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OPTIMAL ORIENTATIONS OF SUBGRAPHS OF COMPLETE BIPARTITE GRAPHS

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Abstract. For a graph G, let $\mathscr{D}(G)$ be the set of all strong digraphs D obtained by the orientations of G. The *orientation number* of G is $\vec{d}(G) = \min\{d(D) \mid D \in \mathscr{D}(G)\}$, where $d(D)$ denotes the diameter of the digraph D. In this paper, we determine the orientation number for some subgraphs of complete bipartite graphs.

1. Introduction

Let G be a simple connected graph with vertex set $V(G)$ and edge set $E(G)$. For $v \in V(G)$, the *eccentricity of* v is $e_G(v) = \max\{d_G(v,x) \mid x \in V(G)\}$, where $d_G(v,x)$ denotes the distance from v to x in G. The *diameter* of G is $d(G) = \max\{e_G(v) \mid v \in V(G)\}$.

A k-*factor* of a graph G is a k-regular spanning subgraph of G. A 2-factor of G is *uniform* if its components are of cycles of length r for some r.

For a graph H, nH denotes n disjoint copies of H.

Let D be a digraph with vertex set $V(D)$ and arc set $A(D)$ which has no loops and no two of its arcs have same tail and same head. The notion $e_D(v)$ for $v \in V(D)$, and $d(D)$ are defined as in the undirected graph.

Let D be a digraph. If (x,y) is an arc in D, we say that x is adjacent to y in D and we denote it by $x \to y$ or $y \leftarrow x$. More generally, for $X, Y \subseteq V(D)$ with $X \cap Y = \phi$, write, $X \to Y$ if every vertex of X is adjacent to every vertex of Y. For simplicity, write $x \to Y$ for $\{x\} \to Y$ and $X \to y$ for $X \to \{y\}$.

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An *orientation* of a graph G is a digraph D obtained from G by assigning a direction to each of its edges. A vertex v is *reachable* from a vertex u of a digraph D if there is a directed path in D from u to v. An orientation D of G is *strong* if any pair of vertices in D are mutually reachable in D. Robbins' one-way street theorem [11] states that a connected graph G has a strong orientation if and only if G is 2-edge-connected. For a 2-edge-connected graph G, let $\mathscr{D}(G)$ denote the set of all strong digraphs D obtained by the orientations of G. The *orientation number* of G is $\vec{d}(G) = \min\{d(D) \mid D \in \mathscr{D}(G)\}$.

Any orientation D in $\mathscr{D}(G)$ with $d(D) = \vec{d}(G)$ is called an *optimal orientation* of G. The problem of evaluating the orientation number of an arbitrary connected graph is very difficult as Chvátal and Thomassen have shown that the problem of deciding whether a graph admits an orientation of diameter 2 is NP-hard, see [3].

Define $f(d) = \max\{\vec{d}(G) : \kappa'(G) \geq 2 \text{ and } d(G) = d\}$, where κ' denotes the edge-connectivity. Chvátal and Thomassen [3] proved that $\frac{1}{2}d^2 + d \leq f(d) \leq 2d^2 + 2d$ for $d \geq 2$. The known exact values are $f(1) = 3$ and $f(2) = 6$ [3]. For $d = 3$, Kowk, Liu and West [7] improved these bounds to $9 \leq f(3) \leq 11$. For results on orientations of graphs, see a survey by Koh and Tay [6].

The set of integers *modulo* k is denoted by $\mathbb{Z}_k$. Denote the vertices of the partite sets of the complete bipartite graph $K_{n,n}$ by $\{x_0, x_1, x_2, \ldots, x_{n-1}\}$ and $\{y_0, y_1, y_2, \ldots, y_{n-1}\}$. For $\ell \in \mathbb{Z}_n$, the set $B_\ell^{(n)} = \{x_i y_{i+\ell} : i \in \mathbb{Z}_n\}$ of n edges of $K_{n,n}$ are called edges of *length* ℓ in $K_{n,n}$, where the addition $i + \ell$ is performed in $\mathbb{Z}_n$. The subgraph induced by the edges of $B_\ell^{(n)}$ is a 1-factor of $K_{n,n}$ and $\{K_{n,n}[B_0^{(n)}], K_{n,n}[B_1^{(n)}], \ldots, K_{n,n}[B_{n-1}^{(n)}]\}$ is a 1-factorization of $K_{n,n}$, where $K_{n,n}[B_\ell^{(n)}]$ is the subgraph induced by the edges of $B_\ell^{(n)}$.

Optimal orientations of the complete bipartite graphs $K_{p,q}$ were studied by Plesnik [10], Boesch and Tindell [2], Šoltés [12] and Gutin [4]. It is known that $\vec{d}(K_{n,n}) = 3$ for $n \geq 2$. In [8], Lakshmi and Paulraja proved that for $n \geq 5$ and for $0 \leq i \leq n-1$, $\vec{d}(K_{n,n} - B_i^{(n)}) = 3$ and $\vec{d}(K_{3,3} - B_i^{(3)}) = 5 = \vec{d}(K_{4,4} - B_i^{(4)})$. (See Lemma 2.3 of [8].) Consequently, for $n \geq 5$, if M is a matching of $K_{n,n}$, then $\vec{d}(K_{n,n} - M) = 3$.

Notation and terminology not defined here can be seen in [1] or [5].

In this paper, among other results, we show that if F is a 2-factor of $K_{n,n}$ with $n \geq 9$, then $\vec{d}(K_{n,n} - E(F)) = 3$.

2. $\mathbb{Z}_n^2$-sets

The concept of $\mathbb{Z}_n^t$-set, for orientations of graphs, is introduced by Lakshmi and Paulraja in [9]. A subset L of $\mathbb{Z}_n$ is a $\mathbb{Z}_n^2$-set, if $j \in L$, then $(n-j)\ (mod\ n) \notin L$ and every element $i \in \mathbb{Z}_n \backslash \{0\}$ can be written as $(a_1 + a_2)\,(mod\, n)$ for some $a_1, a_2 \in L$. For a $\mathbb{Z}_n^2$-set L, we associate a set $\hat{L}$ as follows: $\hat{L} = L \cup L_1$, where $L_1 = \{n - i : i \in L\}$.

For a subset M of $\mathbb{Z}_n$, a partition (A, B) of M is a $\mathbb{Z}_n^2$-*partition* of M if $\{(a-b)(mod\ n) : a \in A, b \in B\} = \mathbb{Z}_n \backslash \{0\}$.

Observe that if L is a $\mathbb{Z}_n^2$-set, then $(L, \{n - i : i \in L\})$ is a $\mathbb{Z}_n^2$-partition of $\hat{L}$. To see this, if $i \in \mathbb{Z}_n \backslash \{0\}$, then $i = (a_1 + a_2)\,(mod\, n)$ for some $a_1, a_2 \in L$, and hence $i = a_1 - (n - a_2)$.

Example. A $\mathbb{Z}_7^2$-partition of $\{2,3,4,5,6\}$ is $(\{2,6\},\{3,4,5\})$, but neither $\{2,6\}$ nor $\{3,4,5\}$ is a $\mathbb{Z}_7^2$-set.

3. Results

Theorem 3.1. *Let (A,B) be a $\mathbb{Z}_n^2$-partition of a subset M of $\mathbb{Z}_n$, and let G be the subgraph of $K_{n,n}$ induced by the edges of $\bigcup_{\ell \in M} B_\ell^{(n)}$. Then $\vec{d}(G) = 3$.*

Proof. Let (X,Y) be the bipartition of $K_{n,n}$, where $X = \{x_0, x_1, x_2, \ldots, x_{n-1}\}$ and $Y = \{y_0, y_1, y_2, \ldots, y_{n-1}\}$. Orient G so that $\{y_{r+b} : b \in B\} \to x_r \to \{y_{r+a} : a \in A\}$, $r \in \mathbb{Z}_n$. Consequently, $\{x_{r-a} : a \in A\} \to y_r \to \{x_{r-b} : b \in B\}$, $r \in \mathbb{Z}_n$. Let D be the resulting digraph.

Claim. For $i \in \mathbb{Z}_n \setminus \{0\}$, $d_D(x_0, x_i) = 2$ and $d_D(y_0, y_i) = 2$.

By hypothesis, $i = (a-b) \, (mod\, n)$ for some $a \in A$ and $b \in B$. The existence of the directed paths $x_0 \to y_a \to x_i$ and $y_0 \to x_{n-b} \to y_i$ in D proves the claim.

By the claim, for $i, j \in \mathbb{Z}_n$ and $i \neq j$, $d_D(x_i, x_j) = 2$ and $d_D(y_i, y_j) = 2$. This implies that $d(D) \leq 3$. Since G is a spanning subgraph of $K_{n,n}$, $\vec{d}(G) \geq 3$. Thus $\vec{d}(G) = 3$. □

Corollary 3.2. *Let L be a $\mathbb{Z}_n^2$-set, and let G be the subgraph of $K_{n,n}$ induced by the edges of $\bigcup_{\ell \in \hat{L}} B_\ell^{(n)}$. Then $\vec{d}(G) = 3$.* □

Lemma 3.3. *If $n \geq 21$ or $n \in \{19, 17\}$ and $\ell_1, \ell_2 \in \{1, 2, \ldots, \lfloor \frac{n-1}{2} \rfloor\}$ with $\ell_1 \neq \ell_2$, then there exists a $\mathbb{Z}_n^2$-set L contained in $\mathbb{Z}_n \setminus \{\ell_1, \ell_2, n-\ell_2, n-\ell_1\}$.*

Proof. In what follows, for integers i and j with $i < j$, $[i,j]$ denotes the set of integers k such that $i \leq k \leq j$. Also, for integers a, b, c, d with $a < b$ and $c < d$, $[a,b]+[c,d] = \{(p+q)(\text{mod}\, n) : p \in [a,b]$ and $q \in [c,d]\}$.

Case 1. $\ell_1 \geq 3$, $\ell_1 + 3 \leq \ell_2$ and $\ell_2 \leq \lfloor \frac{n-11}{2} \rfloor$.

Consider the subset $L = [1, \ell_1 - 1] \cup [\ell_1 + 1, \ell_2 - 1] \cup [\ell_2 + 2, \lfloor \frac{n-3}{2} \rfloor] \cup \{\lceil \frac{n+1}{2} \rceil, n - \ell_2 - 1\}$ of $\mathbb{Z}_n$. We show that L is a $\mathbb{Z}_n^2$-set, by adding the elements in the intervals of L as follows:

$(\ell_2 + 2) + (n - \ell_2 - 1) = 1$,

$[1, \ell_1 - 1] + [1, \ell_1 - 1] = [2, 2\ell_1 - 2]$,

$[1, \ell_1 - 1] + [\ell_1 + 1, \ell_2 - 1] = [\ell_1 + 2, \ell_1 + \ell_2 - 2]$,

$[\ell_1 + 1, \ell_2 - 1] + [\ell_1 + 1, \ell_2 - 1] = [2\ell_1 + 2, 2\ell_2 - 2]$,

$[1, \ell_1 - 1] + [\ell_2 + 2, \lfloor \frac{n-3}{2} \rfloor] = [\ell_2 + 3, \ell_1 + \lfloor \frac{n-5}{2} \rfloor]$,

$[\ell_1 + 1, \ell_2 - 1] + [\ell_2 + 2, \lfloor \frac{n-3}{2} \rfloor] = [\ell_1 + \ell_2 + 3, \ell_2 + \lfloor \frac{n-5}{2} \rfloor]$,

$[\ell_2 + 2, \lfloor \frac{n-3}{2} \rfloor] + [\ell_2 + 2, \lfloor \frac{n-3}{2} \rfloor]$ is $[2\ell_2 + 4, n-4]$ if n is even and is $[2\ell_2 + 4, n-3]$ if n is odd, and

$[\lfloor \frac{n-7}{2} \rfloor, \lfloor \frac{n-3}{2} \rfloor] + \lceil \frac{n+1}{2} \rceil = [n-3, n-1]$.

This together with $\{1\} \cup [2, 2\ell_1 - 2] \cup [\ell_1 + 2, \ell_1 + \ell_2 - 2] \cup [2\ell_1 + 2, 2\ell_2 - 2] \cup [\ell_2 + 3, \ell_1 + \lfloor \frac{n-5}{2} \rfloor] \cup [\ell_1 + \ell_2 + 3, \ell_2 + \lfloor \frac{n-5}{2} \rfloor] \cup ([2\ell_2 + 4, n-4]$ if n is even $\setminus$ $[2\ell_2 + 4, n-3]$ if n is odd$) \cup [n-3, n-1] = \mathbb{Z}_n \setminus \{0\}$, shows that L is a $\mathbb{Z}_n^2$-set.

In the following cases, we explicitly mention the $\mathbb{Z}_n^2$-set L. The verification of L is an $\mathbb{Z}_n^2$-set is similar to Case 1.

Case 2. $\ell_2 = \ell_1 + 1$.

Subcase 2.1. $\ell_1 = 1$: For $n \geq 23$, $[4, \lfloor \frac{n-3}{2} \rfloor] \cup \{\lceil \frac{n+1}{2} \rceil, n-3\}$ is a $\mathbb{Z}_n^2$-set. $\{4, 6, 7, 8, 9, 12, 17, 19\}$ is a $\mathbb{Z}_{22}^2$-set. $\{3\} \cup [5, 9] \cup \{11, 17\}$ is a $\mathbb{Z}_{21}^2$-set. $[3, 6] \cup [10, 12]$ is a $\mathbb{Z}_{19}^2$-set. $\{6, 7, 9, 12, 13, 14\}$ is a $\mathbb{Z}_{17}^2$-set.

Subcase 2.2. $\ell_1 = 2$: For $n \geq 21$, $\{1\} \cup [5, \lfloor \frac{n-3}{2} \rfloor] \cup \{\lceil \frac{n+1}{2} \rceil, n-4\}$ is a $\mathbb{Z}_n^2$-set. $\{1, 4, 5, 6, 9, 11, 12\}$ is a $\mathbb{Z}_{19}^2$-set. $\{1, 6, 9, 10, 12, 13\}$ is a $\mathbb{Z}_{17}^2$-set.

Subcase 2.3. $\ell_1 = 3$: For $n \geq 25$, $[1, 2] \cup [6, \lfloor \frac{n-3}{2} \rfloor] \cup \{\lceil \frac{n+1}{2} \rceil, n-5\}$ is a $\mathbb{Z}_n^2$-set. $\{1, 6, 7, 8, 9, 10, 11, 19, 22\}$ is a $\mathbb{Z}_{24}^2$-set. $\{1, 2, 7, 8, 9, 10, 12, 17, 18\}$ is a $\mathbb{Z}_{23}^2$-set. $\{1, 6, 7, 8, 9, 10, 17, 20\}$ is a $\mathbb{Z}_{22}^2$-set. $\{1, 2, 6, 7, 9, 11, 13, 16\}$ is a $\mathbb{Z}_{21}^2$-set. $\{1, 5, 6, 7, 10, 11, 17\}$ is a $\mathbb{Z}_{19}^2$-set. $\{1, 5, 9, 10, 11, 15\}$ is a $\mathbb{Z}_{17}^2$-set.

Subcase 2.4. $\ell_1 = 4$: For $n \geq 29$ or $n \in \{25, 24, 19\}$, $[1, 3] \cup [7, \lfloor \frac{n-3}{2} \rfloor] \cup \{\lceil \frac{n+1}{2} \rceil, n-6\}$ is a $\mathbb{Z}_n^2$-set. $\{1, 2, 3, 6, 8, 9, 10, 11, 12, 15, 21\}$ is a $\mathbb{Z}_{28}^2$-set. $\{1, 2, 3, 6, 8, 9, 10, 11, 12, 14, 20\}$ is a $\mathbb{Z}_{27}^2$- set. $\{1, 2, 3, 6, 8, 9, 10, 11, 14, 19\}$ is a $\mathbb{Z}_{26}^2$-set. $\{1, 2, 3, 6, 8, 9, 10, 12, 16\}$ is a $\mathbb{Z}_{23}^2$-set. $\{1, 2, 3, 6, 8, 10, 13, 15\}$ is a $\mathbb{Z}_{22}^2$-set. $\{1, 2, 3, 6, 8, 9, 11, 14\}$ is a $\mathbb{Z}_{21}^2$-set. $\{1, 2, 3, 6, 7, 9\}$ is a $\mathbb{Z}_{17}^2$- set.

Subcase 2.5. $5 \leq \ell_1 \leq \lfloor \frac{n-15}{2} \rfloor$: $[1, \ell_1 - 1] \cup [\ell_1 + 3, \lfloor \frac{n-3}{2} \rfloor] \cup \{\lceil \frac{n+1}{2} \rceil, n - \ell_1 - 2\}$ is a $\mathbb{Z}_n^2$-set.

Subcase 2.6. $\ell_1 = \lfloor \frac{n-13}{2} \rfloor$: By Subcases 2.1 - 2.4, assume that $\ell_1 \geq 5$, and hence $n \geq 23$. $[1, \lfloor \frac{n-15}{2} \rfloor] \cup [\lfloor \frac{n-7}{2} \rfloor, \lfloor \frac{n-3}{2} \rfloor] \cup \{\lceil \frac{n+1}{2} \rceil, \lceil \frac{n+9}{2} \rceil\}$ is a $\mathbb{Z}_n^2$-set.

Subcase 2.7. $\ell_1 = \lfloor \frac{n-11}{2} \rfloor$: Since $\ell_1 \geq 5$, $n \geq 21$. $[1, \lfloor \frac{n-13}{2} \rfloor] \cup \{\lfloor \frac{n-5}{2} \rfloor, \lfloor \frac{n-3}{2} \rfloor, \lceil \frac{n+1}{2} \rceil, \lceil \frac{n+7}{2} \rceil\}$ is a $\mathbb{Z}_n^2$-set.

Subcase 2.8. $\ell_1 = \lfloor \frac{n-9}{2} \rfloor$: As $\ell_1 \geq 5$, $n \geq 19$. For $n \geq 21$, $[1, \lfloor \frac{n-15}{2} \rfloor] \cup \{\lfloor \frac{n-11}{2} \rfloor\} \cup [\lfloor \frac{n-5}{2} \rfloor, \lfloor \frac{n-1}{2} \rfloor] \cup \{\lceil \frac{n+13}{2} \rceil\}$ is a $\mathbb{Z}_n^2$-set. $[1, 4] \cup \{7, 8\} \cup \{10\}$ is a $\mathbb{Z}_{19}^2$-set.

Subcase 2.9. $\ell_1 = \lfloor \frac{n-7}{2} \rfloor$: Since $\ell_1 \geq 5$, $n \geq 17$. For $n \geq 21$ and $n \neq 22$, $[1, \lfloor \frac{n-13}{2} \rfloor] \cup \{\lfloor \frac{n-9}{2} \rfloor, \lfloor \frac{n-3}{2} \rfloor, \lfloor \frac{n-1}{2} \rfloor, \lceil \frac{n+11}{2} \rceil\}$ is a $\mathbb{Z}_n^2$-set. $\{1, 2, 5, 6, 9, 12, 18, 19\}$ is a $\mathbb{Z}_{22}^2$-set. $[1, 4] \cup \{8, 10, 14\}$ is a $\mathbb{Z}_{19}^2$-set. $\{2, 3, 4, 8, 10, 16\}$ is a $\mathbb{Z}_{17}^2$-set.

Subcase 2.10. $\ell_1 = \lfloor \frac{n-5}{2} \rfloor$: For $n \geq 19$ and $n \neq 20$, $[1, \lfloor \frac{n-11}{2} \rfloor] \cup \{\lfloor \frac{n-7}{2} \rfloor, \lceil \frac{n+1}{2} \rceil, \lceil \frac{n+9}{2} \rceil\}$ is a $\mathbb{Z}_n^2$-set. $\{4, 5, 8, 14, 15, 16\}$ is a $\mathbb{Z}_{17}^2$-set.

Subcase 2.11. $\ell_1 = \lfloor \frac{n-3}{2} \rfloor$: For $n \geq 21$ and $n \neq 22$, $[1, \lfloor \frac{n-9}{2} \rfloor] \cup \{\lfloor \frac{n-5}{2} \rfloor, \lceil \frac{n+7}{2} \rceil\}$ is a $\mathbb{Z}_n^2$-set. $\{3, 5, 6, 7, 8, 18, 20, 21\}$ is a $\mathbb{Z}_{22}^2$-set. $\{2, 3, 4, 6, 12, 14, 18\}$ is a $\mathbb{Z}_{19}^2$-set. $\{3, 4, 6, 12, 15, 16\}$ is a $\mathbb{Z}_{17}^2$-set.

Case 3. $\ell_2 = \ell_1 + 2$.

Subcase 3.1. $\ell_1 = 1$: For $n \geq 23$, $\{2\} \cup [5, \lfloor \frac{n-3}{2} \rfloor] \cup \{\lceil \frac{n+1}{2} \rceil, n-4\}$ is a $\mathbb{Z}_n^2$-set. $\{4, 8, 9, 10, 15, 16, 17, 20\}$ is a $\mathbb{Z}_{22}^2$-set. $\{4, 5, 6, 7, 8, 9, 11, 19\}$ is a $\mathbb{Z}_{21}^2$-set. $\{2, 5, 10, 11, 12, 13, 15\}$ is a $\mathbb{Z}_{19}^2$-set. $\{5, 6, 9, 10, 13, 15\}$ is a $\mathbb{Z}_{17}^2$-set.

Subcase 3.2. $\ell_1 = 2$: For $n \geq 21$, $\{1, 3\} \cup [5, \lfloor \frac{n-9}{2} \rfloor] \cup \{\lfloor \frac{n-1}{2} \rfloor, \lceil \frac{n+3}{2} \rceil, \lceil \frac{n+5}{2} \rceil, \lceil \frac{n+7}{2} \rceil\}$ is a $\mathbb{Z}_n^2$-set. $\{1, 3, 5, 6, 9, 11, 12\}$ is a $\mathbb{Z}_{19}^2$-set. $\{1, 7, 9, 11, 12, 14\}$ is a $\mathbb{Z}_{17}^2$-set.

Subcase 3.3. $\ell_1 = 3$: For $n \geq 21$ or $n = 19$, $[1, 2] \cup \{4\} \cup [6, \lfloor \frac{n-5}{2} \rfloor] \cup \{\lfloor \frac{n-1}{2} \rfloor, \lceil \frac{n+3}{2} \rceil\}$ is a $\mathbb{Z}_n^2$-set. $\{1, 2, 4, 6, 7, 9\}$ is a $\mathbb{Z}_{17}^2$-set.

Subcase 3.4. $4 \leq \ell_1 \leq \lfloor \frac{n-19}{2} \rfloor$: $[1, \ell_1 - 1] \cup \{\ell_1 + 1\} \cup [\ell_1 + 4, \lfloor \frac{n-3}{2} \rfloor] \cup \{\lceil \frac{n+1}{2} \rceil, n - \ell_1 - 3\}$ is a $\mathbb{Z}_n^2$-set.

Subcase 3.5. $\ell_1 = \lfloor \frac{n-17}{2} \rfloor$: By Subcases 3.1 - 3.3, assume that $\ell_1 \geq 4$, and hence $n \geq 25$. $[1, \lfloor \frac{n-19}{2} \rfloor] \cup \{\lfloor \frac{n-15}{2} \rfloor\} \cup [\lfloor \frac{n-11}{2} \rfloor, \lfloor \frac{n-5}{2} \rfloor] \cup \{\lfloor \frac{n-1}{2} \rfloor, \lceil \frac{n+3}{2} \rceil\}$ is a $\mathbb{Z}_n^2$-set.

Subcase 3.6. $\ell_1 = \lfloor \frac{n-15}{2} \rfloor$: As $\ell_1 \geq 4$, $n \geq 23$. $[1, \lfloor \frac{n-17}{2} \rfloor] \cup \{\lfloor \frac{n-13}{2} \rfloor\} \cup [\lfloor \frac{n-9}{2} \rfloor, \lfloor \frac{n-5}{2} \rfloor] \cup \{\lfloor \frac{n-1}{2} \rfloor, \lceil \frac{n+3}{2} \rceil\}$

is a $\mathbb{Z}_n^2$-set.

Subcase 3.7. $\ell_1 = \lfloor \frac{n-13}{2} \rfloor$: As $\ell_1 \geq 4$, $n \geq 21$. $[1, \lfloor \frac{n-15}{2} \rfloor] \cup \{\lfloor \frac{n-11}{2} \rfloor, \lfloor \frac{n-7}{2} \rfloor, \lfloor \frac{n-3}{2} \rfloor, \lfloor \frac{n-1}{2} \rfloor, \lceil \frac{n+5}{2} \rceil\}$ is a $\mathbb{Z}_n^2$-set.

Subcase 3.8. $\ell_1 = \lfloor \frac{n-11}{2} \rfloor$: As $\ell_1 \geq 4$, $n \geq 19$. For $n \geq 21$, $[1, \lfloor \frac{n-17}{2} \rfloor] \cup \{\lfloor \frac{n-13}{2} \rfloor, \lfloor \frac{n-9}{2} \rfloor\} \cup [\lfloor \frac{n-5}{2} \rfloor, \lfloor \frac{n-1}{2} \rfloor] \cup \{\lceil \frac{n+15}{2} \rceil\}$ is a $\mathbb{Z}_n^2$-set. $\{1,3,5,7,8,9,17\}$ is a $\mathbb{Z}_{19}^2$-set.

Subcase 3.9. $\ell_1 = \lfloor \frac{n-9}{2} \rfloor$: Since $\ell_1 \geq 4$, $n \geq 17$. For $n \geq 21$, $[1, \lfloor \frac{n-17}{2} \rfloor] \cup \{\lfloor \frac{n-13}{2} \rfloor, \lfloor \frac{n-11}{2} \rfloor, \lfloor \frac{n-7}{2} \rfloor, \lfloor \frac{n-3}{2} \rfloor, \lfloor \frac{n-1}{2} \rfloor, \lceil \frac{n+15}{2} \rceil\}$ is a $\mathbb{Z}_n^2$-set. $\{1,2,4,6,8,9,16\}$ is a $\mathbb{Z}_{19}^2$- set. $\{1,2,3,7,9,12\}$ is a $\mathbb{Z}_{17}^2$-set.

Subcase 3.10. $\ell_1 = \lfloor \frac{n-7}{2} \rfloor$: For $n \geq 23$ or $n \in \{21,19\}$, $[1, \lfloor \frac{n-13}{2} \rfloor] \cup \{\lfloor \frac{n-9}{2} \rfloor, \lfloor \frac{n-5}{2} \rfloor, \lceil \frac{n+1}{2} \rceil, \lceil \frac{n+11}{2} \rceil\}$ is a $\mathbb{Z}_n^2$-set. $\{1,2,3,5,6,8,12,18\}$ is a $\mathbb{Z}_{22}^2$-set. $\{3,4,6,8,15,16\}$ is a $\mathbb{Z}_{17}^2$-set.

Subcase 3.11. $\ell_1 = \lfloor \frac{n-5}{2} \rfloor$: For $n \geq 23$ or $n = 21$, $[1, \lfloor \frac{n-13}{2} \rfloor] \cup \{\lfloor \frac{n-9}{2} \rfloor, \lfloor \frac{n-7}{2} \rfloor, \lfloor \frac{n-3}{2} \rfloor, \lceil \frac{n+11}{2} \rceil\}$ is a $\mathbb{Z}_n^2$-set. $\{1,2,4,6,7,13,17,19\}$ is a $\mathbb{Z}_{22}^2$-set. $\{1,2,4,6,8,14,16\}$ is a $\mathbb{Z}_{19}^2$-set. $\{4,7,12,14,15,16\}$ is a $\mathbb{Z}_{17}^2$-set.

Hence, assume that $\ell_2 \geq \ell_1 + 3$.

Case 4. $\ell_1 = 1$ ($\ell_2 \geq 4$).

Subcase 4.1. $\ell_2 = 4$: For $n \geq 27$ or $n \in \{23,22\}$, $\{2,3\} \cup [6, \lfloor \frac{n-3}{2} \rfloor] \cup \{\lceil \frac{n+1}{2} \rceil, n-5\}$ is a $\mathbb{Z}_n^2$-set. $\{2,3,5\} \cup [7,12] \cup \{20\}$ is a $\mathbb{Z}_{26}^2$-set. $\{2,3\} \cup [6,12] \cup \{20\}$ is a $\mathbb{Z}_{25}^2$-set. $\{2,3,5\} \cup [7,11] \cup \{18\}$ is a $\mathbb{Z}_{24}^2$-set. $\{2\} \cup [6,10] \cup \{16,18\}$ is a $\mathbb{Z}_{21}^2$-set. $\{2,3,5,6,9,11,12\}$ is a $\mathbb{Z}_{19}^2$-set. $\{2,3,5,6,9,10\}$ is a $\mathbb{Z}_{17}^2$-set.

Subcase 4.2. $5 \leq \ell_2 \leq \lfloor \frac{n-11}{2} \rfloor$: $[2, \ell_2 - 1] \cup [\ell_2 + 2, \lfloor \frac{n-3}{2} \rfloor] \cup \{\lceil \frac{n+1}{2} \rceil, n - \ell_2 - 1\}$ is a $\mathbb{Z}_n^2$-set.

Subcase 4.3. $\ell_2 = \lfloor \frac{n-9}{2} \rfloor$: Since $\ell_2 \geq 5$, $n \geq 19$. For $n \geq 25$ or $n = 23$, $[2, \lfloor \frac{n-13}{2} \rfloor] \cup [\lfloor \frac{n-5}{2} \rfloor, \lfloor \frac{n-1}{2} \rfloor] \cup \{\lceil \frac{n+7}{2} \rceil, \lceil \frac{n+11}{2} \rceil\}$ is a $\mathbb{Z}_n^2$-set. $\{2\} \cup [4,6] \cup [8,11] \cup \{21\}$ is a $\mathbb{Z}_{24}^2$-set. $\{2,4,5\} \cup [8,10] \cup \{15,19\}$ is a $\mathbb{Z}_{22}^2$-set. $[2,4] \cup [7,9] \cup \{11,16\}$ is a $\mathbb{Z}_{21}^2$-set. $\{2,3,4,6,9,11,12\}$ is a $\mathbb{Z}_{19}^2$-set.

Subcase 4.4. $\ell_2 = \lfloor \frac{n-7}{2} \rfloor$: As $\ell_2 \geq 5$, $n \geq 17$. For $n \geq 21$, $[2, \lfloor \frac{n-17}{2} \rfloor] \cup [\lfloor \frac{n-13}{2} \rfloor, \lfloor \frac{n-9}{2} \rfloor] \cup [\lfloor \frac{n-5}{2} \rfloor, \lfloor \frac{n-1}{2} \rfloor] \cup \{\lceil \frac{n+15}{2} \rceil\}$ is a $\mathbb{Z}_n^2$-set. $\{2,3,4,7,8,10,14\}$ is a $\mathbb{Z}_{19}^2$-set. $\{2,3,4,6,9,10\}$ is a $\mathbb{Z}_{17}^2$-set.

Subcase 4.5. $\ell_2 = \lfloor \frac{n-5}{2} \rfloor$: For $n \geq 21$ or $n = 19$, $[2, \lfloor \frac{n-15}{2} \rfloor] \cup [\lfloor \frac{n-11}{2} \rfloor, \lfloor \frac{n-7}{2} \rfloor] \cup \{\lfloor \frac{n-3}{2} \rfloor, \lfloor \frac{n-1}{2} \rfloor, \lceil \frac{n+13}{2} \rceil\}$ is a $\mathbb{Z}_n^2$-set. $\{4,7,8,12,14,15\}$ is a $\mathbb{Z}_{17}^2$-set.

Subcase 4.6. $\ell_2 = \lfloor \frac{n-3}{2} \rfloor$: For $n \geq 21$ or $n = 19$, $[2, \lfloor \frac{n-13}{2} \rfloor] \cup [\lfloor \frac{n-9}{2} \rfloor, \lfloor \frac{n-5}{2} \rfloor] \cup \{\lfloor \frac{n-1}{2} \rfloor, \lceil \frac{n+11}{2} \rceil\}$ is a $\mathbb{Z}_n^2$-set. $\{3,5,6,9,13,15\}$ is a $\mathbb{Z}_{17}^2$-set.

Subcase 4.7. $\ell_2 = \lfloor \frac{n-1}{2} \rfloor$: $[2, \lfloor \frac{n-11}{2} \rfloor] \cup [\lfloor \frac{n-7}{2} \rfloor, \lfloor \frac{n-3}{2} \rfloor] \cup \{\lceil \frac{n+9}{2} \rceil\}$ is a $\mathbb{Z}_n^2$-set.

Case 5. $\ell_1 = 2$ ($\ell_2 \geq 5$).

Subcase 5.1. $\ell_2 \leq \lfloor \frac{n-11}{2} \rfloor$: $\{1\} \cup [3, \ell_2 - 1] \cup [\ell_2 + 2, \lfloor \frac{n-3}{2} \rfloor] \cup \{\lceil \frac{n+1}{2} \rceil, n - \ell_2 - 1\}$ is a $\mathbb{Z}_n^2$-set.

Subcase 5.2. $\ell_2 = \lfloor \frac{n-9}{2} \rfloor$: Since $\ell_2 \geq 5$, $n \geq 19$. For $n \geq 23$ or $n = 21$, $\{1\} \cup [3, \lfloor \frac{n-15}{2} \rfloor] \cup \{\lfloor \frac{n-11}{2} \rfloor\} \cup [\lfloor \frac{n-5}{2} \rfloor, \lfloor \frac{n-1}{2} \rfloor] \cup \{\lceil \frac{n+7}{2} \rceil, \lceil \frac{n+13}{2} \rceil\}$ is a $\mathbb{Z}_n^2$-set. $\{1,3,4,5,8,10,13,15\}$ is a $\mathbb{Z}_{22}^2$-set. $\{1,3,4,7,8,9,13\}$ is a $\mathbb{Z}_{19}^2$-set.

Subcase 5.3. $\ell_2 = \lfloor \frac{n-7}{2} \rfloor$: For $n \geq 23$ or $n = 21$, $\{1\} \cup [3, \lfloor \frac{n-11}{2} \rfloor] \cup \{\lfloor \frac{n-3}{2} \rfloor, \lfloor \frac{n-1}{2} \rfloor, \lceil \frac{n+5}{2} \rceil, \lceil \frac{n+9}{2} \rceil\}$ is a $\mathbb{Z}_n^2$-set. $\{1,3,4,6,8,9,10,17\}$ is a $\mathbb{Z}_{22}^2$-set. $\{1,3,4,7,8,10,14\}$ is a $\mathbb{Z}_{19}^2$-set. $\{1,6,7,9,13,14\}$ is a $\mathbb{Z}_{17}^2$-set.

Subcase 5.4. $\ell_2 = \lfloor \frac{n-5}{2} \rfloor$: For $n \geq 21$, $\{1\} \cup [3, \lfloor \frac{n-9}{2} \rfloor] \cup \{\lfloor \frac{n-1}{2} \rfloor, \lceil \frac{n+3}{2} \rceil, \lceil \frac{n+7}{2} \rceil\}$ is a $\mathbb{Z}_n^2$-set. $\{3,4,5,6,8,10,18\}$ is a $\mathbb{Z}_{19}^2$-set. $\{4,5,7,8,14,16\}$ is a $\mathbb{Z}_{17}^2$-set.

Subcase 5.5. $\ell_2 = \lfloor \frac{n-3}{2} \rfloor$: For $n \geq 23$ or $n \in \{21, 17\}$, $\{1\} \cup [3, \lfloor \frac{n-9}{2} \rfloor] \cup \{\lfloor \frac{n-5}{2} \rfloor, \lfloor \frac{n-1}{2} \rfloor, \lceil \frac{n+7}{2} \rceil\}$ is a $\mathbb{Z}_n^2$- set. $\{1, 3, 4, 7, 8, 12, 16, 17\}$ is a $\mathbb{Z}_{22}^2$-set. $\{1, 3, 4, 5, 6, 10, 12\}$ is a $\mathbb{Z}_{19}^2$-set.

Subcase 5.6. $\ell_2 = \lfloor \frac{n-1}{2} \rfloor$: For $n \geq 21$ or $n = 19$, $\{1\} \cup [4, \lfloor \frac{n-5}{2} \rfloor] \cup \{\lceil \frac{n+3}{2} \rceil, n-3\}$ is a $\mathbb{Z}_n^2$-set. $\{5, 7, 11, 13, 14, 16\}$ is a $\mathbb{Z}_{17}^2$-set.

Hence, assume that $\ell_1 \geq 3$.

Case 6. $\ell_2 = \lfloor \frac{n-9}{2} \rfloor$.

Subcase 6.1. $\ell_1 \leq \lfloor \frac{n-19}{2} \rfloor$: $[1, \ell_1 - 1] \cup [\ell_1 + 1, \lfloor \frac{n-15}{2} \rfloor] \cup \{\lfloor \frac{n-11}{2} \rfloor\} \cup [\lfloor \frac{n-7}{2} \rfloor, \lfloor \frac{n-1}{2} \rfloor] \cup \{\lceil \frac{n+13}{2} \rceil\}$ is a $\mathbb{Z}_n^2$-set.

Subcase 6.2. $\ell_1 = \lfloor \frac{n-17}{2} \rfloor$: As $\ell_1 \geq 3$, $n \geq 23$. $[1, \lfloor \frac{n-19}{2} \rfloor] \cup [\lfloor \frac{n-15}{2} \rfloor, \lfloor \frac{n-11}{2} \rfloor] \cup \{\lfloor \frac{n-7}{2} \rfloor\} \cup [\lfloor \frac{n-3}{2} \rfloor, \lfloor \frac{n-1}{2} \rfloor] \cup \{\lceil \frac{n+5}{2} \rceil\}$ is a $\mathbb{Z}_n^2$-set.

Subcase 6.3. $\ell_1 = \lfloor \frac{n-15}{2} \rfloor$: As $\ell_1 \geq 3$, $n \geq 21$. $[1, \lfloor \frac{n-17}{2} \rfloor] \cup \{\lfloor \frac{n-13}{2} \rfloor, \lfloor \frac{n-11}{2} \rfloor, \lfloor \frac{n-7}{2} \rfloor, \lfloor \frac{n-3}{2} \rfloor, \lfloor \frac{n-1}{2} \rfloor, \lceil \frac{n+5}{2} \rceil\}$ is a $\mathbb{Z}_n^2$-set.

Case 7. $\ell_2 = \lfloor \frac{n-7}{2} \rfloor$.

Subcase 7.1. $\ell_1 \leq \lfloor \frac{n-17}{2} \rfloor$: $[1, \ell_1 - 1] \cup [\ell_1 + 1, \lfloor \frac{n-13}{2} \rfloor] \cup \{\lfloor \frac{n-9}{2} \rfloor\} \cup [\lfloor \frac{n-5}{2} \rfloor, \lfloor \frac{n-1}{2} \rfloor] \cup \{\lceil \frac{n+11}{2} \rceil\}$ is a $\mathbb{Z}_n^2$-set.

Subcase 7.2. $\ell_1 = \lfloor \frac{n-15}{2} \rfloor$: As $\ell_1 \geq 3$, $n \geq 21$. $[1, \lfloor \frac{n-17}{2} \rfloor] \cup \{\lfloor \frac{n-13}{2} \rfloor, \lfloor \frac{n-9}{2} \rfloor\} \cup [\lfloor \frac{n-5}{2} \rfloor, \lfloor \frac{n-1}{2} \rfloor] \cup \{\lceil \frac{n+11}{2} \rceil\}$ is a $\mathbb{Z}_n^2$-set.

Subcase 7.3. $\ell_1 = \lfloor \frac{n-13}{2} \rfloor$: As $\ell_1 \geq 3$, $n \geq 19$. For $n \geq 23$, $[1, \lfloor \frac{n-19}{2} \rfloor] \cup \{\lfloor \frac{n-15}{2} \rfloor, \lfloor \frac{n-11}{2} \rfloor, \lfloor \frac{n-9}{2} \rfloor\} \cup [\lfloor \frac{n-5}{2} \rfloor, \lfloor \frac{n-1}{2} \rfloor] \cup \{\lceil \frac{n+17}{2} \rceil\}$ is a $\mathbb{Z}_n^2$-set. $\{1, 2, 3, 6, 8, 9, 12, 17\}$ is a $\mathbb{Z}_{22}^2$-set. $\{1, 2, 3, 5, 6, 8, 11, 12\}$ is a $\mathbb{Z}_{21}^2$-set. $\{1, 2, 4, 5, 7, 10, 11\}$ is a $\mathbb{Z}_{19}^2$-set.

Case 8. $\ell_2 = \lfloor \frac{n-5}{2} \rfloor$.

Subcase 8.1. $\ell_1 \leq \lfloor \frac{n-15}{2} \rfloor$: $[1, \ell_1 - 1] \cup [\ell_1 + 1, \lfloor \frac{n-11}{2} \rfloor] \cup \{\lfloor \frac{n-7}{2} \rfloor, \lfloor \frac{n-3}{2} \rfloor, \lfloor \frac{n-1}{2} \rfloor, \lceil \frac{n+9}{2} \rceil\}$ is a $\mathbb{Z}_n^2$-set.

Subcase 8.2. $\ell_1 = \lfloor \frac{n-13}{2} \rfloor$: As $\ell_1 \geq 3$, $n \geq 19$. $[1, \lfloor \frac{n-15}{2} \rfloor] \cup \{\lfloor \frac{n-11}{2} \rfloor, \lfloor \frac{n-7}{2} \rfloor, \lfloor \frac{n-3}{2} \rfloor, \lfloor \frac{n-1}{2} \rfloor, \lceil \frac{n+9}{2} \rceil\}$ is a $\mathbb{Z}_n^2$-set.

Subcase 8.3. $\ell_1 = \lfloor \frac{n-11}{2} \rfloor$: For $n \geq 21$, $[1, \lfloor \frac{n-17}{2} \rfloor] \cup \{\lfloor \frac{n-13}{2} \rfloor, \lfloor \frac{n-9}{2} \rfloor, \lfloor \frac{n-7}{2} \rfloor, \lfloor \frac{n-3}{2} \rfloor, \lfloor \frac{n-1}{2} \rfloor, \lfloor \frac{n+15}{2} \rfloor\}$ is a $\mathbb{Z}_n^2$-set. $\{1, 2, 3, 6, 8, 10, 14\}$ is a $\mathbb{Z}_{19}^2$-set. $\{1, 2, 4, 5, 8, 10\}$ is a $\mathbb{Z}_{17}^2$-set.

Case 9. $\ell_2 = \lfloor \frac{n-3}{2} \rfloor$.

Subcase 9.1. $\ell_1 \leq \lfloor \frac{n-19}{2} \rfloor$: $[1, \ell_1 - 1] \cup [\ell_1 + 1, \lfloor \frac{n-9}{2} \rfloor] \cup \{\lfloor \frac{n-5}{2} \rfloor, \lfloor \frac{n-1}{2} \rfloor, \lceil \frac{n+7}{2} \rceil\}$ is a $\mathbb{Z}_n^2$-set.

Subcase 9.2. $\ell_1 = \lfloor \frac{n-17}{2} \rfloor$: As $\ell_1 \geq 3$, $n \geq 23$. $[1, \lfloor \frac{n-19}{2} \rfloor] \cup [\lfloor \frac{n-15}{2} \rfloor, \lfloor \frac{n-11}{2} \rfloor] \cup \{\lfloor \frac{n-7}{2} \rfloor, \lfloor \frac{n-5}{2} \rfloor, \lfloor \frac{n-1}{2} \rfloor, \lceil \frac{n+9}{2} \rceil\}$ is a $\mathbb{Z}_n^2$-set.

Subcase 9.3. $\ell_1 = \lfloor \frac{n-15}{2} \rfloor$: As $\ell_1 \geq 3$, $n \geq 21$. For $n \geq 23$, $[1, \lfloor \frac{n-21}{2} \rfloor] \cup \{\lfloor \frac{n-17}{2} \rfloor\} \cup [\lfloor \frac{n-13}{2} \rfloor, \lfloor \frac{n-5}{2} \rfloor] \cup \{\lceil \frac{n+1}{2} \rceil, \lceil \frac{n+19}{2} \rceil\}$ is a $\mathbb{Z}_n^2$-set. $\{1, 2, 4, 5, 6, 8, 10, 15\}$ is a $\mathbb{Z}_{22}^2$-set. $\{1, 2, 4, 5, 6, 8, 11, 14\}$ is a $\mathbb{Z}_{21}^2$- set.

Subcase 9.4. $\ell_1 = \lfloor \frac{n-13}{2} \rfloor$: As $\ell_1 \geq 3$, $n \geq 19$. For $n \geq 21$, $[1, \lfloor \frac{n-19}{2} \rfloor] \cup \{\lfloor \frac{n-15}{2} \rfloor\} \cup [\lfloor \frac{n-11}{2} \rfloor, \lfloor \frac{n-5}{2} \rfloor] \cup \{\lceil \frac{n+1}{2} \rceil, \lceil \frac{n+17}{2} \rceil\}$ is a $\mathbb{Z}_n^2$-set. $\{1, 2, 4, 5, 7, 9, 13\}$ is a $\mathbb{Z}_{19}^2$-set.

Subcase 9.5. $\ell_1 = \lfloor \frac{n-11}{2} \rfloor$: For $n \geq 23$ or $n = 21$, $[1, \lfloor \frac{n-17}{2} \rfloor] \cup \{\lfloor \frac{n-13}{2} \rfloor\} \cup [\lfloor \frac{n-9}{2} \rfloor, \lfloor \frac{n-5}{2} \rfloor] \cup \{\lfloor \frac{n-1}{2} \rfloor, \lceil \frac{n+15}{2} \rceil\}$ is a $\mathbb{Z}_n^2$- set. $\{1, 2, 3, 4, 6, 8, 12, 15\}$ is a $\mathbb{Z}_{22}^2$-set. $\{1, 2, 3, 5, 7, 10, 13\}$ is a $\mathbb{Z}_{19}^2$-set. $\{6, 8, 12, 13, 15, 16\}$ is a $\mathbb{Z}_{17}^2$-set.

Subcase 9.6. $\ell_1 = \lfloor \frac{n-9}{2} \rfloor$: For $n \geq 21$ or $n = 19$, $[1, \lfloor \frac{n-15}{2} \rfloor] \cup \{\lfloor \frac{n-11}{2} \rfloor, \lfloor \frac{n-7}{2} \rfloor, \lfloor \frac{n-5}{2} \rfloor, \lfloor \frac{n-1}{2} \rfloor, \lceil \frac{n+13}{2} \rceil\}$ is a $\mathbb{Z}_n^2$-set. $\{1, 2, 3, 6, 8, 12\}$ is a $\mathbb{Z}_{17}^2$-set.

Case 10. $\ell_2 = \lfloor \frac{n-1}{2} \rfloor$.
Subcase 10.1. $\ell_1 \le \lfloor \frac{n-17}{2} \rfloor$: $[1, \ell_1 - 1] \cup [\ell_1 + 1, \lfloor \frac{n-7}{2} \rfloor] \cup \{\lfloor \frac{n-3}{2} \rfloor, \lceil \frac{n+5}{2} \rceil\}$ is a $\mathbb{Z}_n^2$-set.
Subcase 10.2. $\ell_1 = \lfloor \frac{n-15}{2} \rfloor$: As $\ell_1 \ge 3$, $n \ge 21$. $[1, \lfloor \frac{n-17}{2} \rfloor] \cup [\lfloor \frac{n-13}{2} \rfloor, \lfloor \frac{n-9}{2} \rfloor] \cup \{\lfloor \frac{n-5}{2} \rfloor, \lfloor \frac{n-3}{2} \rfloor, \lceil \frac{n+7}{2} \rceil\}$ is a $\mathbb{Z}_n^2$-set.
Subcase 10.3. $\ell_1 = \lfloor \frac{n-13}{2} \rfloor$: As $\ell_1 \ge 3$, $n \ge 19$. For $n \ge 25$ or $n = 23$, $[1, \lfloor \frac{n-15}{2} \rfloor] \cup \{\lfloor \frac{n-11}{2} \rfloor, \lfloor \frac{n-9}{2} \rfloor, \lfloor \frac{n-3}{2} \rfloor, \lceil \frac{n+5}{2} \rceil, \lceil \frac{n+7}{2} \rceil\}$ is a $\mathbb{Z}_n^2$-set. $\{1, 2, 3, 4, 6, 7, 8, 10, 15\}$ is a $\mathbb{Z}_{24}^2$-set. $\{1, 2, 3, 5, 6, 7, 9, 14\}$ is a $\mathbb{Z}_{22}^2$-set. $\{1, 2, 3, 5, 7, 8, 12, 15\}$ is a $\mathbb{Z}_{21}^2$-set. $\{1, 2, 4, 5, 7, 8, 13\}$ is a $\mathbb{Z}_{19}^2$-set.
Subcase 10.4. $\ell_1 = \lfloor \frac{n-11}{2} \rfloor$: For $n \ge 23$ or $n = 21$, $[1, \lfloor \frac{n-17}{2} \rfloor] \cup \{\lfloor \frac{n-13}{2} \rfloor\} \cup [\lfloor \frac{n-9}{2} \rfloor, \lfloor \frac{n-3}{2} \rfloor] \cup \{\lceil \frac{n+15}{2} \rceil\}$ is a $\mathbb{Z}_n^2$-set. $\{1, 2, 3, 6, 7, 9, 14, 18\}$ is a $\mathbb{Z}_{22}^2$-set. $\{1, 2, 3, 5, 6, 8, 12\}$ is a $\mathbb{Z}_{19}^2$-set. $\{1, 2, 4, 5, 7, 11\}$ is a $\mathbb{Z}_{17}^2$-set.
Subcase 10.5. $\ell_1 = \lfloor \frac{n-9}{2} \rfloor$: $[1, \lfloor \frac{n-13}{2} \rfloor] \cup \{\lfloor \frac{n-7}{2} \rfloor, \lfloor \frac{n-3}{2} \rfloor, \lceil \frac{n+5}{2} \rceil, \lceil \frac{n+11}{2} \rceil\}$ is a $\mathbb{Z}_n^2$- set.
Subcase 10.6. $\ell_1 = \lfloor \frac{n-7}{2} \rfloor$: $[1, \lfloor \frac{n-13}{2} \rfloor] \cup \{\lfloor \frac{n-9}{2} \rfloor, \lfloor \frac{n-5}{2} \rfloor, \lfloor \frac{n-3}{2} \rfloor, \lceil \frac{n+11}{2} \rceil\}$ is a $\mathbb{Z}_n^2$-set. □

Lemma 3.4. *If* $\{\ell_1, \ell_2\} \in \{\{1,3\}, \{1,4\}, \{1,5\}, \{1,6\}, \{1,7\}, \{1,9\}, \{2,3\}, \{2,7\}, \{2,8\}, \{3,5\}, \{3,7\}, \{3,8\}, \{3,9\}, \{4,6\}, \{4,9\}, \{5,7\}, \{5,9\}, \{6,9\}, \{7,8\}, \{7,9\}\}$, *then there exists a* $\mathbb{Z}_{20}^2$*-set* L *contained in* $\mathbb{Z}_{20} \setminus \{\ell_1, \ell_2, 20 - \ell_2, 20 - \ell_1\}$.

Proof. For $\{\ell_1, \ell_2\} = \{1, 3\}$, a $\mathbb{Z}_{20}^2$-set is $\{2, 5, 6, 11, 12, 13, 16\}$. Similarly, for $\{\ell_1, \ell_2\} = \{1, 4\}$, $\{1, 5\}$, $\{1, 6\}$, $\{1, 7\}$, $\{1, 9\}$, $\{2, 3\}$, $\{2, 7\}$, $\{2, 8\}$, $\{3, 5\}$, $\{3, 7\}$, $\{3, 8\}$, $\{3, 9\}$, $\{4, 6\}$, $\{4, 9\}$, $\{5, 7\}$, $\{5, 9\}$, $\{6, 9\}$, $\{7, 8\}$, $\{7, 9\}$, respective $\mathbb{Z}_{20}^2$-sets are $\{2, 3, 5, 7, 8, 9, 14\}$, $\{2, 8, 9, 13, 14, 16, 17\}$, $\{3, 4, 5, 7, 9, 12, 18\}$, $\{2, 4, 5, 6, 8, 9, 17\}$, $\{2, 3, 4, 6, 7, 8, 15\}$, $\{4, 5, 6, 8, 9, 13, 19\}$, $\{1, 3, 4, 6, 8, 9, 15\}$, $\{1, 3, 4, 5, 7, 9, 14\}$, $\{1, 2, 4, 6, 8, 9, 13\}$, $\{1, 2, 4, 6, 8, 9, 15\}$, $\{1, 2, 4, 6, 7, 9, 15\}$, $\{1, 2, 4, 5, 6, 8, 13\}$, $\{1, 3, 5, 7, 8, 9, 18\}$, $\{1, 3, 5, 6, 7, 12, 18\}$, $\{1, 2, 4, 6, 8, 9, 17\}$, $\{2, 3, 7, 8, 14, 16, 19\}$, $\{1, 2, 3, 5, 7, 8, 16\}$, $\{1, 2, 4, 5, 6, 11, 17\}$, $\{3, 5, 6, 8, 16, 18, 19\}$. □

Lemma 3.5. *If* $\{\ell_1, \ell_2\} \in \{\{1,2\}, \{1,8\}, \{2,4\}, \{2,5\}, \{2,6\}, \{2,9\}, \{3,4\}, \{3,6\}, \{4,5\}, \{4,7\}, \{4,8\}, \{5,6\}, \{5,8\}, \{6,7\}, \{6,8\}, \{8,9\}\}$, *then there exists a* $\mathbb{Z}_{20}^2$*-partition of* $\mathbb{Z}_{20} \setminus \{0, \ell_1, \ell_2, 20 - \ell_1, 20 - \ell_2\}$.

Proof. For $\{\ell_1, \ell_2\} = \{1, 2\}$, a $\mathbb{Z}_{20}^2$-partition is $(\{3, 4, 5, 8, 12, 15, 16\}, \{6, 7, 9, 11, 13, 14, 17\})$. Similarly, for $\{\ell_1, \ell_2\} = \{1, 8\}$, $\{2, 4\}$, $\{2, 5\}$, $\{2, 6\}$, $\{2, 9\}$, $\{3, 4\}$, $\{3, 6\}$, $\{4, 5\}$, $\{4, 7\}$, $\{4, 8\}$, $\{5, 6\}$, $\{5, 8\}$, $\{6, 7\}$, $\{6, 8\}$, $\{8, 9\}$, respective $\mathbb{Z}_{20}^2$-partitions are $(\{2, 3, 4, 5, 9, 15, 16\}, \{6, 7, 11, 13, 14, 17, 18\})$, $(\{1, 3, 5, 7, 9, 13, 15\}, \{6, 8, 11, 12, 14, 17, 19\})$, $(\{1, 3, 4, 6, 9, 14, 16\}, \{7, 8, 11, 12, 13, 17, 19\})$, $(\{1, 3, 4, 5, 9, 15, 16\}, \{7, 8, 11, 12, 13, 17, 19\})$, $(\{1, 3, 4, 5, 8, 15, 16\}, \{6, 7, 12, 13, 14, 17, 19\})$, $(\{1, 2, 5, 8, 9, 12, 15\}, \{6, 7, 11, 13, 14, 18, 19\})$, $(\{1, 2, 4, 7, 9, 13, 16\}, \{5, 8, 11, 12, 15, 18, 19\})$, $(\{1, 2, 3, 6, 9, 14, 17\}, \{7, 8, 11, 12, 13, 18, 19\})$, $(\{1, 2, 3, 5, 9, 15, 17\}, \{6, 8, 11, 12, 14, 18, 19\})$, $(\{1, 2, 3, 6, 9, 14, 17\}, \{5, 7, 11, 13, 15, 18, 19\})$, $(\{1, 2, 3, 7, 9, 13, 17\}, \{4, 8, 11, 12, 16, 18, 19\})$, $(\{1, 2, 3, 6, 9, 14, 17\}, \{4, 7, 11, 13, 16, 18, 19\})$, $(\{1, 2, 3, 4, 9, 16, 17\}, \{5, 8, 11, 12, 15, 18, 19\})$, $(\{1, 2, 3, 4, 9, 16, 17\}, \{5, 7, 11, 13, 15, 18, 19\})$, $(\{1, 2, 3, 4, 7, 16, 17\}, \{5, 6, 13, 14, 15, 18, 19\})$. □

Lemma 3.6. *If* $\ell_1, \ell_2 \in \{1, 2, 3, 4, 5, 6, 7, 8\}$ *with* $\ell_1 \ne \ell_2$, *then there exists a* $\mathbb{Z}_{18}^2$*-partition of* $\mathbb{Z}_{18} \setminus \{0, \ell_1, \ell_2, 18 - \ell_1, 18 - \ell_2\}$.

Proof. For $\{\ell_1, \ell_2\} = \{1, 2\}$, a $\mathbb{Z}_{18}^2$-partition is $(\{3, 4, 5, 6, 12, 13\}, \{7, 8, 10, 11, 14, 15\})$. Similarly, for $\{\ell_1, \ell_2\} = \{1, 3\}$, $\{1, 4\}$, $\{1, 5\}$, $\{1, 6\}$, $\{1, 7\}$, $\{1, 8\}$, $\{2, 3\}$, $\{2, 4\}$, $\{2, 5\}$, $\{2, 6\}$, $\{2, 7\}$, $\{2, 8\}$, $\{3, 4\}$, $\{3, 5\}$, $\{3, 6\}$, $\{3, 7\}$, $\{3, 8\}$, $\{4, 5\}$, $\{4, 6\}$, $\{4, 7\}$, $\{4, 8\}$, $\{5, 6\}$, $\{5, 7\}$, $\{5, 8\}$, $\{6, 7\}$, $\{6, 8\}$, $\{7, 8\}$, respective $\mathbb{Z}_{18}^2$-partitions are $(\{2, 4, 5, 6, 12, 13\}, \{7, 8, 10, 11, 14, 16\})$,
$(\{2, 3, 5, 6, 12, 13\}, \{7, 8, 10, 11, 15, 16\}), (\{2, 3, 4, 6, 12, 14\}, \{7, 8, 10, 11, 15, 16\})$,
$(\{2, 3, 4, 7, 11, 14\}, \{5, 8, 10, 13, 15, 16\}), (\{2, 3, 4, 5, 13, 14\}, \{6, 8, 10, 12, 15, 16\})$,
$(\{2, 3, 4, 5, 13, 14\}, \{6, 7, 11, 12, 15, 16\}), (\{1, 4, 5, 6, 12, 13\}, \{7, 8, 10, 11, 14, 17\})$,
$(\{1, 3, 5, 6, 12, 13\}, \{7, 8, 10, 11, 15, 17\}), (\{1, 3, 4, 6, 12, 14\}, \{7, 8, 10, 11, 15, 17\})$,
$(\{1, 3, 4, 8, 10, 14\}, \{5, 7, 11, 13, 15, 17\}), (\{1, 3, 4, 5, 13, 14\}, \{6, 8, 10, 12, 15, 17\})$,
$(\{1, 3, 4, 5, 13, 14\}, \{6, 7, 11, 12, 15, 17\}), (\{1, 2, 5, 6, 12, 13\}, \{7, 8, 10, 11, 16, 17\})$,
$(\{1, 2, 4, 6, 12, 14\}, \{7, 8, 10, 11, 16, 17\}), (\{1, 2, 4, 7, 11, 14\}, \{5, 8, 10, 13, 16, 17\})$,
$(\{1, 2, 4, 5, 13, 14\}, \{6, 8, 10, 12, 16, 17\}), (\{1, 2, 4, 5, 13, 14\}, \{6, 7, 11, 12, 16, 17\})$,
$(\{1, 2, 3, 6, 8, 12\}, \{7, 10, 11, 15, 16, 17\}), (\{1, 2, 3, 7, 11, 15\}, \{5, 8, 10, 13, 16, 17\})$,
$(\{1, 2, 3, 5, 13, 15\}, \{6, 8, 10, 12, 16, 17\}), (\{1, 2, 3, 5, 13, 15\}, \{6, 7, 11, 12, 16, 17\})$,
$(\{1, 2, 3, 8, 10, 15\}, \{4, 7, 11, 14, 16, 17\}), (\{1, 2, 3, 6, 12, 15\}, \{4, 8, 10, 14, 16, 17\})$,
$(\{1, 2, 3, 6, 12, 15\}, \{4, 7, 11, 14, 16, 17\}), (\{1, 2, 3, 8, 10, 15\}, \{4, 5, 13, 14, 16, 17\})$,
$(\{1, 2, 3, 4, 14, 15\}, \{5, 7, 11, 13, 16, 17\}), (\{1, 2, 6, 12, 14, 16\}, \{3, 4, 5, 13, 15, 17\})$. □

Lemma 3.7. *If $\ell_1, \ell_2 \in \{1, 2, 3, 4, 5, 6, 7\}$ with $\ell_1 \neq \ell_2$, then there exists a $\mathbb{Z}_{16}^2$-partition of $\mathbb{Z}_{16} \setminus \{0, \ell_1, \ell_2, 16 - \ell_1, 16 - \ell_2\}$.*

Proof. For $\{\ell_1, \ell_2\} = \{1, 2\}$, a $\mathbb{Z}_{16}^2$-partition is $(\{3, 4, 5, 11, 12\}, \{6, 7, 9, 10, 13\})$. Similarly, for $\{\ell_1, \ell_2\} = \{1, 3\}, \{1, 4\}, \{1, 5\}, \{1, 6\}, \{1, 7\}, \{2, 3\}, \{2, 4\}, \{2, 5\}, \{2, 6\}, \{2, 7\}, \{3, 4\}, \{3, 5\}, \{3, 6\}, \{3, 7\}, \{4, 5\}, \{4, 6\}, \{4, 7\}, \{5, 6\}, \{5, 7\}, \{6, 7\}$, respective $\mathbb{Z}_{16}^2$-partitions are
$(\{2, 4, 5, 11, 12\}, \{6, 7, 9, 10, 14\}), (\{2, 3, 5, 10, 11\}, \{6, 7, 9, 13, 14\}), (\{2, 7, 9, 12, 13\}, \{3, 4, 6, 10, 14\})$,
$(\{2, 3, 5, 11, 12\}, \{4, 7, 9, 13, 14\}), (\{2, 5, 11, 12, 13\}, \{3, 4, 6, 10, 14\}), (\{1, 4, 7, 12, 15\}, \{5, 6, 9, 10, 11\})$,
$(\{1, 3, 5, 11, 13\}, \{6, 7, 9, 10, 15\}), (\{1, 3, 6, 10, 13\}, \{4, 7, 9, 12, 15\}), (\{1, 3, 4, 12, 13\}, \{5, 7, 9, 11, 15\})$,
$(\{1, 3, 4, 12, 13\}, \{5, 6, 10, 11, 15\}), (\{1, 2, 5, 11, 15\}, \{6, 7, 9, 10, 14\}), (\{1, 2, 4, 6, 10\}, \{7, 9, 12, 14, 15\})$,
$(\{1, 2, 5, 11, 15\}, \{4, 7, 9, 12, 14\}), (\{1, 2, 4, 12, 14\}, \{5, 6, 10, 11, 15\}), (\{1, 2, 6, 14, 15\}, \{3, 7, 9, 10, 13\})$,
$(\{2, 7, 9, 14, 15\}, \{1, 3, 5, 11, 13\}), (\{2, 3, 5, 11, 13\}, \{1, 6, 10, 14, 15\}), (\{1, 2, 3, 13, 14\}, \{4, 7, 9, 12, 15\})$,
$(\{1, 2, 3, 14, 15\}, \{4, 6, 10, 12, 13\}), (\{1, 2, 3, 14, 15\}, \{4, 5, 11, 12, 13\})$. □

Lemma 3.8. *If $\ell_1, \ell_2 \in \{1, 2, 3, 4, 5, 6, 7\}$ with $\ell_1 \neq \ell_2$, then there exists a $\mathbb{Z}_{15}^2$-partition of $\mathbb{Z}_{15} \setminus \{0, \ell_1, \ell_2, 15 - \ell_1, 15 - \ell_2\}$.*

Proof. For $\{\ell_1, \ell_2\} = \{1, 2\}$, a $\mathbb{Z}_{15}^2$-partition is $(\{3, 5, 6, 9, 11\}, \{4, 7, 8, 10, 12\})$. Similarly, for $\{\ell_1, \ell_2\} = \{1, 3\}$, $\{1, 4\}$, $\{1, 5\}$, $\{1, 6\}$, $\{1, 7\}$, $\{2, 3\}$, $\{2, 4\}$, $\{2, 5\}$, $\{2, 6\}$, $\{2, 7\}$, $\{3, 4\}$, $\{3, 5\}$, $\{3, 6\}$, $\{3, 7\}$, $\{4, 5\}$, $\{4, 6\}$, $\{4, 7\}$, $\{5, 6\}$, $\{5, 7\}$, $\{6, 7\}$, respective $\mathbb{Z}_{15}^2$-partitions are
$(\{4, 5, 6, 9, 13\}, \{2, 7, 8, 10, 11\}), (\{2, 5, 6, 9, 12\}, \{3, 7, 8, 10, 13\}), (\{2, 3, 6, 9, 11\}, \{4, 7, 8, 12, 13\})$,
$(\{2, 3, 5, 10, 11\}, \{4, 7, 8, 12, 13\}), (\{3, 4, 5, 11, 13\}, \{2, 6, 9, 10, 12\}), (\{1, 5, 6, 9, 11\}, \{4, 7, 8, 10, 14\})$,
$(\{1, 5, 6, 9, 12\}, \{3, 7, 8, 10, 14\}), (\{1, 4, 6, 9, 12\}, \{3, 7, 8, 11, 14\}), (\{1, 3, 4, 5, 11\}, \{7, 8, 10, 12, 14\})$,

$(\{3,4,5,10,14\},\{1,6,9,11,12\}),(\{1,5,6,9,13\},\{2,7,8,10,14\}),(\{2,4,6,9,14\},\{1,7,8,11,13\})$, $(\{1,2,5,7,8,10\},\{4,11,13,14\}),(\{1,2,4,5,10\},\{6,9,11,13,14\}),(\{1,3,6,9,14\},\{2,7,8,12,13\})$, $(\{1,3,5,10,14\},\{2,7,8,12,13\}),(\{1,2,3,5,10\},\{6,9,12,13,14\}),(\{1,2,3,13,14\},\{4,7,8,11,12\})$, $(\{1,2,4,11,12\},\{3,6,9,13,14\}),(\{1,2,3,13,14\},\{4,5,10,11,12\})$. □

Corollaries 3.9 and 3.10 follow from Theorem 3.1, Corollary 3.2 and Lemmas 3.3 - 3.8.

Corollary 3.9. *Let $n \geq 16$ be even and let H be the subgraph induced by the edges of $B_0^{(n)} \cup B_{\frac{n}{2}}^{(n)} \cup B_{\ell_1}^{(n)} \cup B_{n-\ell_1}^{(n)} \cup B_{\ell_2}^{(n)} \cup B_{n-\ell_2}^{(n)}$, $\ell_1, \ell_2 \in \{1,2,\ldots,\frac{n-2}{2}\}$ and $\ell_1 \neq \ell_2$. Then $\vec{d}(K_{n,n} - E(H)) = 3$.* □

Corollary 3.10. *Let $n \geq 15$ be odd and let H be the subgraph induced by the edges of $B_0^{(n)} \cup B_{\ell_1}^{(n)} \cup B_{n-\ell_1}^{(n)} \cup B_{\ell_2}^{(n)} \cup B_{n-\ell_2}^{(n)}$, $\ell_1, \ell_2 \in \{1,2,\ldots,\frac{n-1}{2}\}$ and $\ell_1 \neq \ell_2$. Then $\vec{d}(K_{n,n} - E(H)) = 3$.* □

Note that for even $n \geq 16$, we have deleted a 6-factor from $K_{n,n}$, in Corollary 3.9, and for odd $n \geq 15$, we have deleted a 5-factor from $K_{n,n}$, in Corollary 3.10; but the 6-factor and 5-factor are not necessarily arbitrary.

Lemma 3.11. *For $\ell_1 \in \{1,2,3,4,5,6\}$, there is a $\mathbb{Z}_{14}^2$-partition of $\mathbb{Z}_{14} \setminus \{0,\ell_1,14-\ell_1\}$. For $\ell_1 \in \{1,2,3,4,5,6\}$, there is a $\mathbb{Z}_{13}^2$-set L contained in $\mathbb{Z}_{13} \setminus \{0,\ell_1,13-\ell_1\}$. For $\ell_1 \in \{1,2,3,4,5\}$, there is a $\mathbb{Z}_{12}^2$-partition of $\mathbb{Z}_{12} \setminus \{0,\ell_1,12-\ell_1\}$. For $\ell_1 \in \{1,2,3,4,5\}$, there is a $\mathbb{Z}_{11}^2$-partition of $\mathbb{Z}_{11} \setminus \{0,\ell_1,11-\ell_1\}$.*

Proof. For $\ell_1 = 1$, a $\mathbb{Z}_{14}^2$-partition is $(\{3,4,5,6,10\},\{2,8,9,11,12\})$. Similarly, for $\ell_1 = 2,3,4,5,6$, respective $\mathbb{Z}_{14}^2$-partitions are $(\{3,4,5,6,10\},\{1,8,9,11,13\}),(\{1,4,5,6,10\},\{2,8,9,12,13\})$, $(\{1,3,5,6,11\},\{2,8,9,12,13\}),(\{1,3,4,6,8\},\{2,10,11,12,13\}),(\{1,3,4,5,9\},\{2,10,11,12,13\})$.

For $\ell_1 = 1$, a $\mathbb{Z}_{13}^2$-set is $\{3,5,6,9,11\}$. Similarly, for $\ell_1 = 2,3,4,5,6$, respective $\mathbb{Z}_{13}^2$-sets are $\{1,3,4,6,8\},\{1,2,4,5,7\},\{1,5,7,10,11\},\{3,4,6,11,12\},\{1,2,4,5,10\}$.

For $\ell_1 = 1$, a $\mathbb{Z}_{12}^2$-partition is $(\{4,5,7,8\},\{2,3,9,10\})$. Similarly, for $\ell_1 = 2,3,4,5$, respective $\mathbb{Z}_{12}^2$-partitions are $(\{1,3,4,8\},\{5,7,9,11\}),(\{1,2,4,8\},\{5,7,10,11\}),(\{1,2,3,10\},\{5,7,9,11\})$, $(\{2,3,9,10\},\{1,4,8,11\})$.

For $\ell_1 = 1$, a $\mathbb{Z}_{11}^2$-partition is $(\{2,3,6,8\},\{4,5,7,9\})$. Similarly, for $\ell_1 = 2,3,4,5$, respective $\mathbb{Z}_{11}^2$-partitions are $(\{1,4,5,10\},\{3,6,7,8\}),(\{1,2,5,6\},\{4,7,9,10\}),(\{1,3,9,10\},\{2,5,6,8\})$, $(\{1,2,3,10\},\{4,7,8,9\})$. □

Corollaries 3.12 and 3.13 follow from Theorem 3.1, Corollary 3.2 and Lemma 3.11.

Corollary 3.12. *Let $n \in \{14,12\}$ and let H be the subgraph induced by the edges of $B_0^{(n)} \cup B_{\frac{n}{2}}^{(n)} \cup B_{\ell_1}^{(n)} \cup B_{n-\ell_1}^{(n)}$, $\ell_1 \in \{1,2,\ldots,\frac{n-2}{2}\}$. Then $\vec{d}(K_{n,n} - E(H)) = 3$.* □

Corollary 3.13. *Let $n \in \{13,11\}$ and let H be the subgraph induced by the edges of $B_0^{(n)} \cup B_{\ell_1}^{(n)} \cup B_{n-\ell_1}^{(n)}$, $\ell_1 \in \{1,2,\ldots,\frac{n-1}{2}\}$. Then $\vec{d}(K_{n,n} - E(H)) = 3$.* □

Corollary 3.14. *If F is a 2-factor of $K_{n,n}$ with $n \geq 9$, then $\vec{d}(K_{n,n} - E(F)) = 3$.*

Proof. First, we claim that the 3-factor H induced by the edges of $B_0^{(n)} \cup B_1^{(n)} \cup B_{n-1}^{(n)}$ in $K_{n,n}$ contains an isomorphic subgraph of all possible 2-factors of $K_{n,n}$. This follows from the observation that: $x_iy_{i+1}x_{i+2}y_{i+3}x_{i+4}y_{i+5} \cdots y_{i+2k-1}x_{i+2k}y_{i+2k}x_{i+2k-1} \cdots y_{i+4}x_{i+3}y_{i+2}x_{i+1}y_ix_i$ is a cycle of length $4k+2$ and $x_iy_{i+1}x_{i+2}y_{i+3}x_{i+4}y_{i+5} \cdots x_{i+2k-2}y_{i+2k-1}x_{i+2k-1}\ y_{i+2k-2} \cdots y_{i+4}x_{i+3}y_{i+2}x_{i+1}y_ix_i$ is a cycle of length $4k$ contained in H. If $n \geq 11$, the proof follows from the above claim and Corollaries 3.9, 3.10, 3.12 and 3.13. For $n = 10$ and $n = 9$, apply Theorem 3.1 with the $\mathbb{Z}_{10}^2$-partition $(\{3,4,5,6\},\{2,7,8\})$ and the $\mathbb{Z}_9^2$-partition $(\{3,4,5,6\},\{2,7\})$, respectively, and then use the above claim. □

Corollary 3.15. *If F is a uniform 2-factor of $K_{8,8}$, then $\vec{d}(K_{8,8} - E(F)) = 3$.*

Proof. Since F is a uniform 2-factor, $F \in \{C_{16}, 2C_8, 4C_4\}$. If $F = C_{16}$, then let $G = K_{8,8} - (B_0^{(8)} \cup B_1^{(8)})$; apply Theorem 3.1 with the $\mathbb{Z}_8^2$-partition $(\{3,4,5\},\{2,6,7\})$. If $F = 2C_8$, then let $G = K_{8,8} - (B_0^{(8)} \cup B_2^{(8)})$; apply Theorem 3.1 with the $\mathbb{Z}_8^2$-partition $(\{3,4,6\},\{1,5,7\})$. If $F = 4C_4$, then let $G = K_{8,8} - (B_0^{(8)} \cup B_4^{(8)})$; apply Theorem 3.1 with the $\mathbb{Z}_8^2$-partition $(\{1,3,5\},\{2,6,7\})$. The proof follows from the observation that if F_1 and F_2 are any two uniform 2-factors, whose cycle length is k for some integer k, of $K_{8,8}$, then $K_{8,8} - E(F_1)$ and $K_{8,8} - E(F_2)$ are isomorphic. □

Corollary 3.16. *If F is a uniform 2-factor of $K_{7,7}$, then $\vec{d}(K_{7,7} - E(F)) = 3$.*

Proof. As F is a uniform 2-factor, $F = C_{14}$. Let $G = K_{7,7} - (B_0^{(7)} \cup B_1^{(7)})$. Apply Theorem 3.1 with the $\mathbb{Z}_7^2$-partition $\{\{2,6\},\{3,4,5\}\}$. The proof follows from the fact that if H_1 and H_2 are any two hamilton cycles of $K_{7,7}$, then $K_{7,7} - E(H_1)$ and $K_{7,7} - E(H_2)$ are isomorphic. □

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