



PRODUCT-CORDIAL INDEX AND FRIENDLY INDEX OF REGULAR GRAPHS

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ABSTRACT. Let $G = (V, E)$ be a connected simple graph. A labeling $f : V \rightarrow \mathbb{Z}_2$ induces two edge labelings $f^+, f^* : E \rightarrow \mathbb{Z}_2$ defined by $f^+(xy) = f(x) + f(y)$ and $f^*(xy) = f(x)f(y)$ for each $xy \in E$. For $i \in \mathbb{Z}_2$, let $v_f(i) = |f^{-1}(i)|$, $e_{f^+}(i) = |(f^+)^{-1}(i)|$ and $e_{f^*}(i) = |(f^*)^{-1}(i)|$. A labeling f is called friendly if $|v_f(1) - v_f(0)| \leq 1$. For a friendly labeling f of a graph G , the friendly index of G under f is defined by $i_f^+(G) = e_{f^+}(1) - e_{f^+}(0)$. The set $\{i_f^+(G) \mid f \text{ is a friendly labeling of } G\}$ is called the full friendly index set of G . Also, the product-cordial index of G under f is defined by $i_f^*(G) = e_{f^*}(1) - e_{f^*}(0)$. The set $\{i_f^*(G) \mid f \text{ is a friendly labeling of } G\}$ is called the full product-cordial index set of G . In this paper, we find a relation between the friendly index and the product-cordial index of a regular graph. As applications, we will determine the full product-cordial index sets of torus graphs which was asked by Kwong, Lee and Ng in 2010; and those of cycles.

1. Introduction

In this paper, all graphs are simple and connected. All undefined symbols and concepts may be looked up from [1]. Let $G = (V, E)$ be a connected simple graph. A labeling $f : V \rightarrow \mathbb{Z}_2$ induces two edge labelings $f^+, f^* : E \rightarrow \mathbb{Z}_2$ defined by $f^+(xy) = f(x) + f(y)$ and $f^*(xy) = f(x)f(y)$ for each $xy \in E$. For $i \in \mathbb{Z}_2$, let $v_f(i) = |f^{-1}(i)|$, $e_{f^+}(i) = |(f^+)^{-1}(i)|$ and $e_{f^*}(i) = |(f^*)^{-1}(i)|$. A labeling f is called *friendly* if $|v_f(1) - v_f(0)| \leq 1$. For a friendly labeling f of a graph G , the *friendly index* of G under f is defined by $i_f^+(G) = e_{f^+}(1) - e_{f^+}(0)$. The set

$$\text{FFI}(G) = \{i_f^+(G) \mid f \text{ is a friendly labeling of } G\}$$

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is called the *full friendly index set* of G . Also the *product-cordial index* of G under f is defined by $i_f^*(G) = e_{f^*}(1) - e_{f^*}(0)$. The set

$$\text{FPCI}(G) = \{i_f^*(G) \mid f \text{ is a friendly labeling of } G\}$$

is called the *full product-cordial index set* of G . Throughout this paper, we will use the term ‘labeling’ to mean a vertex labeling whose values are taken from \mathbb{Z}_2 . Note that $i_f^+(G)$ and $i_f^*(G)$ can be extended to any labeling.

Friendly index set was initiated by Lee and Ng in 2004 [6]. More about friendly index sets of graphs can be found in [3, 4, 9]. Full friendly index set was first introduced by Shiu and Kwong [10] in 2007 (published in 2008). The friendly index sets or full friendly index sets of the graphs $P_m \times P_n$, $C_m \times C_n$ and $C_m \times P_n$ were found [8, 10, 11, 12, 13, 15]. Recently Gao determined the full friendly index set of $P_m \times P_n$, but he used the terms ‘edge difference set’ instead of ‘full friendly index set’ and ‘direct product’ instead of ‘Cartesian product’ in [2]. Friendly index is related to the Laplacian eigenvalues of a graph (interested readers please see [14]).

Product-cordial set $\text{PC}(G)$ was introduced by Salehi in 2009 [7]. Since this is the multiplicative version of $\text{FI}(G)$, it is also called a *product-cordial index set* and denoted by $\text{PCI}(G)$ in [5].

2. Relationship between friendly index and product-cordial index

For a fixed labeling f , a vertex v is called a k -vertex if $f(v) = k$, and an edge is called an (i, j) -edge if it is incident with an i -vertex and a j -vertex. We define the number of (i, j) -edges by $E_f(i, j)$. Then

$$\begin{aligned} e_{f^+}(1) &= E_f(1, 0) = E_f(0, 1), \\ e_{f^+}(0) &= E_f(1, 1) + E_f(0, 0); \\ e_{f^*}(1) &= E_f(1, 1), \\ e_{f^*}(0) &= E_f(0, 0) + E_f(1, 0). \end{aligned}$$

Since $e_{f^+}(1) + e_{f^+}(0) = e_{f^*}(1) + e_{f^*}(0) = q$, the size of the graph G , we obtain

$$(2.1) \quad i_f^+(G) = 2e_{f^+}(1) - q = 2E_f(1, 0) - q = q - 2e_{f^+}(0);$$

$$(2.2) \quad i_f^*(G) = 2e_{f^*}(1) - q = 2E_f(1, 1) - q = q - 2e_{f^*}(0).$$

Lemma 2.1 ([12]). *Let f be any labeling of a graph G with q edges. If the degree sum of 1-vertices is s , then $i_f^+(G) = 2s - 4E_f(1, 1) - q$.*

Combining Equation (2.2) and Lemma 2.1 we have

Corollary 2.2. *Let f be any labeling of a graph G with q edges. If the degree sum of 1-vertices is s , then $2i_f^*(G) = 2s - 3q - i_f^+(G)$.*

Corollary 2.3. *Let f be a friendly labeling of G . If G is an r -regular graph of even order. Then $i_f^*(G) = -\frac{1}{2}(q + i_f^+(G))$.*

Proof. Let p be the order of G . Then $rp = 2q$, and $s = (\frac{p}{2})r$. The result follows immediately from Corollary 2.2. \square

Corollary 2.4. *Suppose G is an r -regular graph of odd order. Let f be a friendly labeling of G with $v_f(1) = v_f(0) + 1$. Then $i_f^*(G) = -\frac{1}{2}(q - r + i_f^+(G))$.*

Proof. Let p be the order of G . Then $rp = 2q$, and $s = (\frac{p+1}{2})r$. To complete the proof, apply Corollary 2.2. \square

Similarly we have

Corollary 2.5. *Suppose G is an r -regular graph of odd order. Let f be a friendly labeling of G with $v_f(1) = v_f(0) - 1$. Then $i_f^*(G) = -\frac{1}{2}(q + r + i_f^+(G))$.*

3. Application to Torus

A problem proposed in [5] asked readers to determine the exact value of $\text{PCI}(C_m \times C_n)$. We could apply the results in Section 2 to solve this problem. From [12] we have the following results:

$$\text{FFI}(C_{2h+1} \times C_{2k+1}) = \{8hk + 4h + 4k + 6 - 4\ell \mid h + k + 2 \leq \ell \leq 4hk + 2h\}, \text{ for } 1 \leq k \leq h;$$

$$\text{FFI}(C_{2h+1} \times C_{2k}) = \{8hk + 4k - 4\ell \mid k \leq \ell \leq 4hk - 1\}, \text{ for } 2 \leq k \leq h;$$

$$\text{FFI}(C_{2h} \times C_{2k+1}) = \{8hk + 4h - 4\ell \mid h \leq \ell \leq 4hk + 2h - 2k - 1, \ell \neq 4hk + 2h - 2k - 2\},$$

for $1 \leq k < h$;

$$\text{FFI}(C_{2h} \times C_{2k}) = \{8hk - 4\ell \mid 0 \leq \ell \leq 4hk - 2k, \ell \neq 1, 2, 4hk - 2k - 1\}, \text{ for } 2 \leq k \leq h.$$

Note that, in the proofs of those results, ℓ is equal to $E_f(1, 1)$, providing that $v_f(1) \geq v_f(0)$ (see [12]).

Since the torus $C_m \times C_n$ is a 4-regular graph, $i_f^*(C_m \times C_n) = -\frac{1}{2}(2mn + i_f^+(C_m \times C_n))$ when mn is even. Thus

$$\text{FPCI}(C_{2h+1} \times C_{2k}) = \{-8hk - 4k + 2\ell \mid k \leq \ell \leq 4hk - 1\}, \text{ for } 2 \leq k \leq h;$$

$$\text{FPCI}(C_{2h} \times C_{2k+1}) = \{-8hk - 4h + 2\ell \mid h \leq \ell \leq 4hk + 2h - 2k - 1, \ell \neq 4hk + 2h - 2k - 2\},$$

for $1 \leq k < h$;

$$\text{FPCI}(C_{2h} \times C_{2k}) = \{-8hk + 2\ell \mid 0 \leq \ell \leq 4hk - 2k, \ell \neq 1, 2, 4hk - 2k - 1\}, \text{ for } 2 \leq k \leq h.$$

It is clear that

$$\text{FPCI}(G) = \{i_f^*(G) \mid v_f(1) = v_f(0) + 1\} \cup \{i_f^*(G) \mid v_f(1) = v_f(0) - 1\}$$

for G of odd order.

Now we consider the graph $C_{2h+1} \times C_{2k+1}$ for $1 \leq k \leq h$. By Corollary 2.4 and the above result we have

$$\{i_f^*(C_{2h+1} \times C_{2k+1}) \mid v_f(1) = v_f(0) + 1\} = \{-8hk - 4h - 4k - 2 + 2\ell \mid h + k + 2 \leq \ell \leq 4hk + 2h\}.$$

Suppose f is a friendly labeling with $v_f(1) = v_f(0) - 1$. Let $\bar{f} = 1 - f$. Then $v_{\bar{f}}(1) = v_{\bar{f}}(0) + 1$, and $i_{\bar{f}}^+(G) = i_f^+(G)$. Hence $i_{\bar{f}}^*(C_{2h+1} \times C_{2k+1}) = -(8hk + 4h + 4k + 6 - 2\ell)$ if $v_f(1) = v_f(0) - 1$, where

$\ell = E_{\bar{f}}(1, 1)$. Since $f \leftrightarrow \bar{f}$ is an one-to-one correspondence,

$$\{i_f^*(C_{2h+1} \times C_{2k+1}) \mid v_f(1) = v_f(0) - 1\} = \{-8hk - 4h - 4k - 6 + 2\ell \mid h + k + 2 \leq \ell \leq 4hk + 2h\}.$$

Thus

$$\text{FPCI}(C_{2h+1} \times C_{2k+1}) = \{-8hk - 4h - 4k - 2 + 2\ell \mid h + k \leq \ell \leq 4hk + 2h\}.$$

Note that all product-cordial indices of torus are negative. Hence we have

Theorem 3.1. *The product-cordial index sets of torus are:*

$$\text{PCI}(C_{2h+1} \times C_{2k}) = \{8hk + 4k - 2\ell \mid k \leq \ell \leq 4hk - 1\}, \text{ for } 2 \leq k \leq h;$$

$$\text{PCI}(C_{2h} \times C_{2k+1}) = \{8hk + 4h - 2\ell \mid h \leq \ell \leq 4hk + 2h - 2k - 1, \ell \neq 4hk + 2h - 2k - 2\},$$

for $1 \leq k < h$;

$$\text{PCI}(C_{2h} \times C_{2k}) = \{8hk - 2\ell \mid 0 \leq \ell \leq 4hk - 2k, \ell \neq 1, 2, 4hk - 2k - 1\}, \text{ for } 2 \leq k \leq h;$$

$$\text{PCI}(C_{2h+1} \times C_{2k+1}) = \{8hk + 4h + 4k + 2 - 2\ell \mid h + k \leq \ell \leq 4hk + 2h\}.$$

4. Application to Cycles

In [10, Corollary 6], the authors showed that $\text{FFI}(C_n) \subseteq \{4j - n \mid 1 \leq j \leq \lfloor \frac{n}{2} \rfloor\}$ for $n \geq 3$. Note that, in this description, $2j = e_{f^+}(1)$ for some friendly labeling f . We now prove that equality holds.

Theorem 4.1. *For $n \geq 3$, $\text{FFI}(C_n) = \{4j - n \mid 1 \leq j \leq \lfloor \frac{n}{2} \rfloor\}$. Moreover, the friendly labelings f used to obtain these friendly indices have the additional property that $v_f(1) \geq v_f(0)$.*

Proof. Induct on n . It is easy to show that $\text{FFI}(C_3) = \{1\}$ and $\text{FFI}(C_4) = \{0, 4\}$, and that the friendly labelings satisfy the additional requirement. Now we assume that the theorem holds for all k with $3 \leq k \leq n$, where $n \geq 4$.

Let f be a friendly labeling of C_{n-1} such that $e_{f^+}(1) = 2j$ for $1 \leq j \leq \lfloor \frac{n-1}{2} \rfloor$. Since $j \geq 1$, there is a $(0, 1)$ -edge $xy \in E(C_{n-1})$. By inserting two new vertices u and v on the edge xy , we subdivide it into a path $xuvy$ so as to generate the cycle C_{n+1} . Define two labelings g and h on C_{n+1} according to

$$g(z) = \begin{cases} f(z) & \text{if } z \notin \{u, v\}; \\ f(x) & \text{if } z = u; \\ f(y) & \text{if } z = v, \end{cases} \quad h(z) = \begin{cases} f(z) & \text{if } z \notin \{u, v\}; \\ f(y) & \text{if } z = u; \\ f(x) & \text{if } z = v. \end{cases}$$

Then $v_g(1) = v_h(1) = v_f(1) + 1$, $v_g(0) = v_h(0) = v_f(0) + 1$, $e_{g^+}(1) = 2j$, and $e_{h^+}(1) = 2j + 2$, thereby completing the induction. \square

Applying the results from Section 2, we obtain

Theorem 4.2. *For $n \geq 2$, $\text{FPCI}(C_{2n}) = \{-2j \mid 1 \leq j \leq n\}$.*

Proof. Suppose f is a friendly labeling with $i_f^+(C_{2n}) = 4j - 2n$ for $1 \leq j \leq n$. Then, because of Corollary 2.3, we have $i_f^*(C_{2n}) = -\frac{1}{2}(2n + 4j - 2n) = -2j$, which is what we want to prove. \square

Theorem 4.3. For $n \geq 1$, $\text{FPCI}(C_{2n+1}) = \{-2j - 1 \mid 0 \leq j \leq n\}$.

Proof. Suppose f is a friendly labeling with $v_f(1) = v_f(0) + 1$ and $i_f^+(C_{2n}) = 4j - 2n$ for $1 \leq j \leq n$. Then by Corollary 2.4 we have $i_f^*(C_{2n+1}) = -\frac{1}{2}(2n + 1 - 2 + 4j - 2n - 1) = -2j + 1$. Hence we have $\{i_f^*(C_{2n+1}) \mid v_f(1) = v_f(0) + 1\} = \{-2j + 1 \mid 1 \leq j \leq n\}$.

Along the same line of discussion in Section 3, we also find

$$\{i_f^*(C_{2n+1}) \mid v_f(1) = v_f(0) - 1\} = \{-2j - 1 \mid 1 \leq j \leq n\}.$$

Hence $\text{FPCI}(C_{2n+1}) = \{-2j - 1 \mid 0 \leq j \leq n\}$. □

Corollary 4.4. For $n \geq 1$, $\text{PCI}(C_{2n+1}) = \{2j + 1 \mid 0 \leq j \leq n\}$ and for $n \geq 2$, $\text{PCI}(C_{2n}) = \{2j \mid 1 \leq j \leq n\}$.

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