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RESTRAINED ROMAN DOMINATION IN GRAPHS

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Abstract. A *Roman dominating function* (RDF) on a graph $G = (V, E)$ is defined to be a function $f : V \to \{0, 1, 2\}$ satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. A set $S \subseteq V$ is a *Restrained dominating set* if every vertex not in S is adjacent to a vertex in S and to a vertex in $V - S$. We define a *Restrained Roman dominating function* on a graph $G = (V, E)$ to be a function $f : V \to \{0, 1, 2\}$ satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$ and at least one vertex w for which $f(w) = 0$. The *weight* of a Restrained Roman dominating function is the value $f(V) = \sum_{u \in V} f(u)$. The minimum weight of a Restrained Roman dominating function on a graph G is called the *Restrained Roman domination number* of G and denoted by $\gamma_{rR}(G)$. In this paper, we initiate a study of this parameter.

1. **Introduction**

In the 4th century A. D., when the Roman Empire was under attack during the period of Emperor Constantine the Great, he had the requirement that an army or a legion could be sent from its home to defend a neighbouring location only if there was a second army which would stay and protect the home. Thus there are two types of armies, stationary and traveling. Each vertex with no army must have a neighbouring vertex with a traveling army. Stationary armies then dominate their own vertices and a vertex with two armies are dominated by its stationary army and its open neighbourhood is dominated by the traveling army. The objective, of course, is to minimize the total number of legions needed. The problem generalizes to arbitrary graphs.

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Cockayne et al. [7] defined a *Roman dominating function* (RDF) on a graph $G = (V, E)$ to be a function $f : V \to \{0, 1, 2\}$ satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The weight of f is $f(V) = \sum_{v \in V} f(v)$. The *Roman domination number*, denoted by $\gamma_R(G)$, is the minimum weight of an RDF in G. An RDF of weight $\gamma_R(G)$ is called a $\gamma_R(G)$-function. Roman domination in graphs has been studied, for example in [7, 10, 13, 15, 16, 21, 22, 23, 24, 25]. This definition of a Roman dominating function was motivated by an article in Scientific American by Ian Stewart entitled "Defend the Roman Empire!" [29].

By a graph $G = (V, E)$, we mean a simple, finite, undirected, connected graph with $|V| = n$. For graph theoretic terminology we refer to Charatrand and Lesniak. [1]. A set of vertices S is a *dominating set* if $N[S] = V$, or equivalently, every vertex in $V \setminus S$ is adjacent to at least one vertex in S. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set in G, and a dominating set S of minimum cardinality is called a γ-set of G. The literature on Domination and its variations in graphs has been surveyed and detailed in the two books by Haynes et al. [11, 12]. A set $S \subseteq V$ is a *Restrained dominating set* if every vertex not in S is adjacent to a vertex in S and to a vertex in $V - S$. The *Restrained domination number* of G, denoted by $\gamma_r(G)$, is the smallest cardinality of a Restrained dominating set of G. Restrained domination in graphs has been in studied, for example in [2, 3, 4, 5, 14].

We relate the Roman domination and Restrained domination of a graph as follows. A *Restrained Roman dominating function* on a graph $G = (V, E)$ is a function $f : V \to \{0, 1, 2\}$ satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$ and at least one vertex w for which $f(w) = 0$. The *weight* of a Restrained Roman dominating function is the value $f(V) = \sum_{u \in V} f(u)$. The minimum weight of a Restrained Roman dominating function on a graph G is called the *Restrained Roman domination number* of G and denoted by $\gamma_{rR}(G)$. A Restrained Roman dominating function of weight $\gamma_{rR}(G)$ is called a $\gamma_{rR}(G)$-function. In this paper, we initiate a study of this parameter. In section 3, we give some specific values of Restrained Roman domination number of some standard graphs. In section 4, we characterize the class of graphs for which $\gamma_{rR}(G) = n$ and we also characterize graphs for which $\gamma_{rR}(G) = \gamma_r(G)$. In section 5, we characterize graphs for which $\gamma_{rR}(G) = \gamma_r(G) + 1$. In section 6, we characterize graphs for which $\gamma_{rR}(G) = \gamma_r(G) + 2$. In section 7, we study some properties of split graphs with respect to the parameter $\gamma_{rR}(G)$.

2. **Notation**

The *degree* of a vertex v in a graph G is the number of edges of G incident with v and is denoted by $\deg(v)$. A vertex of degree zero in G is called an *isolated vertex*, while a vertex of degree one is called a *leaf* vertex or a *pendant vertex* of G. The *minimum degree* of G is the minimum degree among the vertices of G and is denoted by $\delta(G)$. The *maximum degree* of G is defined as the maximum degree among the vertices of G and is denoted by $\triangle(G)$.

A graph G is *bipartite* if the vertex set can be partitioned into two disjoint subsets A and B such that the vertices in A are only adjacent to vertices in B and vice versa. $K_{r,s}$ denotes the *complete bipartite graph* where $V = A \cup B$, $|A| = r, |B| = s$, A and B are independent sets and every vertex in A is adjacent to every vertex in B.

A complete bipartite graph of the form $K_{1,n}$ is called a *star* graph. we call the vertex of degree n, the head vertex and the vertices of degree one, the end vertices. We consider K_2 also to be a star with one vertex as the head and the other vertex as the end vertex.

A connected graph having no cycle is called a *tree.* A *split graph* is a graph $G = (V, E)$ whose vertices can be partitioned into two sets X and Y where the vertices in X are independent and vertices in Y form a complete graph. A *unicyclic graph* is a graph with exactly one cycle. A *caterpillar* is a tree with the property that the removal of end vertices leaves a path called the *spine* of the caterpillar. A *lobster* is a tree with the property that the removal of end vertices leaves a caterpillar. The spine of the lobster is the spine of the corresponding caterpillar.

In a connected graph G, the *distance between two vertices* u and v is the number of edges in a shortest path joining u and v if any; and is denoted $d(u, v)$. If $u \in V$ and $S \subset V$, then $d(u, S)$ denotes the minimum distance between u and any vertex of S. The *eccentricity* of a vertex v is $ecc(v) = max\{d(u, w); w \in V\}$. The *radius* of a graph G is $rad(G) = min\{ecc(v) : v \in V\}$ and the *diameter* of the graph G is $diam(G) = max\{ecc(v) : v \in V\}$.

A set S of vertices is called *independent* if no two vertices in S are adjacent.

For any set $S \subseteq V$, the *induced subgraph* S is the maximal subgraph of G with vertex set S and is denoted by $G[S]$.

A *support* is a vertex which is adjacent to at least one leaf vertex. A *weak support* is a vertex which is adjacent to exactly one leaf vertex. A *strong support* is a vertex which is adjacent to at least two leaf vertices.

For any vertex $v \in V$, the *open neighbourhood* of v is the set $N(v) = \{u \in V/uv \in E\}$ and the *closed neighbourhood* is the set $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V$, the *open neighbourhood* is $N(S) = \cup_{v \in S} N(v)$ and the *closed neighbourhood* is $N[S] = N(S) \cup S$.

3. Specific Values of Restrained Roman Domination

For a graph $G = (V, E)$, let $f \to \{0, 1, 2\}$, and let (V_0, V_1, V_2) be the ordered partition of V induced by f, where $V_i = \{v \in V : f(v) = i\}$ for $i = 0, 1, 2$. Note that there exists a 1-1 correspondence between the functions $f \to \{0, 1, 2\}$ and the ordered partitions (V_0, V_1, V_2) of V. Thus, we will write $f = (V_0, V_1, V_2)$. In this section, we obtain the value of $\gamma_{rR}(G)$ for some standard graphs.

It is obvious that $\gamma_{rR}(P_2) = 2$ and $\gamma_{rR}(P_3) = 3$.

Theorem 3.1. *For $n \geq 4$, $\gamma_{rR}(P_n) = \frac{2n+3+r}{3}$, where $n \equiv r(mod\, 3)$ and $r \in \{1, 2, 3\}$.*

Proof. Let $G = P_n$ and $V(G) = \{v_1, v_2, \ldots, v_n\}$, where v_i is adjacent to v_{i+1} for $i = 1, 2, \ldots, n-1$. Let $f = (V_0, V_1, V_2)$ be a $\gamma_{rR}(P_n)$-function. Clearly each component of $G[V_0]$ is a K_2. Let m be the number of components of $G[V_0]$. Then $|V_2| \geq m+1$. It follows that $2m + m + 1 \leq n$ implying that $m \leq \frac{n-1}{3}$. Now $\gamma_{rR}(G) \geq 2(n-2m) = 2n-4m \geq \frac{2n+4}{3}$. Thus for $r = 1$, $\gamma_{rR}G \geq \frac{2n+3+r}{3}$. If $r = 2$, then $2m+m+1 < n$ implying that $m \leq \frac{n-2}{3}$. Thus $\gamma_{rR}(G) \geq 2(n-2m)-1 = 2n-4m-1 \geq \frac{2n+5}{3} = \frac{2n+3+r}{3}$. Similarly, if $r = 3$, then $\gamma_{rR}(G) \geq \frac{2n+3+r}{3}$.
On the other hand, for $r = 1$, $f_1 = (V(G) \setminus \{v_{3i+1} : 0 \leq i \leq \lfloor \frac{n}{3} \rfloor\}, \emptyset, \{v_{3i+1} : 0 \leq i \leq \lfloor \frac{n}{3} \rfloor\})$ is a restrained roman dominating function for G, for $r = 2$, $f_2 = (V(G) \setminus \{v_n, v_{3i+1} : 0 \leq i \leq \lfloor \frac{n}{3} \rfloor\}, \{v_n\}, \{v_{3i+1} : 0 \leq i \leq \lfloor \frac{n}{3} \rfloor\})$ is a restrained roman dominating function for G, for $r = 3$, $f_3 = (V(G) \setminus \{v_n, v_{n-1}, v_{3i+1} : 0 \leq i \leq \lfloor \frac{n}{3} \rfloor\}, \{v_n, v_{n-1}\}, \{v_{3i+1} : 0 \leq i \leq \lfloor \frac{n}{3} \rfloor\})$ is a restrained roman dominating function for G. □

Similarly, we obtain the following Theorem.

Theorem 3.2. *For cycles C_n,*

$$\gamma_{rR}(C_n) = \begin{cases} \frac{2n+3+r}{3}, & n \equiv r(mod\, 3), \quad r \in \{1, 2\} \\ \frac{2n}{3}, & n \equiv 0(mod\, 3) \end{cases}$$

Theorem 3.3. *For complete graphs K_n,*

$$\gamma_{rR}(K_n) = 2.$$

Theorem 3.4. *For complete bipartite graphs $K_{m,n}$,*

$$\gamma_{rR}(K_{m,n}) = 4.$$

4. Properties of Restrained Roman Dominating Sets

We have the following inequality chain.

Observation 4.1. *For any graph G, $\gamma_r(G) \leq \gamma_{rR}(G) \leq 2\gamma_r(G)$.*

Observation 4.2. *If G is a graph of order n, which contains a vertex of degree $n-1$ and $\delta(G) > 1$, then $\gamma_r(G) = 1$ and $\gamma_{rR}(G) = 2$.*

Proof. Let $v \in V(G)$ be such that $\deg(v) = n-1$. Then clearly $S = \{v\}$ is a Restrained dominating set of G and hence $\gamma_r(G) = 1$. Now we define $f = (V_0, V_1, V_2)$ where $V_0 = V - \{v\}$; $V_1 = \phi$; $V_2 = \{v\}$. Clearly f is a $\gamma_{rR}(G)$- function and $\gamma_{rR}(G) = 2$. □

Observation 4.3. *If G contains a triangle, then $\gamma_{rR}(G) < n$.*

Proof. Let u, v, w form a triangle in G. Define $f : V \to \{0, 1, 2\}$ by $f(u) = 2, f(v) = f(w) = 0$ and $f(x) = 1$ for all $x \in V \setminus \{u, v, w\}$. Hence $\gamma_{rR}(G) < n$ □

Theorem 4.4. *For any graph G which is not a tree $\gamma_{rR}(G) = n$ if and only if G is isomorphic to C_4 or C_5 or G_1 or G_2 where G_1 and G_2 are given in Figure* 1.

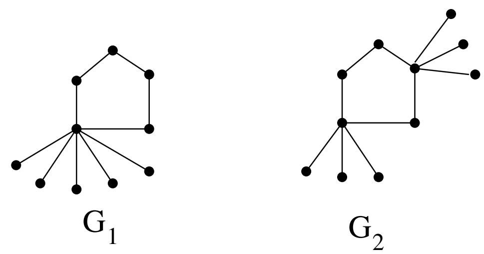

Figure 1.

Proof. Let f be a $\gamma_{rR}(G)$-function with $\gamma_{rR}(G) = n$ and let l be the length of a longest path P in G. Let $P = (v_1, v_2, \ldots, v_{l+1})$. First we claim that $l \leq 5$. Suppose $l \geq 6$. Define $g : V \to \{0, 1, 2\}$ by

$$g(v_i) = \begin{cases} 2, & i = 1, 4, 7. \\ 0, & i = 2, 3, 5, 6. \end{cases}$$

and $g(x) = 1$ for all $x \in V \setminus \{v_i : 1 \leq i \leq 7\}$, then $g(V) < f(V)$, which is a contradiction. Hence $l \leq 5$.

Since G is not a tree, it contains at least one cycle say $C_k = (v_1, v_2, \ldots v_k, v_1)$. Since $l \leq 5$, $k \leq 6$. If $k = 6$, then clearly the function $g : V \to \{0, 1, 2\}$ defined by $g(v_1) = g(v_4) = 2$ and $g(v_i) = 0$, $i = 2, 3, 5, 6$ and $g(u) = 1$ for every $u \in V(G) \setminus V(C_6)$ imply that $f(V) > g(V)$, which is a contradiction.

Let $k = 5$ and $V(G) = V(C_5)$. If $G \cong C_5$, we are through. Otherwise G will have a triangle, which implies that $\gamma_{rR}(G) < n$, which is a contradiction. Let $V(G) \neq V(C_5)$. Since $l \leq 5$, each vertex w in $V(G) \setminus V(C_5)$ is adjacent to a vertex in C_5. Now we claim that w has exactly one neighbour in C_5. Suppose w has two neighbours say $v_i, v_j \in C_5$. If v_i, v_j are adjacent, then w, v_i, v_j will form a triangle so that $\gamma_{rR}(G) < n$, which is a contradiction.

If v_i and v_j are not adjacent, without loss of generality $i = 1$, $j = 3$.

Define a function g by $g(w) = 0$, $g(v_1) = g(v_4) = 2$, $g(v_2) = g(v_3) = 0$ and $g(v_5) = 1$, $g(u) = 1$ for all $u \in V(G) \setminus V(C_5), u \neq w$. We see that $g(V) < f(V)$, which is a contradiction. Hence w is adjacent to exactly one vertex of C_5. Since $l \leq 5$, no two consecutive vertices in C_5 are of degree at least 3. Hence $G \cong G_1$ *or* G_2. Let $k = 4$ and $V(G) = V(C_4)$. If $G \cong C_4$ we are through. Otherwise G will have a triangle, which implies that $\gamma_{rR}(G) < n$, which is a contradiction. Let $V(G) \neq V(C_4)$. Now there exists at least one vertex $w \in V(G) \setminus V(C_4)$ such that w is adjacent to a vertex in C_4. Without loss of generality, let w be adjacent to v_1. Define g by $g(w) = g(v_3) = 2$, $g(v_i) = 0$, $i = 1, 2, 4$ and $g(u) = 1$ for all $u \notin V(C_4), u \neq w$. Then $g(V) < f(V)$, which is a contradiction. If $k = 3$, clearly $\gamma_{rR}(G) < n$, which is a contradiction.

Conversely, if G is isomorphic to the graph given in the theorem, then clearly $\gamma_{rR}(G) = n$. □

Theorem 4.5. *For any tree T, $\gamma_{rR}(T) = n$ if and only if T is either a star or a caterpillar with spine of length at most 3 or a lobster with spine of length at most 3 and diameter at most 5.*

Proof. Suppose $\gamma_{rR}(T) = n$. Let f be a $\gamma_{rR}(T)$-function. Since $\gamma_{rR}(T) = n$, each vertex in V_2 is adjacent to exactly one vertex in V_0 . Let $P = v_1, v_2, \ldots, v_k$ be a longest path in T. Then by Theorem 4.4, $l \leq 5$ where l is the length of P.

If $l = 2$, then T is a star.

If $3 \leq l \leq 5$ and each vertex in $V(T) \setminus V(P)$ is adjacent to vertices of P, then T reduces to a caterpillar with spine length at most 3.

If $3 \leq l \leq 5$ and there exists a vertex w in $V(T) \setminus V(P)$ which is not adjacent to a vertex in P, then we claim that $d(w, P) \leq 2$. Suppose not, then there exists a vertex $v \in V(P)$ such that $d(w, v) \geq 3$. Without loss of generality, let the length of the (v_1, v) section of P be less than or equal to that of the (v, v_k) section of P. Then the (w, v_k) path in T is of length greater than or equal to l, which is a contradiction. Hence $d(w, P) \leq 2$. Hence T reduces to a lobster with spine length at most 3 and diameter at most 5.

Conversely, for any tree T, which satisfies the given conditions, corresponding to any $\gamma_{rR}(G)$ function $f = (V_0, V_1, V_2)$, either $V_2 \neq \emptyset$ and every vertex in V_2 is adjacent to exactly one vertex in V_0 or $V_1 = V$. Hence $\gamma_{rR}(T) = n$. □

Theorem 4.6. *For any graph G, $\gamma_{rR}(G) = \gamma_r(G)$ if and only if each component of G is either a star or a K_1.*

Proof. Suppose $\gamma_{rR}(G) = \gamma_r(G)$, let f be a $\gamma_{rR}(G)$-function. Now

$$\begin{aligned} &\gamma_{rR}(G) = \gamma_r(G) \\ &|V_1| + |V_2| = |V_1| + 2|V_2| \\ &\Rightarrow |V_2| = 0 \end{aligned}$$

Therefore, $|V_1| = n = V(G) = \gamma_r(G)$. Hence by Theorem 4.5, G is a star. The converse follows immediately. □

5. Graphs with $\gamma_{rR}(G) = \gamma_r(G) + 1$

In this section, we characterize unicyclic graphs G for which $\gamma_{rR}(G) = \gamma_r(G) + 1$.

Theorem 5.1. *For any graph G of order n, $\gamma_{rR}(G) = \gamma_r(G) + 1$ if and only if there exits a vertex $v \in V$ such that the number of non - isolates in $N(v)$ is $n - \gamma_r$.*

Proof. Suppose G has a vertex $v \in V$ such that the number of non-isolates in $N(v)$ is $n - \gamma_r$. Let A be the set of non isolates in $N(v)$. Define $f = (V_0, V_1, V_2)$ by $V_2 = \{v\}$,$V_0 = A$, $V_1 = V \setminus (V_0 \cup V_2)$. Clearly f is a $\gamma_{rR}(G)$-function. Now

$$\begin{aligned} \gamma_{rR}(G) &= 2 + n - (|A| + |V_2|) \\ &= 2 + n - (n - \gamma_r(G) + 1) \\ &= \gamma_r(G) + 1 \end{aligned}$$

Conversely, let $f = (V_0, V_1, V_2)$ be a $\gamma_{rR}(G)$-function. Suppose $\gamma_{rR}(G) = \gamma_r(G) + 1$. This is true only when

(i) $|V_1| = \gamma_r(G) + 1$ and $|V_2| = 0$.
(ii) $|V_1| = \gamma_r(G) - 1$ and $|V_2| = 1$.

In all other cases $|V_1| + |V_2| < \gamma_r(G)$.

case(i): $|V_1| = \gamma_r(G) + 1$ and $|V_2| = 0$.
Here $V = V_1$. Hence $\gamma_{rR}(G) = n$. Therefore $\gamma_r(G) = n - 1$, which is a contradiction to the definition of a $\gamma_r(G)$-set. Hence this case does not arise.

case(ii): $|V_1| = \gamma_r(G) - 1$ and $|V_2| = 1$.
Let $V_2 = \{v\}$. Let n_1 be the number of non-isolates in $N(v)$ and n_2 be the number of isolates in $N(v)$. Then

$$\begin{aligned}
\gamma_{rR}(G) &= \gamma_r(G) + 1 \\
2 + n_2 + n - (1 + n_1 + n_2) &= \gamma_r(G) + 1 \\
n_1 &= n - \gamma_r(G)
\end{aligned}$$

Hence there exists a vertex $v \in V$ such that the number of non-isolates in $N(v)$ is $n - \gamma_r$. □

Corollary 5.2. *For any graph G with $\gamma_{rR}(G) = \gamma_r(G) + 1$, $1 \le diam(G) \le 4$ and $1 \le rad(G) \le 2$.*

Proof. By Theorem 5.1, G has a vertex $v \in V$ such that the number of non - isolates in $N(v)$ is $n - \gamma_r$. Hence we see that every vertex in $V \setminus \{v\}$ is adjacent to a member of $N[v]$. Therefore $diam(G) \le 4$. Clearly $1 \le rad(G) \le 2$. □

Theorem 5.3. *For any unicyclic graph G with cycle C_k, $\gamma_{rR}(G) = \gamma_r(G) + 1$ if and only if*

(a) *$k = 3$.*
(b) *At least one vertex in C_3 is of degree 2.*
(c) *Every vertex not in C_3 is a leaf.*

Proof. Let $f = (V_0, V_1, V_2)$ be a $\gamma_{rR}(G)$-function. Suppose $\gamma_{rR}(G) = \gamma_r(G) + 1$. This is true only when

(i) $|V_1| = \gamma_r(G) + 1$ and $|V_2| = 0$.
(ii) $|V_1| = \gamma_r(G) - 1$ and $|V_2| = 1$.

In all other cases, $|V_1| + |V_2| < \gamma_r(G)$.

When $|V_1| = \gamma_r(G) + 1$ and $|V_2| = 0$, $V = V_1$ and hence $\gamma_{rR}(G) = n$ which implies that $\gamma_r(G) = n - 1$, which is a contradiction.

When $|V_1| = \gamma_r(G) - 1$ and $|V_2| = 1$, by Theorem 5.1, there exists a vertex $v \in G$ such that the number of non-isolates in $N(v)$ is $n - \gamma_r$. In this case, clearly $v \in V(C_k)$ and $n - \gamma_r \le 2$ and hence $k = 3$.

We claim that every vertex not in C_3 is adjacent to a vertex in C_3. Let $x, y \in N(v)$ which are non-isolates. Suppose there exists a vertex u such that u is not adjacent to any vertex in C_3 . Let Q be a $v-u$ path in G. Let x_1 be a vertex in Q which is adjacent to a vertex in C_3 say x. [The possibility that $v = x$ is not ruled out]. Let y be the third vertex in C_3. Then $S \setminus \{x_1, v\} \cup \{y\}$ or $S \setminus \{x_1\}$ is a $\gamma_r(G)$-set according as $x \neq v$ or $x = v$, which is a contradiction to the minimality of S. Finally, we claim that at least one vertex in C_3 is of degree 2. Suppose not, we see that $\gamma_{rR}(G) = |L| + 2$ and $\gamma_r(G) = |L|$ where L is the set of leaves in G which implies that $\gamma_{rR}(G) = \gamma_r(G) + 2$, which is a contradiction.
Conversely, suppose the conditions of the theorem are true. Then $\gamma_{rR}(G) = |L|+2$ and $\gamma_r(G) = |L|+1$ which implies that $\gamma_{rR}(G) = \gamma_r(G) + 1$. □

6. **Graphs with** $\gamma_{rR}(G) = \gamma_r(G) + 2$

In this section, we prove that, for any tree T which is not a star, $\gamma_{rR}(T) \geq \gamma_r(T)+2$ and characterize trees for which $\gamma_{rR}(T) = \gamma_r(T) + 2$. We also characterize unicyclic graphs G for which $\gamma_{rR}(G) = \gamma_r(G) + 2$.

Theorem 6.1. *For any graph G of order n, then $\gamma_{rR}(G) = \gamma_r(G) + 2$ if and only if*

(a) *G does not have a vertex v such that the number of non-isolates in $N(v)$ is $n - \gamma_r$ and*

(b) *G has two vertices v and w such that the number of non-isolates in $N(v) \cup N(w)$ is $n - \gamma_r$.*

Proof. Suppose conditions (a) and (b) hold. By condition (a) and Theorem 5.1, $\gamma_{rR}(G) \geq \gamma_r(G) + 2$. Also by condition (b), G has two vertices v and w such that the number of non-isolates in $N(v)\cup N(w)$ is $n-\gamma_r$. We define $V_2 = \{v, w\}$, $V_1 = V - \{N[v]\cup N[w]\}$ and $V_0 = N(v)\cup N(w)$. Then $f = (V_0, V_1, V_2)$ is a $\gamma_{rR}(G)$ - function with weight less than or equal to $\gamma_r(G) + 2$.
Conversely, let $\gamma_{rR}(G) = \gamma_r(G) + 2$ and $f = (V_0, V_1, V_2)$ be a $\gamma_{rR}(G)$ - function. Let S be a $\gamma_r(G)$-set. By Theorem 5.1, condition (a) holds. Also $\gamma_{rR}(G) = \gamma_r(G) + 2$ only when

(i) $|V_1| = \gamma_r(G) + 2$ and $|V_2| = 0$.
(ii) $|V_1| = \gamma_r(G)$ and $|V_2| = 1$.
(iii) $|V_1| = \gamma_r(G) - 2$ and $|V_2| = 2$.

case(i): $|V_1| = \gamma_r(G) + 2$ and $|V_2| = 0$.
Since $|V_2| = 0$, $V = V_1$ and $\gamma_{rR}(G) = n$. Hence $\gamma_r(G) = n - 2$. Therefore there exists exactly two vertices say $x, y \in V \setminus S$ such that x and y are adjacent. Now there exists a vertex $v \in S$, such that $x \in N(v)$. Since $|V_2| = 0$, $y \notin N(v)$. Therefore there exists another vertex $w \in S$ such that $y \in N(w)$. Hence the number of non-isolates in $N(v) \cup N(w)$ is $n - \gamma_r$.

case(ii): $|V_1| = \gamma_r(G)$ and $|V_2| = 1$.
Let $V_2 = \{v\}$. Let n_1 be the number of non-isolates in $N(v)$ and n_2 be the remaining vertices in $N(v)$. Then $\gamma_{rR}(G) = 2 + n_2 + n - (1 + n_1 + n_2) = n - n_1 + 1$. Now $\gamma_{rR}(G) = \gamma_r(G) + 2$ implies that $n_1 = n - \gamma_r(G) - 1$. Hence there are $n - \gamma_r(G) - 1$ non-isolate vertices in $V \setminus S$ which are adjacent to

v. Let z be the vertex in $V \setminus S$ which is not adjacent to v. Now z is adjacent to some $w \in S$. Clearly z is also adjacent to some vertex in $V \setminus S$. Now v and w are two vertices such that the number of non-isolates of $N(v) \cup N(w)$ is $n - \gamma_r$.

case(iii): $|V_1| = \gamma_r(G) - 2$ and $|V_2| = 2$.

Let $V_2 = \{v, w\}$. Let n_1 be the number of non-isolates in $N(v) \cup N(w)$ and n_2 be the remaining vertices in $N(v) \cup N(w)$. Then $\gamma_{rR}(G) = 2 \times 2 + n_2 + n - (2 + n_1 + n_2) = n - n_1 + 2$. Now $\gamma_{rR}(G) = \gamma_r(G) + 2$ implies that $n_1 = n - \gamma_r(G)$. Hence v and w are two vertices such that the number of non-isolates in $N(v) \cup N(w)$ is $n - \gamma_r$. □

Corollary 6.2. *For any graph G with $\gamma_{rR}(G) = \gamma_r(G) + 2$, $2 \leq diam(G) \leq 7$ and $1 \leq rad(G) \leq 4$.*

Proof. If $diam(G) = 1$, then G is a complete graph and $\gamma_{rR}(G) = \gamma_r(G) + 1$. Hence $diam(G) \geq 2$. Clearly for the cycle C_4, $\gamma_{rR}(G) = 4$ and $\gamma_r(G) = 2$. By Theorem 6.1, G has two vertices v and w such that the number of non-isolates in $N(v) \cup N(w)$ is $n - \gamma_r$. Hence we see that every vertex in $V \setminus \{v, w\}$ is adjacent to a member of $N[v] \cup N[w]$. Therefore $diam(G) \leq 7$.
Clearly $1 \leq rad(G) \leq 4$. □

Theorem 6.3. *For any tree T which is not a star, $\gamma_{rR}(T) \geq \gamma_r(T) + 2$.*

Proof. Suppose $\gamma_{rR}(T) = \gamma_r(T) + 1$. Then by Theorem 5.1, there exists a vertex $v \in T$ such that $N(v)$ contains $n - \gamma_r$ non-isolate vertices. Clearly, these non-isolate vertices, say k in number, are in V_0 and $k \geq 2$. Let w_1 and w_2 be two non-isolate vertices in $N(v)$. Since T is a tree, w_1 and w_2 are not adjacent. Hence there exist vertices z_1 and z_2 which are adjacent to w_1 and w_2 with $\{w_1, w_2, z_1, z_2\} \subseteq V_0$. Further, there exist vertices x_1 and x_2 such that $x_1, x_2 \in V_2$ and x_1 and x_2 are adjacent to z_1 and z_2 respectively. Let H be the subgraph induced by the vertices $\{w_1, w_2, z_1, z_2, x_1, x_2, v\}$. Then $\gamma_{rR}(H) = 6$ and $\gamma_r(H) = 3$ which imply that $\gamma_{rR}(T) \geq \gamma_r(T) + 3$, which is a contradiction. Hence $\gamma_{rR}(T) \geq \gamma_r(T) + 2$. □

Theorem 6.4. *For any tree T, $\gamma_{rR}(T) = \gamma_r(T) + 2$ if and only if T is a caterpillar with spine of length at most 3 and the internal vertices of the spine are of degree 2.*

Proof. Let $f = (V_0, V_1, V_2)$ be a $\gamma_{rR}(T)$-function and $\gamma_{rR}(T) = \gamma_r(T) + 2$. This is true only when

(i) $|V_1| = \gamma_r(T)$ and $|V_2| = 1$.
(ii) $|V_1| = \gamma_r(T) + 2$ and $|V_2| = 0$.
(iii) $|V_1| = \gamma_r(T) - 2$ and $|V_2| = 2$.

case (i) : $|V_1| = \gamma_r(T)$ and $|V_2| = 1$.
Since T is a tree, $|V_2| \geq 2$ and hence this case is not possible.

case(ii): $|V_1| = \gamma_r(T) + 2$ and $|V_2| = 0$.
Since $|V_2| = 0$, $V = V_1$ and hence $\gamma_{rR}(T) = n$. By Theorem 4.5, T is either a star or a caterpillar with spine of length at most 3 or a lobster with spine of length at most 3 and diameter at most 5.

If G is a star, then $\gamma_{rR}(T) = \gamma_r(T) = n$, which is a contradiction.

If G is a lobster, then $\gamma_{rR}(T) = |L| + x$ where L is the set of all leaves in T and x is the number of non-leaves in T. If l is the length of the spine of T, then $l + 2 \leq x$. Now $\gamma_r(T) = |L| + i,\ 0 \leq i \leq 2$. Hence we get a contradiction in all the cases. Hence T is not a lobster. Hence T must be a caterpillar with spine of length at most 3. Finally, we claim that the internal vertices of the spine are of degree 2. Suppose not, then

$$\gamma_{rR}(T) = \begin{cases} |L| + 4, \text{ if } l = 3. \\ |L| + 3, \text{ if } l = 2. \end{cases}$$

and

$$\gamma_r(T) = \begin{cases} |L| + 1, & \text{if } l = 3 \text{ and one internal vertex of the spine is} \\ & \text{of degree greater than 2.} \\ |L|, & \text{otherwise.} \end{cases}$$

In all the cases, we get a contradiction. Hence our claim.

case(iii): $|V_1| = \gamma_r(G) - 2$ and $|V_2| = 2$.
Now $\gamma_{rR}(G) = 2 \times 2 + n - 4 = n$. Then as in case(ii) we see that G reduces to a graph given in the theorem.
Conversely, suppose T is a caterpillar satisfying the conditions of the theorem. Let l be the length of the spine. Clearly $\gamma_{rR}(T) = n$ where n is the order of T and

$$\gamma_{rR}(T) = \begin{cases} |L| + 2, \text{ if } l = 3. \\ |L| + 1, \text{ if } l = 2. \\ |L|, \text{ if } l = 1. \end{cases}$$

Hence we see that $\gamma_r(T) = n - 2$ and therefore $\gamma_{rR}(T) = \gamma_r(T) + 2$. □

We define a set of unicyclic graphs G_i, $1 \leq i \leq 7$, such that G_i is a graph containing H_i (Refer Figure 2) as an induced subgraph such that each vertex in $V(G_i) \setminus V(H_i)$ is a leaf subject to the following conditions:

1. In G_1, at most two vertices on C_3 are supports and if there are two supports in C_3 , then v is a support and $\deg(w_1) = 2$.
2. In G_2, v_3 is not a support and at most two vertices on C_3 are supports.
3. In G_3, not all v, v_1, v_2, v_3 are supports.
4. In G_4, not all v, v_1, v_2 are supports.
5. In G_5, $\deg(v) = 3$, $\deg(v_1) = \deg(v_2) = 2$.
6. In G_6, at most three vertices are supports and when w is a support, each vertex $z \neq w$ in G is a non support.
7. In G_7, no three consecutive vertices are supports.

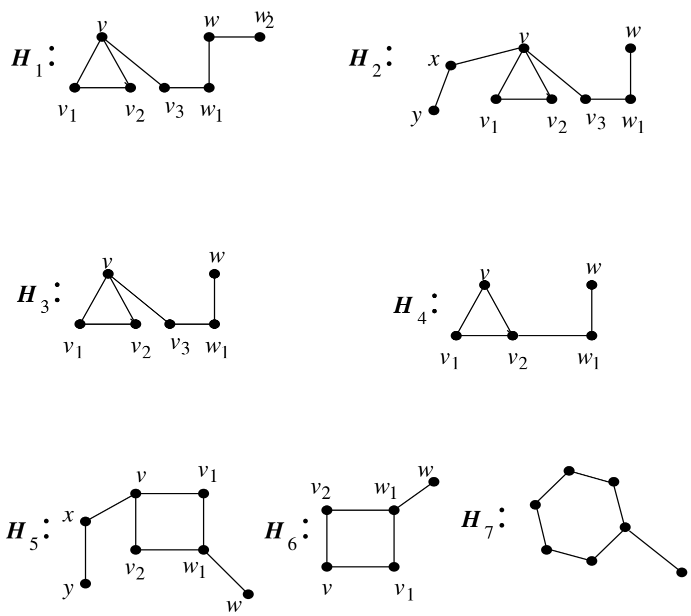

FIGURE 2.

Theorem 6.5. *Let G be a unicyclic graph with cycle C_k, $3 \leq k \leq 6$. Then $\gamma_{rR}(G) = \gamma_r(G) + 2$ if and only if either $G \cong C_k, 4 \leq k \leq 6$ or $G \cong G_i$, $1 \leq i \leq 7$.*

Proof. Let $\gamma_{rR}(G) = \gamma_r(G) + 2$. Then by Theorem 6.1(a), G does not have a vertex v such that the number of non-isolates in $N(v)$ is $n - \gamma_r$ and by Theorem 6.1(b), G has two vertices v and w such that the number of non-isolates in $N(v) \cup N(w)$ is $n - \gamma_r$. Let S be a $\gamma_r(G)$- set. Clearly $v, w \in S$. Now each vertex in $V \setminus S$ is adjacent to either v or w. Let L be the the set of all leaves in G.
First we claim that every vertex in $S \setminus \{v, w\}$ is either adjacent to $N[v]$ or $N[w]$. Suppose there exists a vertex $z \in S$ such that $z \notin N[v] \cup N[w]$. Let Q be a (z, v) path in G. Let x, y, t be the vertices in Q such that v, x, y, t form a path in that order. Then $S \setminus \{x, y\}$ or $S \setminus \{y\}$ is a $\gamma_r(G)$-set according as $x \in S$ or $x \notin S$, which is a contradiction to the minimality of S.
Further $2 \leq |V \setminus S| \leq 4$. Otherwise G will have more than one cycle which is a contradiction.

Case (i) : v has three neighbours and w has one neighbour in $V \setminus S$.

Let v_1, v_2, v_3 be three neighbours of v and w_1 be the neighbour of w in $V \setminus S$. Now since w_1 is a non-isolate in $V \setminus S$, w_1 must be adjacent to either v_1, v_2 or v_3. Without loss of generality, let w_1 be adjacent to v_3 and since $v_1$and v_2 are non-isolates and G is unicyclic, $v_1$and v_2 are adjacent.

Subcase(a) : deg $(w) > 1$.

First we claim that no vertex in S is at a distance 2 from either v or w or both. Suppose not. We see that $\gamma_{rR}(G) = |L| + 5$ and $\gamma_r(G) \leq |L| + 2$ which implies that $\gamma_{rR}(G) \geq \gamma_r(G) + 3$, which is a contradiction.
Secondly, we claim that at most two vertices in C_k are supports.

Suppose not. We define a function $f : V \to \{0, 1, 2\}$ such that

$$f(u) = \begin{cases} 2, & \text{if } u = v \textit{ or } u = w. \\ 1, & \text{if } u \text{ is a leaf.} \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\gamma_r(G) = \begin{cases} |L| & , \quad \text{if } v, v_1, v_2, v_3, w_1 \text{ are supports.} \\ |L| + 1 & , \quad \text{otherwise.} \end{cases}$$

and $\gamma_{rR}(G) = |L| + 4$. which implies that $\gamma_{rR}(G) \geq \gamma_r(G) + 3$, which is a contradiction.

Third, we claim that if C_k has two supports, then v is a support. Suppose not. As before we see that $\gamma_{rR}(G) = |L| + 4$ and $\gamma_r(G) = |L| + 1$ which implies that $\gamma_{rR}(G) = \gamma_r(G) + 3$, which is a contradiction.

Finally we claim that $\deg(w_1) = 2$. Suppose not, then we see that $\gamma_{rR}(G) = |L| + 4$ and $\gamma_r(G) = |L| + 1$ which implies that $\gamma_{rR}(G) = \gamma_r(G) + 3$, which is a contradiction. Hence $G \cong G_1$

Subcase(b): $\deg(w) = 1$.

We claim that when there is a vertex in S at a distance 2 from v, v_3 is not a support. Suppose not. Then

$$\gamma_r(G) = \begin{cases} |L| & , \text{ if all the vertices on } C_3 \text{ are supports.} \\ |L| + 1 & , \text{ otherwise.} \end{cases}$$

and $\gamma_{rR}(G) = |L| + 4$. which implies that $\gamma_{rR}(G) \geq \gamma_r(G) + 3$, which is a contradiction.
Next we claim that at most two vertices on C_3 are supports. Suppose not. Then $\gamma_r(G) = |L| + 1$ and $\gamma_{rR}(G) = |L| + 4$, which implies that $\gamma_{rR}(G) = \gamma_r(G) + 3$, which is a contradiction. Hence $G \cong G_2$.
Finally, we claim that, when there is no vertex in S at a distance 2 from v, not all v, v_1, v_2, v_3 are supports. Suppose not, we see that $\gamma_{rR}(G) \geq |L| + 3$ and $\gamma_r(G) = |L|$ which implies that $\gamma_{rR}(G) \geq \gamma_r(G) + 3$, which is a contradiction. Hence $G \cong G_3$.

Case(ii): v has two neighbours and w has one neighbour in $V \setminus S$.

Let v_1, v_2, be two neighbours of v and w_1 be the neighbour of w in $V \setminus S$. Since w_1 is a non-isolate in $V \setminus S$, it is adjacent to either v_1 or v_2.

Subcase(a): v_1 and v_2 are adjacent and $\deg(w) = 1$.

In this case, w_1 is adjacent to v_2. First we claim that, no vertex is at a distance two from v. Suppose not. Then

$$\gamma_r(G) = \begin{cases} |L| & \text{, if all the vertices on } C_3 \text{ are supports.} \\ |L|+1 & \text{, otherwise.} \end{cases}$$

and $\gamma_{rR}(G) = |L|+4$, which implies that $\gamma_{rR}(G) \geq \gamma_r(G)+3$, which is a contradiction.
Next we claim that, not all the vertices v, v_1, v_2 are supports. Suppose not. Then $\gamma_r(G) = |L|$ and $\gamma_{rR}(G) = |L|+3$, which implies that $\gamma_{rR}(G) = \gamma_r(G)+3$ Hence $G \cong G_4$.

Subcase(b): v_1 and v_2 are adjacent and $\deg(w) > 1$.
This case reduces to subcase(b) of case (i).

Subcase(c): v_1 and v_2 are not adjacent and $\deg(w) = 1$.
In this case, w_1 is adjacent to v_1 and v_2.
When there is no vertex at a distance 2 from v, we claim that not all v, v_1, v_2 are supports. Suppose not. We see that $\gamma_{rR}(G) = |L|+3$ and $\gamma_r(G) = |L|$ which implies that $\gamma_{rR}(G) = \gamma_r(G)+3$, which is a contradiction. Hence $G \cong G_6$.
When there is a vertex at a distance 2 from v, we claim that none of the vertices v, v_1, v_2 are supports. Suppose not. Then

$$\gamma_r(G) = \begin{cases} |L| & \text{, if all the vertices on } C_3 \text{ are supports.} \\ |L|+1 & \text{, otherwise.} \end{cases}$$

and we see that $\gamma_{rR}(G) = |L|+4$, which implies that $\gamma_{rR}(G) \geq \gamma_r(G)+3$, which is a contradiction. Hence $\deg(v) = 3$ and $\deg(v_1) = \deg(v_2) = 2$. Hence $G \cong G_5$.

Subcase(d): v_1 and v_2 are not adjacent and $\deg(w) > 1$.
In this case, w_1 is adjacent to v_2.
Now we claim that there is no vertex at a distance two from v. Suppose not. Then

$$\gamma_r(G) = \begin{cases} |L|+2 & \text{, either } v \text{ or } w_1 \text{ or both } v \text{ and } w_1 \text{ are supports or} \\ & \quad \text{none of } v, v_1, v_2, w_1 \text{ are supports.} \\ |L| & \text{, if all the vertices } v, v_1, v_2, w_1 \text{ are supports.} \\ |L|+1 & \text{, otherwise.} \end{cases}$$

and we see that $\gamma_{rR}(G) = |L|+5$, which implies that $\gamma_{rR}(G) \geq \gamma_r(G)+3$, which is a contradiction. Hence there is no vertex at a distance two from v. Now when none of v, v_1, v_2, w_1 are supports, $G \cong G_6$. When v alone is a support, $G \cong G_5$. In all other cases $\gamma_r(G) = |L|+1$ and $\gamma_{rR}(G) = |L|+4$, which implies that $\gamma_{rR}(G) = \gamma_r(G)+3$, which is a contradiction.

Case(iii): v and w has one neighbour in $V \setminus S$.
Let v_1 be the neighbour of v and w_1 be the neighbour of w in $V \setminus S$. Clearly v_1 and w_1 are adjacent as

they are non-isolates in $V \setminus S$. Further v and w are either adjacent or they have at most one common neighbour.

Subcase(a): v and w are adjacent.
We see that the subgraph induced by $\{v, v_1, w_1, w, v\}$ is a cycle C_4 say H. If $G = H$, then $G \cong C_4$. When $G \neq H$, not all v, v_1, w_1, w are supports, otherwise $\gamma_{rR}(G) = \gamma_r(G)+3$, which is a contradiction. Now we claim that, when there is a vertex say x at a distance two from $w \in V(C_4)$ in H, deg(w)=3. Suppose not. Let u be a vertex which is adjacent to w. Then we see that $\gamma_{rR}(G) = |L| + 4$ and $\gamma_r(G) = |L| + 1$ which implies that $\gamma_{rR}(G) = \gamma_r(G) + 3$, which is a contradiction.
Next we claim that, when there is a vertex say x at a distance two from $w \in V(C_4)$, then the two vertices in C_4 which are adjacent to w are of degree 2. Suppose not. We see that $\gamma_{rR}(G) = |L| + 4$ and $\gamma_r(G) = |L|+1$ which implies that $\gamma_{rR}(G) = \gamma_r(G)+3$, which is a contradiction. Hence $G \cong G_5$.
Subcase(b): v and w have at most one common neighbour say u.
We see that the subgraph induced by $\{v, v_1, w_1, w, u\}$ is a cycle C_5 say H. If $G = H$, then $G \cong C_5$. If $G \neq H$, we claim that there is no vertex at a distance two from v or both from v and w. Suppose not.Then

$$\gamma_{rR}(G) = \begin{cases} |L| + 5 \text{ , if } v \text{ has a vertex at a distance two.} \\ |L| + 6 \text{ , if both } v \text{and } w \text{ have a vertex at a distance two.} \end{cases}$$

and we see that $\gamma_r(G) = |L| + 2$, which implies that $\gamma_{rR}(G) \geq \gamma_r(G) + 3$, which is a contradiction.
Next we claim that no vertices on the cycle are supports. Suppose not. We see that

$$\gamma_{rR}(G) = \begin{cases} |L| + 5 \text{ ,if one vertex on } C_5 \text{ is a support.} \\ |L| + 4 \text{ ,otherwise.} \end{cases}$$

and

$$\gamma_r(G) = \begin{cases} |L| \quad \text{, if all the vertices of } C_5 \text{ are supports.} \\ |L| + 2, \text{ if one vertex on } C_5 \text{ is a support.} \\ |L| + 1, \text{ otherwise.} \end{cases}$$

In all the cases, $\gamma_{rR}(G) \geq \gamma_r(G) + 3$, which is a contradiction.

Case(iv): v and w both have two neighbours in $V \setminus S$.
Let v_1, v_2 and w_1, w_2 respectively be the neighbours of v and w. As G is unicyclic and v_1, v_2 and w_1 ,w_2 are non-isolates, v_1 is adjacent to w_2 and v_2 is adjacent to w_1. Now the subgraph induced by $\{v, w, v_1, v_2, w_1, w_2\}$ is a cycle C_6 say H. If $G = H$, then $G \cong C_6$.
If $G \neq H$, we claim that every vertex not in H is a leaf. Suppose not. Then there exists a vertex say x which is not a leaf. We clearly see that $\gamma_{rR}(G) \geq |L| + 5$ and $\gamma_r(G) = |L| + 2$ which implies that $\gamma_{rR}(G) \geq \gamma_r(G) + 3$, which is a contradiction.
Now, we claim that, no three consecutive vertices on the cycle C_6 are supports. Suppose not. We

see that $\gamma_{rR}(G) = |L| + 4$ and $\gamma_r(G) = |L| + 1$ which implies that $\gamma_{rR}(G) = \gamma_r(G) + 3$, which is a contradiction. Hence $G \cong G_7$.

Conversely, if $G \equiv C_k, 4 \leq k \leq 6$, then clearly $\gamma_{rR}(G) = \gamma_r(G) + 2$.
Suppose $G \equiv G_i, 1 \leq i \leq 7$. Let L_1 denote the set of all leaves in $V(H_i)$ and L_2 denote the set of all leaves in $V(G_i) \setminus V(H_i),\ 1 \leq i \leq 7$.
When $G \equiv G_i, i = 3, 4, 6$, $\gamma_{rR}(G_i) = |L_1| + 3 + |L_2|$ and $\gamma_r(G_i) = |L_1| + 1 + |L_2|$ which implies that $\gamma_{rR}(G) = \gamma_r(G) + 2$.
When $G \equiv G_i, i = 1, 2, 5, 7$, $\gamma_{rR}(G_i) = |L_1| + 4 + |L_2|$ and $\gamma_r(G_i) = |L_1| + 2 + |L_2|$ which implies that $\gamma_{rR}(G) = \gamma_r(G) + 2$. □

7. **Split Graphs**

Theorem 7.1. *Let G be a split graph with bipartition (X, Y) where X is independent, $G[Y]$ is complete and deg(x)=1 for all $x \in X$.*

(i) *If deg(y) $= |Y| - 1$ for at least one $y \in Y$, then $\gamma_{rR}(G) = \gamma_r(G) + 1$.*
(ii) *If deg(y) $\geq |Y|$ for every $y \in Y$, then $\gamma_{rR}(G) = \gamma_r(G) + 2$.*

Proof. If G satisfies condition (i), then $\gamma_r(G) =| X | +1$ and $\gamma_{rR}(G) =| X | +2$. Hence $\gamma_{rR}(G) = \gamma_r(G) + 1$.

If G satisfies condition (ii), then $\gamma_r(G) =| X |$ and $\gamma_{rR}(G) =| X | +2$. Hence $\gamma_{rR}(G) = \gamma_r(G)+2$. □

Theorem 7.2. *Let G be a split graph with bipartition (X, Y) where X is independent and $G[Y]$ is complete and deg$(x) \geq 2$ for at least one $x \in X$. Then $\gamma_{rR}(G) = \gamma_r(G) + k,\ k \geq 1$ if and only if there exist exactly k vertices $y_i \in Y, 1 \leq i \leq k$ with deg$(y_1) = \triangle(G)$ and for each i, $2 \leq i \leq k$, the number of non-pendant vertices in the set $H_i = N_X(y_i) \setminus \left\{ \bigcup_{j=1}^{i-1} N_X(y_j) \right\}$ is at least two and $|H_{i-1}| \geq |H_i|$.*

Proof. Let p be the number of pendant vertices in X. Let

$$W = \Big\{ y_i \in Y : i \neq 1, \text{ and the number of non-pendant vertices in the set } H_i = N_X(y_i) \setminus \left\{ \bigcup_{j=1}^{i-1} N_X(y_j) \right\} \text{ is at least two and } |H_{i-1}| \geq |H_i|. \Big\}.$$

Now each vertex $y \in B = Y \setminus (W \cup \{y_1\})$ is such that the number of non pendant vertices in the set $N_X(y) \setminus \left\{ \bigcup_{j=1}^{k} N_X(y_j) \right\}$ is at most one. Let

$$B_1 = \Big\{ y \in B : \textit{the number of non pendant vertices in the set}$$
$$N_X(y) \setminus \left\{ \bigcup_{j=1}^{k} N_X(y_j) \right\} \textit{ is exactly one} \Big\}$$

Let $|W| = l_1$ and $|B_1| = l_2$. Now $\gamma_r(G) = l_1 + l_2 + p + 1$ and $\gamma_{rR}(G) = 2l_1 + l_2 + p + 2$. Suppose $\gamma_{rR}(G) = \gamma_r(G) + k$, then

$$2l_1 + l_2 + p + 2 = l_1 + l_2 + p + 1 + k$$
$$l_1 = k - 1$$

which proves $k = |W| + 1$. Hence there exist k vertices in Y satisfying the given conditions.
Conversely, let there exist exactly k vertices in Y satisfying the given conditions. Then $\gamma_r(G) = k+p+l_2$ and $\gamma_{rR}(G) = 2k + p + l_2$.
Hence $\gamma_{rR}(G) = k + (k + p + l_2) = \gamma_r(G) + k$ □

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