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KERNELS IN CIRCULANT DIGRAPHS

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ABSTRACT. A kernel J of a digraph D is an independent set of vertices of D such that for every vertex $w \in V(D) \setminus J$ there exists an arc from w to a vertex in J . In this paper, among other results, a characterization of 2-regular circulant digraph having a kernel is obtained. This characterization is a partial solution to the following problem: Characterize circulant digraphs which have kernels; it appeared in the book *Digraphs - theory, algorithms and applications*, Second Edition, Springer-Verlag, 2009, by J. Bang-Jensen and G. Gutin.

1. Introduction

For notation and terminology, in general, we follow [1].

A set J of vertices in a digraph D is a *kernel* if J is independent (i.e., the subdigraph induced by J has no arcs) and absorbent (i.e., the first closed in-neighborhood of J , $N^-[J]$, is equal to $V(D)$). A digraph D is called *kernel-less* if it has no kernel.

The notion of kernel was introduced by von Neumann in [4]. Kernels have found many applications, for instance in game theory, logic, computational complexity, coding theory and in list edge-colouring of graphs, see p. 119 of [1]. Finding kernels in special classes of digraphs seems to be difficult. Chvátal (see [3], p. 204) proved that the problem of deciding whether a given digraph has a kernel is NP-complete.

A digraph $D = (V, A)$ is *semicomplete* if for any pair of distinct vertices $u, v \in V$, either $(u, v) \in A$ or $(v, u) \in A$. A digraph D is *locally semicomplete* if the in-neighbors (out-neighbors) of every vertex induce a semicomplete digraph. In [2], Gutin et al. proved that the kernel problem is polynomial time solvable for locally semicomplete digraphs.

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For an integer $n \geq 2$ and a set $S \subseteq \{1, 2, \dots, n-1\}$, the *circulant digraph* $C_n(S)$ is a digraph with vertex set $\mathbb{Z}_n = \{0, 1, \dots, n-1\}$ and arc set $\{(i, (i+j)(\text{mod } n)) : i \in \mathbb{Z}_n \text{ and } j \in S\}$.

Following problem is posed in [1] (see Problem 3.8.5, p. 121).

Problem 1.1. *Characterize circulant digraphs which have kernels.*

In this paper, we find some classes of circulant digraphs which have kernels and some classes of circulant digraphs which do not have kernels. In particular, we have answered Problem 1.1 for 2-regular circulant digraphs.

2. Results

Proposition 2.1. *Let $S \subseteq \{1, 2, \dots, n-1\}$ be such that $n-j \in S$ whenever $j \in S$. Then the circulant digraph $C_n(S)$ has a kernel.*

Proof. By hypothesis, $C_n(S)$ is symmetric (i.e., (y, x) is an arc in $C_n(S)$ whenever (x, y) is an arc in $C_n(S)$). Hence $C_n(S)$ has a kernel, since every maximal independent set of a symmetric digraph is a kernel. \square

The following are two examples for the previous proposition.

- (i) n is even and S is the set of all odd numbers in $\{1, 2, \dots, n-1\}$;
- (ii) n is even and S is the set of all even numbers in $\{1, 2, \dots, n-1\}$.

Hence, unless otherwise mentioned, assume that “there exists a $j \in S$ such that $n-j \notin S$ ”

Proposition 2.2. *If n is even and if S is any nonempty set of odd numbers in $\{1, 2, \dots, n-1\}$, then $C_n(S)$ has a kernel.*

Proof. Let $J = \{0, 2, 4, \dots, n-2\}$. As n is even, J is independent. Since S is nonempty, there exists an odd number $s \in S$. For any $i \in \mathbb{Z}_n \setminus J$, $(i+s)(\text{mod } n) \in J$ and $(i, (i+s)(\text{mod } n))$ is an arc in $C_n(S)$. Then, J is a kernel of $C_n(S)$. \square

Note that, for $C_n(S)$ in Proposition 2.2, $J' = \{1, 3, 5, \dots, n-1\}$ is also a kernel.

Proposition 2.3. *If n is odd and if S_1 and S_2 are, respectively, the set of all odd numbers and the set of all even numbers in $\{1, 2, \dots, n-1\}$, then both $C_n(S_1)$ and $C_n(S_2)$ are kernel-less.*

Proof. By hypothesis, both $C_n(S_1)$ and $C_n(S_2)$ are regular tournaments and hence they are kernel-less. \square

Theorem 2.4. *Let $n \geq 4$ be even and $S \subseteq \{1, 2, \dots, n-1\}$ be such that $|S| = \frac{n}{2}$, $\frac{n}{2} \in S$ and for every $j \in S \setminus \{\frac{n}{2}\}$, $n-j \notin S$. Then $C_n(S)$ is kernel-less.*

Proof. Suppose $C_n(S)$ has a kernel, say, J . By hypothesis, $C_n(S)$ is a semicomplete digraph and hence its independence number is 1. Thus $J = \{i\}$ for some $i \in \mathbb{Z}_n$. As $|N^-[\{i\}]| = \frac{n}{2}$, J is not a kernel of $C_n(S)$, a contradiction. \square

Lemma 2.5. *Let $n \equiv 0 \pmod{k}$, $k \geq 3$, and let $\{i_1, i_2, \dots, i_{k-1}\} \subseteq \{1, 2, \dots, n-1\}$.*

If $\{i_1 \pmod{k}, i_2 \pmod{k}, \dots, i_{k-1} \pmod{k}\} = \{1, 2, \dots, k-1\}$, then $C_n(\{i_1, i_2, \dots, i_{k-1}\})$ has a kernel.

Proof. For convenience, assume that $i_j \pmod{k} = j$; and hence $i_j = kx_j + j$, for some x_j . Set $J = \{0, k, 2k, 3k, \dots, n-k\}$. First, we observe that J is an independent set of $C_n(\{i_1, i_2, \dots, i_{k-1}\})$. Otherwise, there exist $ak, bk \in J$ with an arc $ak \rightarrow bk$. Then $bk = ak + i_j = (a + x_j)k + j$ for some $j \in \{1, 2, \dots, k-1\}$, a contradiction. Next, we observe that J is an absorbent set of $C_n(\{i_1, i_2, \dots, i_{k-1}\})$. For, if $z \notin J$, then $z = \ell k + j$ for some $\ell \in \{0, 1, 2, \dots, \frac{n-k}{k}\}$ and $j \in \{1, 2, \dots, k-1\}$. Now, in $C_n(\{i_1, i_2, \dots, i_{k-1}\})$, $z = \ell k + j \rightarrow \ell k + j + i_{k-j} = (\ell + x_{k-j} + 1)k$. Hence, J is a kernel of $C_n(\{i_1, i_2, \dots, i_{k-1}\})$. □

Theorem 2.6. *Let $S \subseteq \{1, 2, \dots, n-1\}$ be such that $|S| = 1$; and let $S = \{i\}$. (i) If $n \geq 3$ is odd, then $C_n(S)$ is kernel-less. (ii) If $n \geq 4$ is even and if i is odd, then $C_n(S)$ has a kernel. (iii) If both $n \geq 4$ and i are even, then $C_n(S)$ has a kernel if and only if there exists a positive integer k such that $2^{k+1}|n$, $2^k|i$, and $2^{k+1} \nmid i$.*

Proof. By hypothesis, $C_n(S) = \text{gcd}(n, i) \vec{C}_{\frac{n}{\text{gcd}(n, i)}}$, disjoint union of $\text{gcd}(n, i)$ copies of directed cycles of length $\frac{n}{\text{gcd}(n, i)}$. Observe that any directed odd cycle is kernel-less and any maximum independent set of a directed even cycle is a kernel of it. Consequently, if $n \geq 3$ is odd, then $\frac{n}{\text{gcd}(n, i)}$ is odd and hence $C_n(S)$ is kernel-less. Also, if $n \geq 4$ is even and if i is odd, then $\frac{n}{\text{gcd}(n, i)}$ is even and therefore $C_n(S)$ has a kernel. So assume that both $n \geq 4$ and i are even; now $C_n(S)$ has a kernel if and only if $\frac{n}{\text{gcd}(n, i)}$ is even. This completes the proof since in this case $\frac{n}{\text{gcd}(n, i)}$ is even if and only if there exists a positive integer k such that $2^{k+1}|n$, $2^k|i$, and $2^{k+1} \nmid i$. □

A digraph $D = (V, A)$ is an *oriented graph* if $(u, v) \in A$ then $(v, u) \notin A$. By Proposition 2.1, we have:

Lemma 2.7. *If $i + j = n$, then the symmetric digraph $C_n(\{i, j\})$ has a kernel.*

Theorem 2.8. *Let $i + j \neq n$ and let $m = \text{gcd}(i + j, n)$. The oriented graph $C_n(\{i, j\})$ has a kernel if and only if $i \not\equiv 0 \pmod{m}$, $j \not\equiv 0 \pmod{m}$ and $m \neq 1$.*

Proof. First, assume that $C_n(\{i, j\})$ has a kernel, say, J . Without loss of generality assume that $0 \in J$. Since $0 \in J$, we have $i, j, n-i, n-j \notin J$. $n-i, n-j \notin J$ implies that $n-(i+j) \in J$. (Otherwise $n-(i+j) \notin J$. As $n-(i+j)$ dominates only the two vertices $n-i$ and $n-j$, either $n-i \in J$ or $n-j \in J$, a contradiction.) Thus $0 \in J$ implies $n-(i+j) \in J$. Similarly, $n-(i+j) \in J$ implies $n-2(i+j) \in J$. Continuing in this way, we get $0, n-(i+j), n-2(i+j), n-3(i+j), \dots \in J$. Note that $\{0, n-(i+j), n-2(i+j), n-3(i+j), \dots\} = \mathbb{Z}_n$ when $m = 1$, and $\{0, n-(i+j), n-2(i+j), n-3(i+j), \dots\} = \{0, m, 2m, \dots, n-m\}$ is a proper subset of \mathbb{Z}_n when $m \neq 1$. As $J = \mathbb{Z}_n$ is impossible, $m \neq 1$. As $\{0, m, 2m, \dots, n-m\} \subseteq J$, $\{0, m, 2m, \dots, n-m\}$ is independent, and hence $i \not\equiv 0 \pmod{m}$ and $j \not\equiv 0 \pmod{m}$.

Next, assume that $i \not\equiv 0 \pmod{m}$ and $j \not\equiv 0 \pmod{m}$ and $m \neq 1$. Let $P = \{0, m, 2m, \dots, (\frac{n}{m} - 1)m\}$ and consider the circulant digraph $C_m(\{i \pmod{m}, j \pmod{m}\})$. Since $i + j \equiv 0 \pmod{m}$,

by Lemma 2.7, $C_m(\{i \pmod m, j \pmod m\})$ has a kernel, say, A . Set $J = \bigcup_{\ell \in A} (P + \ell)$, where $P + \ell = \{\ell, m + \ell, 2m + \ell, \dots, (\frac{n}{m} - 1)m + \ell\}$.

Claim 1. J is independent in $C_n(\{i, j\})$.

As $i \not\equiv 0 \pmod m$ and $j \not\equiv 0 \pmod m$, we have: P is independent. So, for any $\ell \in A$, $P + \ell$ is independent. Suppose J is not independent, then there exist $am + \ell, bm + \ell' \in J$ with an arc $am + \ell \rightarrow bm + \ell'$, where $a, b \in \{0, 1, 2, \dots, \frac{n}{m} - 1\}$ and $\ell, \ell' \in A$. As $P + \ell$ is independent, $\ell \neq \ell'$. The existence of the arc $am + \ell \rightarrow bm + \ell'$ in $C_n(\{i, j\})$ implies the existence of the arc $\ell \rightarrow \ell'$ in $C_m(\{i \pmod m, j \pmod m\})$, a contradiction to the fact that ℓ and ℓ' are vertices of the kernel A of $C_m(\{i \pmod m, j \pmod m\})$.

Claim 2. J is absorbent in $C_n(\{i, j\})$.

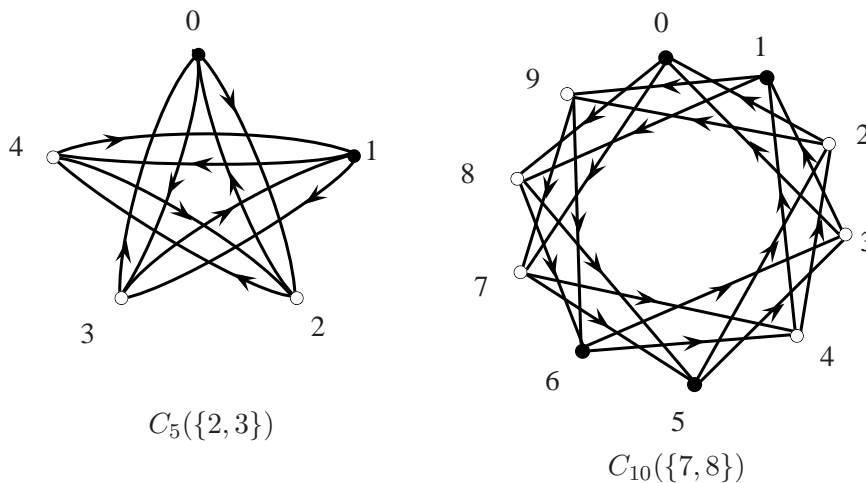
For, if $z \notin J$, then $z = cm + \ell_0$ for some $c \in \{0, 1, 2, \dots, \frac{n}{m} - 1\}$, $\ell_0 \in \{0, 1, 2, \dots, m - 1\}$ and $\ell_0 \notin A$. As $\ell_0 \notin A$ and A is a kernel of $C_m(\{i \pmod m, j \pmod m\})$, there exists $\ell'_0 \in A$ with an arc $\ell_0 \rightarrow \ell'_0$ in $C_m(\{i \pmod m, j \pmod m\})$. Now, in $C_n(\{i, j\})$, we have an arc with tail $cm + \ell_0$ and head in $P + \ell'_0$.

By Claims 1 and 2, J is a kernel of $C_n(\{i, j\})$.

This completes the proof. □

Illustration of Theorem 2.8

Consider $C_{10}(\{7, 8\})$. Then $n = 10, i = 7, j = 8, i + j \neq n$, and $m = 5$. Also, $i \not\equiv 0 \pmod m, j \not\equiv 0 \pmod m$ and $m \neq 1$. Furthermore, $P = \{0, 5\}$. $A = \{0, 1\}$ is a kernel of $C_5(\{2, 3\})$. Hence, $\{0, 5\} \cup \{1, 6\}$ is a kernel of $C_{10}(\{7, 8\})$.



In each of the figures above, the set of bold vertices forms a kernel of it.

Note:

In “H. Jacob, *Etude théorique du noyau d’un graphe*, Thèse de Doctorat de 3^{ème} année, Paris VI, France, 1979”, Jacob has mentioned the following result of Gervacio: $C_n(\{1, k\})$ admits a kernel if and only if $\gcd(n, k + 1) \neq 1$. This result is a corollary to Theorem 2.8. Also, in the above thesis Jacob

proved that $C_n(\{1, 2, \dots, k-1\})$ admits a kernel if and only if k divides n . If part of this result is a consequence of Lemma 2.5.

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REFERENCES

- [1] J. Bang-Jensen and G. Gutin, *Digraphs - theory, algorithms and applications*, Second Edition, Springer-Verlag, 2009.
- [2] J. Bang-Jensen, Y. Guo, G. Gutin and L. Volkmann, A classification of locally semi-complete digraphs, *Discrete Math.*, **167/168** (1997) 101–114.
- [3] M. R. Garey and D. S. Johnson, *Computers and intractability*, A Series of Books in the Mathematical Sciences. W. H. Freeman and Co., San Francisco, Calif., 1979.
- [4] J. von Neumann and O. Morgenstern, *Theory of Games and Economic Behavior*, Princeton University Press, Princeton, NJ, 1944.

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