $1 \in \text{In}_{\mathbb{Z}_3}(\mathbb{F}_4)$ . Since  $d(u) = 4$ , four edges incident with  $u$  have value  $\{2, 2, 2, 1\}$  or  $\{1, 1, 1, 1\}$ . Since,  $d(u_1) = d(u_4) = 2$ , thus,  $l(uu_1) \neq 1$  and  $l(uu_4) \neq 1$ . So, 4 edges incident with  $u$  have value  $\{2, 2, 2, 1\}$ . Therefore,  $l(uu_1) = l(uu_4) = 2$ . With no loss of generality suppose that  $l(uu_2) = 2$ . So, we have  $l(u_1u_2) = 2$ . Therefore,  $l(u_2u_3) = 0$ , a contradiction. Also, if  $2 \in \text{In}_{\mathbb{Z}_3}(\mathbb{F}_4)$  and we multiply the values of all edges of  $\mathbb{F}_4$  in 2, then we obtain that  $1 \in \text{In}_{\mathbb{Z}_3}(\mathbb{F}_4)$ , a contradiction. Therefore,  $2 \notin \text{In}_{\mathbb{Z}_3}(\mathbb{F}_4)$  and so,  $\text{In}_{\mathbb{Z}_3}(\mathbb{F}_4) = \{0\}$ .

**Case 2.** Let  $3 \neq h = 2k + 1$  and  $x$  be a non-zero element of  $\mathbb{Z}_h$ . If  $x = 2t$  and  $x \neq -2, 2, 4$ , then Figure 3 Part (a) shows that  $x \in \text{In}_{\mathbb{Z}_h}(\mathbb{F}_4)$  and if  $x = 2t + 1$  and  $x \neq -1, 1$ , then Figure 3 Part (b) shows that  $x \in \text{In}_{\mathbb{Z}_h}(\mathbb{F}_4)$ .

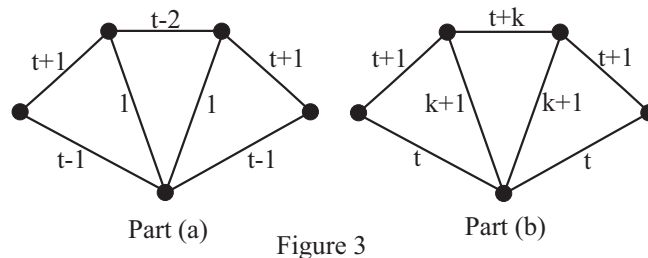


Figure 3

If  $x = 1$ , then Figure 4, Part (a) shows that  $1 \in \text{In}_{\mathbb{Z}_h}(\mathbb{F}_n)$  and if we multiply all values of edges in value  $-1$ , then we obtain that  $-1 \in \text{In}_{\mathbb{Z}_h}(\mathbb{F}_4)$ . If  $x = 2$ , then Figure 4, Part (b) shows that  $2 \in \text{In}_{\mathbb{Z}_h}(\mathbb{F}_4)$  and if we multiply all values of edges in value  $-1$  and  $2$ , respectively, then we obtain that  $-2 \in \text{In}_{\mathbb{Z}_h}(\mathbb{F}_4)$  and  $4 \in \text{In}_{\mathbb{Z}_h}(\mathbb{F}_4)$ , respectively. Therefore,  $\text{In}_{\mathbb{Z}_h}(\mathbb{F}_4) = \mathbb{Z}_h$ , as desired.

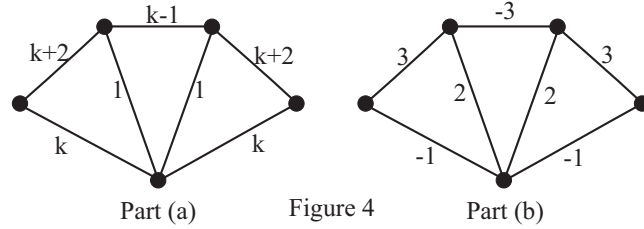


Figure 4

**Case 3.** Let  $h = 2k$ . If  $c \in \mathbb{Z}_h \setminus \{0\}$  is odd, then by Remark 2.2,  $c \notin \text{In}_{\mathbb{Z}_h}(F_4)$ . Let  $x = 2t$  be a non-zero element of  $\mathbb{Z}_h$ . Figure 5 shows that  $x \in \text{In}_{\mathbb{Z}_h}(F_4)$ . Therefore,  $\text{In}_{\mathbb{Z}_h}(F_4) = 2\mathbb{Z}_h$ .

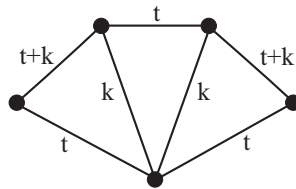


Figure 5

The proof is complete. □

**Lemma 3.5.** For a positive integer  $h \geq 3$ ,  $\text{In}_{\mathbb{Z}_h}(F_5) = \begin{cases} \emptyset & h = 3; \\ \mathbb{Z}_h & h \neq 3. \end{cases}$

*Proof.* By Theorem 3.1, we have  $0 \notin \text{In}_{\mathbb{Z}_3}(F_5)$  and if  $h \neq 3$ , then  $0 \in \text{In}_{\mathbb{Z}_h}(F_5)$ . Let  $1 \in \text{In}_{\mathbb{Z}_3}(F_5)$ . Since  $d(u_1) = d(u_5) = 2$ , then  $l(uu_1) \neq 1$  and  $l(uu_5) \neq 1$ . Therefore,  $l(uu_1) = l(uu_5) = 2$ . So, for  $2 \leq i \leq 4$ ,  $uu_i$  have value 1 or 2. First suppose that for  $2 \leq i \leq 4$ ,  $l(uu_i) = 1$ . Then we have  $l(u_1u_2) = l(u_4u_5) = 2$  and  $l(u_iu_{i+1}) = 1$  ( $i = 2, 3$ ) and so  $l^+(u_3) = 0 \pmod{3}$ , a contradiction. Also, if suppose that for  $i = 2, 3, 4$ ,  $l(uu_i) = 2$ , then we have  $l(u_1u_2) = l(u_4u_5) = 2$ , so  $l(u_2u_3) = l(u_3u_4) = 0$ , a contradiction. Also, if  $2 \in \text{In}_{\mathbb{Z}_3}(F_5)$  and we multiply the value of all edges of  $F_5$  in 2, then we find that  $1 \in \text{In}_{\mathbb{Z}_3}(F_5)$ , a contradiction.

Now, let  $h \neq 3$  and  $x \neq 2$  be a non-zero element of  $\mathbb{Z}_h$ . Figure 6 Part (a) shows that  $x \in \text{In}_{\mathbb{Z}_h}(F_5)$  and Figure 6 Part (b) shows that,  $2 \in \text{In}_{\mathbb{Z}_h}(F_5)$ .

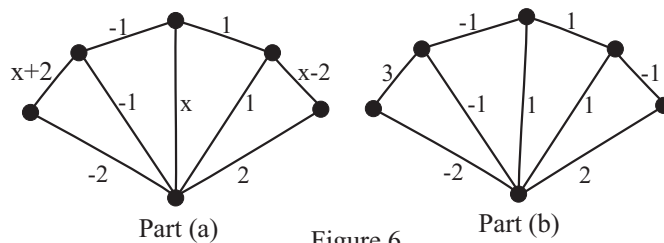


Figure 6

Thus,  $\text{In}_{\mathbb{Z}_h}(F_5) = \mathbb{Z}_h$  and the proof is complete. □

**Theorem 3.6.** If  $n \geq 6$  and  $h \geq 4$  are positive integers, then

$$\text{In}_{\mathbb{Z}_h}(F_n) = \begin{cases} 2\mathbb{Z}_h & n \text{ and } h \text{ are both even;} \\ \mathbb{Z}_h & \text{otherwise.} \end{cases}$$

*Proof.* By Theorem 3.1,  $0 \in \text{In}_{\mathbb{Z}_h}(F_n)$ . Let  $x$  be a non-zero element of  $\mathbb{Z}_h$ . We would like to define a function  $l : E(F_n) \rightarrow \mathbb{Z}_h \setminus \{0\}$  such that  $l$  is an edge magic labeling of  $F_n$ . We consider three cases:

**Case 1.** Let  $n = 2r + 1$  and  $r$  be even. First suppose that  $x \neq 2, -2$ . We label the edges of  $F_n$  as follows:

$-l(uu_1) = l(uu_n) = 2, l(uu_{r+1}) = x$  and  $-l(uu_i) = l(uu_{n-i+1}) = 1$ , for  $2 \leq i \leq r$ . Also, define:

$$l(u_i u_{i+1}) = \begin{cases} x + 2 & 1 \leq i \leq r \text{ and } i \text{ is odd;} \\ -1 & 1 \leq i \leq r \text{ and } i \text{ is even;} \\ 1 & r + 1 \leq i \leq n \text{ and } i \text{ is odd;} \\ x - 2 & r + 1 \leq i < n \text{ and } i \text{ is even.} \end{cases}$$

So,  $l^+(u) = 2 - 2 + x + (r - 1)(1) + (r - 1)(-1) = x \pmod{h}$  and obviously, for every  $i, l^+(u_i) = x$ .

Now, assume that  $x = 2$ . Label all edges of  $F_n$  as follows:

$l(uu_{n-1}) = l(uu_i) = -1$ , for  $2 \leq i \leq r - 2, l(uu_{r+1}) = -2$  and assign value 1 to the remaining edges.

Also, define:

$$l(u_i u_{i+1}) = \begin{cases} 1 & 1 \leq i \leq r - 2 \text{ and } i \text{ is odd;} \\ 2 & r + 1 \leq i \leq n - 2 \text{ and } i \text{ is odd or } 1 \leq i \leq r - 2 \text{ and } i \text{ is even;} \\ -1 & r + 1 \leq i \leq n - 2 \text{ and } i \text{ is even,} \end{cases}$$

and  $l(u_{r-1}u_r) = -1, l(u_r u_{r+1}) = 1$  and  $l(u_{n-1}u_n) = 1$ .

So,  $l^+(u) = -2 - 1 + (r - 3)(-1) + (r + 2)(1) = 2 \pmod{h}$  and obviously, for every  $i, l^+(u_i) = x$ . Also, if we multiply all values of edges in  $-1$ , then we obtain that  $-2 \in \text{In}_{\mathbb{Z}_h}(F_n)$ , as desired.

**Case 2.** Let  $n = 2r + 1$  and  $r$  be odd. First suppose that  $x \neq 2$ . Label all edges of  $F_n$  as follows:

$l(uu_1) = 2, l(uu_n) = x - 2, l(uu_{n-1}) = -2, l(uu_i) = 1$ , for  $2 \leq i \leq r + 1$  and  $l(uu_i) = -1$ , for  $r + 2 \leq i \leq n - 2$ . Also, define:

$$l(u_i u_{i+1}) = \begin{cases} 1 & 1 \leq i \leq n - 2 \text{ and } i \text{ is even;} \\ x - 2 & 1 \leq i \leq r \text{ and } i \text{ is odd;} \\ x & r + 1 \leq i \leq n - 1 \text{ and } i \text{ is odd,} \end{cases}$$

and  $l(u_{n-1}u_n) = 2$ .

So,  $l^+(u) = x - 2 + 2 - 2 + r + (r - 2)(-1) = x \pmod{h}$  and obviously, for every  $i, l^+(u_i) = x$ .

Now, assume that  $x = 2$ . Label all edges of  $F_n$  as follows:

$l(uu_1) = l(uu_n) = -l(uu_2) = -l(uu_{n-1}) = 1, -l(uu_r) = l(uu_{r+1}) = l(uu_{r+2}) = 2, l(uu_i) = 2$ , for  $3 \leq i \leq r - 1$  and  $l(uu_i) = -2$ , for  $r + 3 \leq i \leq n - 2$ . Also, define:

$l(u_1 u_2) = l(u_{n-1}u_n) = 1, l(u_{r+1}u_{r+2}) = -2$  and for  $2 \leq i \leq r - 1$ ,

$$l(u_i u_{i+1}) = \begin{cases} 2 & i \text{ is even;} \\ -2 & i \text{ is odd.} \end{cases}$$

Also, assign value 2 to the remaining edges and obtain the result.

So,  $l^+(u) = 2 - 2 + 4 - 2 + (r - 3)(2) + (r - 3)(-2) = 2 \pmod{h}$  and obviously, for every  $i, l^+(u_i) = x$ .

**Case 3.** Let  $n = 2r$ . First suppose that  $h = 2k + 1$ . We consider two subcases:

**Subcase (i)** Suppose that  $r$  is odd. Define:

$$l(uu_1) = l(uu_n) = k + 1, l(uu_2) = l(uu_{n-1}) = k \text{ and } l(uu_i) = \begin{cases} k + 1 & 3 \leq i \leq r + 1; \\ k & r + 2 \leq i \leq n - 2. \end{cases}$$

Also, define:

$$l(u_1u_2) = k + 1 \text{ and } l(u_iu_{i+1}) = \begin{cases} k & 2 \leq i \leq r + 1 \text{ and } i \text{ is odd;} \\ k + 1 & r + 2 \leq i \leq n \text{ and } i \text{ is odd;} \\ 1 & 2 \leq i \leq n \text{ and } i \text{ is even.} \end{cases}$$

So, we have  $l^+(u) = 2(k + 1) + 2k + (r - 1)(k + 1) + (r - 3)k = 1 \pmod{h}$  and obviously, for every  $i$ ,  $l^+(u_i) = 1$ . Thus,  $1 \in \text{In}_{\mathbb{Z}_h}(\mathbb{F}_n)$  and therefore,  $\text{In}_{\mathbb{Z}_h}(\mathbb{F}_n) = \mathbb{Z}_h$ .

**Subcase (ii)** Suppose that  $r$  is even. Assign the value  $k + 1$  to  $uu_1$  and  $uu_n$  and value 1 to  $uu_2$  and  $uu_{n-1}$ . Also define,

$$l(uu_i) = \begin{cases} k & 3 \leq i \leq r + 2; \\ k + 1 & r + 3 \leq i \leq n - 2. \end{cases}$$

Also, define:

$$l(u_1u_2) = l(u_{n-1}u_n) = k + 1 \text{ and } l(u_iu_{i+1}) = \begin{cases} 2 & 2 \leq i \leq r + 2 \text{ and } i \text{ is odd;} \\ 1 & r + 3 \leq i \leq n - 2 \text{ and } i \text{ is odd;} \\ k & 2 \leq i \leq n - 2 \text{ and } i \text{ is even.} \end{cases}$$

So,  $l^+(u) = 2(k + 1) + 2 + rk + (r - 4)(k + 1) = 1 \pmod{h}$  and obviously, for every  $i$ ,  $l^+(u_i) = 1$ . Thus,  $1 \in \text{In}_{\mathbb{Z}_h}(\mathbb{F}_n)$  and so,  $\text{In}_{\mathbb{Z}_h}(\mathbb{F}_n) = \mathbb{Z}_h$ .

Now, assume that  $h = 2k$ . If  $c \in \mathbb{Z}_h \setminus \{0\}$  is odd, then by Remark 2.2,  $c \notin \text{In}_{\mathbb{Z}_h}(\mathbb{F}_n)$ . Let  $x = 2t$ .

Define:

$l(uu_1) = l(uu_n) = t + k$  and  $l(uu_i) = k$ , for  $2 \leq i \leq n - 1$ . Also, define:

$$l(u_iu_{i+1}) = \begin{cases} t + k & i \text{ is odd;} \\ t & i \text{ is even.} \end{cases}$$

Therefore,  $x \in \text{In}_{\mathbb{Z}_h}(\mathbb{F}_n)$  and so,  $\text{In}_{\mathbb{Z}_h}(\mathbb{F}_n) = 2\mathbb{Z}_h$ . The proof is complete. □

Finally, we have the following remark.

**Remark 3.7.** If  $h = 3$  and  $n \geq 6$  are positive integers, then Theorem 3.1 and the proof of Theorem 3.6, imply that:

$$\text{In}_{\mathbb{Z}_3}(\mathbb{F}_n) = \begin{cases} \mathbb{Z}_3 \setminus \{0\} & n \not\equiv 1 \pmod{3}; \\ \mathbb{Z}_3 & n \equiv 1 \pmod{3}. \end{cases}$$



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