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PRODUCT-CORDIAL INDEX AND FRIENDLY INDEX OF REGULAR GRAPHS

W. C. SHIU* AND H. KWONG

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Abstract. Let $G = (V, E)$ be a connected simple graph. A labeling $f : V \to \mathbb{Z}_2$ induces two edge labelings $f^+, f^* : E \to \mathbb{Z}_2$ defined by $f^+(xy) = f(x)+f(y)$ and $f^*(xy) = f(x)f(y)$ for each $xy \in E$. For $i \in \mathbb{Z}_2$, let $v_f(i) = |f^{-1}(i)|$, $e_{f^+}(i) = |(f^+)^{-1}(i)|$ and $e_{f^*}(i) = |(f^*)^{-1}(i)|$. A labeling f is called friendly if $|v_f(1) - v_f(0)| \le 1$. For a friendly labeling f of a graph G, the friendly index of G under f is defined by $i^+_f(G) = e_{f^+}(1) - e_{f^+}(0)$. The set $\{i^+_f(G) \mid f \text{ is a friendly labeling of} G\}$ is called the full friendly index set of G. Also, the product-cordial index of G under f is defined by $i^*_f(G) = e_{f^*}(1) - e_{f^*}(0)$. The set $\{i^*_f(G) \mid f \text{ is a friendly labeling of} G\}$ is called the full product-cordial index set of G. In this paper, we find a relation between the friendly index and the product-cordial index of a regular graph. As applications, we will determine the full product-cordial index sets of torus graphs which was asked by Kwong, Lee and Ng in 2010; and those of cycles.

1. Introduction

In this paper, all graphs are simple and connected. All undefined symbols and concepts may be looked up from [1]. Let $G = (V, E)$ be a connected simple graph. A labeling $f : V \to \mathbb{Z}_2$ induces two edge labelings $f^+, f^* : E \to \mathbb{Z}_2$ defined by $f^+(xy) = f(x) + f(y)$ and $f^*(xy) = f(x)f(y)$ for each $xy \in E$. For $i \in \mathbb{Z}_2$, let $v_f(i) = |f^{-1}(i)|$, $e_{f^+}(i) = |(f^+)^{-1}(i)|$ and $e_{f^*}(i) = |(f^*)^{-1}(i)|$. A labeling f is called *friendly* if $|v_f(1) - v_f(0)| \le 1$. For a friendly labeling f of a graph G, the *friendly index* of G under f is defined by $i^+_f(G) = e_{f^+}(1) - e_{f^+}(0)$. The set

$$\mathrm{FFI}(G) = \{i^+_f(G) \mid \text{is a friendly labeling of } G\}$$

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*Corresponding author.

is called the *full friendly index set* of G. Also the *product-cordial index* of G under f is defined by $i^*_f(G) = e_{f^*}(1) - e_{f^*}(0)$. The set

$$\mathrm{FPCI}(G) = \{i^*_f(G) \mid f \text{ is a friendly labeling of } G\}$$

is called the *full product-cordial index set* of G. Throughout this paper, we will use the term 'labeling' to mean a vertex labeling whose values are taken from $\mathbb{Z}_2$. Note that $i^+_f(G)$ and $i^*_f(G)$ can be extended to any labeling.

Friendly index set was initiated by Lee and Ng in 2004 [6]. More about friendly index sets of graphs can be found in [3, 4, 9]. Full friendly index set was first introduced by Shiu and Kwong [10] in 2007 (published in 2008). The friendly index sets or full friendly index sets of the graphs $P_m \times P_n$, $C_m \times C_n$ and $C_m \times P_n$ were found [8, 10, 11, 12, 13, 15]. Recently Gao determined the full friendly index set of $P_m \times P_n$, but he used the terms 'edge difference set' instead of 'full friendly index set' and 'direct product' instead of 'Cartesian product' in [2]. Friendly index is related to the Laplacian eigenvalues of a graph (interested readers please see [14]).

Product-cordial set $\mathrm{PC}(G)$ was introduced by Salehi in 2009 [7]. Since this is the multiplicative version of $\mathrm{FI}(G)$, it is also called a *product-cordial index set* and denoted by $\mathrm{PCI}(G)$ in [5].

2. Relationship between friendly index and product-cordial index

For a fixed labeling f, a vertex v is called a *k-vertex* if $f(v) = k$, and an edge is called an *(i,j)-edge* if it is incident with an i-vertex and a j-vertex. We define the number of (i,j)-edges by $E_f(i,j)$. Then

$$\begin{aligned}
e_{f^+}(1) &= E_f(1,0) = E_f(0,1),\\
e_{f^+}(0) &= E_f(1,1) + E_f(0,0);\\
e_{f^*}(1) &= E_f(1,1),\\
e_{f^*}(0) &= E_f(0,0) + E_f(1,0).
\end{aligned}$$

Since $e_{f^+}(1) + e_{f^+}(0) = e_{f^*}(1) + e_{f^*}(0) = q$, the size of the graph G, we obtain

$$i^+_f(G) = 2e_{f^+}(1) - q = 2E_f(1,0) - q = q - 2e_{f^+}(0); \tag{2.1}$$

$$i^*_f(G) = 2e_{f^*}(1) - q = 2E_f(1,1) - q = q - 2e_{f^*}(0). \tag{2.2}$$

Lemma 2.1 ([12])**.** *Let f be any labeling of a graph G with q edges. If the degree sum of* 1*-vertices is s, then $i^+_f(G) = 2s - 4E_f(1,1) - q$.*

Combining Equation (2.2) and Lemma 2.1 we have

Corollary 2.2. *Let f be any labeling of a graph G with q edges. If the degree sum of* 1*-vertices is s, then $2i^*_f(G) = 2s - 3q - i^+_f(G)$.*

Corollary 2.3. *Let f be a friendly labeling of G. If G is an r-regular graph of even order. Then $i^*_f(G) = -\frac{1}{2}(q + i^+_f(G))$.*

Proof. Let p be the order of G. Then $rp = 2q$, and $s = (\frac{p}{2})r$. The result follows immediately from Corollary 2.2. $\square$

Corollary 2.4. *Suppose G is an r-regular graph of odd order. Let f be a friendly labeling of G with $v_f(1) = v_f(0) + 1$. Then $i^*_f(G) = -\frac{1}{2}(q - r + i^+_f(G))$.*

Proof. Let p be the order of G. Then $rp = 2q$, and $s = (\frac{p+1}{2})r$. To complete the proof, apply Corollary 2.2. $\square$

Similarly we have

Corollary 2.5. *Suppose G is an r-regular graph of odd order. Let f be a friendly labeling of G with $v_f(1) = v_f(0) - 1$. Then $i^*_f(G) = -\frac{1}{2}(q + r + i^+_f(G))$.*

3. **Application to Torus**

A problem proposed in [5] asked readers to determine the exact value of $\mathrm{PCI}(C_m \times C_n)$. We could apply the results in Section 2 to solve this problem. From [12] we have the following results:

$$\begin{aligned}
\mathrm{FFI}(C_{2h+1} \times C_{2k+1}) &= \{8hk + 4h + 4k + 6 - 4\ell \mid h + k + 2 \le \ell \le 4hk + 2h\}, \text{ for } 1 \le k \le h; \\
\mathrm{FFI}(C_{2h+1} \times C_{2k}) &= \{8hk + 4k - 4\ell \mid k \le \ell \le 4hk - 1\}, \text{ for } 2 \le k \le h; \\
\mathrm{FFI}(C_{2h} \times C_{2k+1}) &= \{8hk + 4h - 4\ell \mid h \le \ell \le 4hk + 2h - 2k - 1, \ell \ne 4hk + 2h - 2k - 2\}, \\
&\qquad \text{for } 1 \le k < h; \\
\mathrm{FFI}(C_{2h} \times C_{2k}) &= \{8hk - 4\ell \mid 0 \le \ell \le 4hk - 2k, \ell \ne 1, 2, 4hk - 2k - 1\}, \text{ for } 2 \le k \le h.
\end{aligned}$$

Note that, in the proofs of those results, ℓ is equal to $E_f(1,1)$, providing that $v_f(1) \ge v_f(0)$ (see [12]).

Since the torus $C_m \times C_n$ is a 4-regular graph, $i^*_f(C_m \times C_n) = -\frac{1}{2}(2mn + i^+_f(C_m \times C_n))$ when mn is even. Thus

$$\begin{aligned}
\mathrm{FPCI}(C_{2h+1} \times C_{2k}) &= \{-8hk - 4k + 2\ell \mid k \le \ell \le 4hk - 1\}, \text{ for } 2 \le k \le h; \\
\mathrm{FPCI}(C_{2h} \times C_{2k+1}) &= \{-8hk - 4h + 2\ell \mid h \le \ell \le 4hk + 2h - 2k - 1, \ell \ne 4hk + 2h - 2k - 2\}, \\
&\qquad \text{for } 1 \le k < h; \\
\mathrm{FPCI}(C_{2h} \times C_{2k}) &= \{-8hk + 2\ell \mid 0 \le \ell \le 4hk - 2k, \ell \ne 1, 2, 4hk - 2k - 1\}, \text{ for } 2 \le k \le h.
\end{aligned}$$

It is clear that

$$\mathrm{FPCI}(G) = \{i^*_f(G) \mid v_f(1) = v_f(0) + 1\} \cup \{i^*_f(G) \mid v_f(1) = v_f(0) - 1\}$$

for G of odd order.

Now we consider the graph $C_{2h+1} \times C_{2k+1}$ for $1 \le k \le h$. By Corollary 2.4 and the above result we have

$$\{i^*_f(C_{2h+1} \times C_{2k+1}) \mid v_f(1) = v_f(0) + 1\} = \{-8hk - 4h - 4k - 2 + 2\ell \mid h + k + 2 \le \ell \le 4hk + 2h\}.$$

Suppose f is a friendly labeling with $v_f(1) = v_f(0) - 1$. Let $\overline{f} = 1 - f$. Then $v_{\overline{f}}(1) = v_{\overline{f}}(0) + 1$, and $i^+_f(G) = i^+_{\overline{f}}(G)$. Hence $i^*_f(C_{2h+1} \times C_{2k+1}) = -(8hk + 4h + 4k + 6 - 2\ell)$ if $v_f(1) = v_f(0) - 1$, where

$\ell = E_{\overline{f}}(1,1)$. Since $f \leftrightarrow \overline{f}$ is an one-to-one correspondence,

$$\{i_f^*(C_{2h+1} \times C_{2k+1}) \mid v_f(1) = v_f(0) - 1\} = \{-8hk - 4h - 4k - 6 + 2\ell \mid h + k + 2 \le \ell \le 4hk + 2h\}.$$

Thus

$$\mathrm{FPCI}(C_{2h+1} \times C_{2k+1}) = \{-8hk - 4h - 4k - 2 + 2\ell \mid h + k \le \ell \le 4hk + 2h\}.$$

Note that all product-cordial indices of torus are negative. Hence we have

Theorem 3.1. *The product-cordial index sets of torus are:*

$$\begin{aligned}
\mathrm{PCI}(C_{2h+1} \times C_{2k}) &= \{8hk + 4k - 2\ell \mid k \le \ell \le 4hk - 1\}, \ \textit{for } 2 \le k \le h;\\
\mathrm{PCI}(C_{2h} \times C_{2k+1}) &= \{8hk + 4h - 2\ell \mid h \le \ell \le 4hk + 2h - 2k - 1, \ell \ne 4hk + 2h - 2k - 2\},\\
&\qquad\qquad \textit{for } 1 \le k < h;\\
\mathrm{PCI}(C_{2h} \times C_{2k}) &= \{8hk - 2\ell \mid 0 \le \ell \le 4hk - 2k, \ell \ne 1, 2, 4hk - 2k - 1\}, \ \textit{for } 2 \le k \le h;\\
\mathrm{PCI}(C_{2h+1} \times C_{2k+1}) &= \{8hk + 4h + 4k + 2 - 2\ell \mid h + k \le \ell \le 4hk + 2h\}.
\end{aligned}$$

4. Application to Cycles

In [10, Corollary 6], the authors showed that $\mathrm{FFI}(C_n) \subseteq \{4j - n \mid 1 \le j \le \frac{n}{2}\}$ for $n \ge 3$. Note that, in this description, $2j = e_{f^+}(1)$ for some friendly labeling f. We now prove that equality holds.

Theorem 4.1. *For $n \ge 3$, $\mathrm{FFI}(C_n) = \{4j - n \mid 1 \le j \le \lfloor \frac{n}{2} \rfloor\}$. Moreover, the friendly labelings f used to obtain these friendly indices have the additional property that $v_f(1) \ge v_f(0)$.*

Proof. Induct on n. It is easy to show that $\mathrm{FFI}(C_3) = \{1\}$ and $\mathrm{FFI}(C_4) = \{0, 4\}$, and that the friendly labelings satisfy the additional requirement. Now we assume that the theorem holds for all k with $3 \le k \le n$, where $n \ge 4$.

Let f be a friendly labeling of C_{n-1} such that $e_{f^+}(1) = 2j$ for $1 \le j \le \lfloor \frac{n-1}{2} \rfloor$. Since $j \ge 1$, there is a $(0,1)$-edge $xy \in E(C_{n-1})$. By inserting two new vertices u and v on the edge xy, we subdivide it into a path $xuvy$ so as to generate the cycle C_{n+1}. Define two labelings g and h on C_{n+1} according to

$$g(z) = \begin{cases} f(z) & \text{if } z \notin \{u, v\};\\ f(x) & \text{if } z = u;\\ f(y) & \text{if } z = v, \end{cases} \qquad h(z) = \begin{cases} f(z) & \text{if } z \notin \{u, v\};\\ f(y) & \text{if } z = u;\\ f(x) & \text{if } z = v. \end{cases}$$

Then $v_g(1) = v_h(1) = v_f(1) + 1$, $v_g(0) = v_h(0) = v_f(0) + 1$, $e_{g^+}(1) = 2j$, and $e_{h^+}(1) = 2j + 2$, thereby completing the indcution. $\square$

Appying the results from Section 2, we obtain

Theorem 4.2. *For $n \ge 2$, $\mathrm{FPCI}(C_{2n}) = \{-2j \mid 1 \le j \le n\}$.*

Proof. Suppose f is a friendly labeling with $i_f^+(C_{2n}) = 4j - 2n$ for $1 \le j \le n$. Then, because of Corollary 2.3, we have $i_f^*(C_{2n}) = -\frac{1}{2}(2n + 4j - 2n) = -2j$, which is what we want to prove. $\square$

Theorem 4.3. *For $n \geq 1$,* $\mathrm{FPCI}(C_{2n+1}) = \{-2j-1 \mid 0 \leq j \leq n\}$*.*

Proof. Suppose f is a friendly labeling with $v_f(1) = v_f(0)+1$ and $i_f^+(C_{2n}) = 4j-2n$ for $1 \leq j \leq n$. Then by Corollary 2.4 we have $i_f^*(C_{2n+1}) = -\frac{1}{2}(2n+1-2+4j-2n-1) = -2j+1$. Hence we have $\{i_f^*(C_{2n+1}) \mid v_f(1) = v_f(0)+1\} = \{-2j+1 \mid 1 \leq j \leq n\}$.

Along the same line of discussion in Section 3, we also find

$$\{i_f^*(C_{2n+1}) \mid v_f(1) = v_f(0)-1\} = \{-2j-1 \mid 1 \leq j \leq n\}.$$

Hence $\mathrm{FPCI}(C_{2n+1}) = \{-2j-1 \mid 0 \leq j \leq n\}$. $\square$

Corollary 4.4. *For $n \geq 1$,* $\mathrm{PCI}(C_{2n+1}) = \{2j+1 \mid 0 \leq j \leq n\}$ *and for $n \geq 2$,* $\mathrm{PCI}(C_{2n}) = \{2j \mid 1 \leq j \leq n\}$*.*

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Wai Chee Shiu

Department of Mathematics, Hong Kong Baptist University, 224 Waterloo Road, Kowloon Tong, Hong Kong, China

`Email: wcshiu@hkbu.edu.hk`

Harris Kwong

Department of Mathematical Sciences, State University of New York at Fredonia, Fredonia, NY 14063, U.S.A.

Email: kwong@fredonia.edu