



SEMI-STRONG SPLIT DOMINATION IN GRAPHS

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ABSTRACT. Given a graph $G = (V, E)$, a dominating set $D \subseteq V$ is called a *semi-strong split dominating set* of G if $|V \setminus D| \geq 1$ and the maximum degree of the induced subgraph $\langle V \setminus D \rangle$ is 1. The cardinality of a minimum semi-strong split dominating set (SSSDS) of G is the *semi-strong split domination number* of G , denoted $\gamma_{sss}(G)$. In this paper, we introduce the concept and prove several results regarding it.

1. Introduction

By a *graph* $G = (V, E)$ we mean a finite and undirected graph with neither loops nor multiple edges. The *order* of G is $n = |V|$, and the *size* of G is $m = |E|$. For any vertex $v \in V$, the *open neighborhood* of v is the set $N(v) = \{u \in V : uv \in E\}$, and the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. If $u \in N(v)$, we say that u and v are *adjacent*. The *degree* of v in G is $\deg_G(v) = |N(v)|$. When G is clear we will write simply $\deg(v)$. The *maximum degree* of G is $\Delta(G) = \max\{\deg(v) : v \in V\}$, and the *minimum degree* of G is $\delta(G) = \min\{\deg(v) : v \in V\}$. A *pendant vertex* is a vertex $v \in V$ such that $\deg(v) = 1$. The *stem* of G is the set of vertices adjacent to at least one pendant vertex. Given a set $S \subseteq V$ and $v \in S$, a vertex $u \in V \setminus S$ is an (*external*) *private neighbor* of v (with respect to S) if $N(u) \cap S = \{v\}$. Given a graph $G = (V, E)$ and $\{u, v\} \subseteq V$, the *distance* between u and v is the minimum length of a path (u, \dots, v) , and is denoted $d(u, v)$. A *connected component* of G is a maximal (by inclusion) connected subgraph of G . From now on it will be called just *component*. A graph G is *totally disconnected* if the cardinality of each component of G is 1 and G has at least two components.

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For a given positive integer $t \geq 2$, a *wounded spider* is a star $K_{1,t}$ with at most $t - 1$ of its edges subdivided once, and a *healthy spider* is a star $K_{1,t}$ with all of its edges subdivided once.

Given two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ such that $V_1 \cap V_2 = \emptyset$, the *join* of G_1 and G_2 is $G_1 + G_2 = (V_1 \cup V_2, E')$, where $E' = E_1 \cup E_2 \cup \{uv : u \in V_1, v \in V_2\}$. Let G_1 and G_2 be two graphs such that $V(G_1) \cap V(G_2) = \emptyset$, with $|V(G_1)| = n_1$. The *corona* $G_1 \circ G_2$ of G_1 and G_2 is the graph obtained by taking one copy of G_1 and n_1 copies of G_2 , and then joining the i^{th} vertex of G_1 to every vertex in the i^{th} copy of G_2 . If G_2 is isomorphic to K_1 , we say that G is a *corona graph*.

Given an arbitrary graph G , the *trestled graph of index k* , denoted by $T_k(G)$, is the graph obtained from G by adding k -copies of K_2 for each edge uv of G , and joining u and v to the respective end vertices of each K_2 . Let $G = (V, E)$ be a graph. A set $D \subseteq V$ is a *dominating set* of G if for every vertex v in $V \setminus D$, there exists a vertex u in D such that $uv \in E$. The *domination number* $\gamma(G)$ of G is the minimum cardinality of a dominating set of G . The concept of domination in graphs and several of its variations are well studied in graph theory. A thorough review of domination appears in [5, 6]. A set $S \subseteq V$ is called *independent* if no two vertices in S are adjacent, and $\beta_0(G)$ denotes the maximum cardinality among the independent sets of G . The *independent domination number* $i(G)$ is the minimum cardinality of a set $I \subseteq V$ which is both independent and dominating in G . A set $C \subseteq V$ is a *vertex covering* of G if every edge has at least one vertex in C . The *vertex covering number* $\alpha_0(G)$ denotes the minimum cardinality of a vertex covering of G . A set $F \subseteq E$ is a *matching* of G if for every $\{e, f\} \subseteq F$ we have $e \cap f = \emptyset$. The *edge independence number* $\beta_1(G)$ denotes the maximum cardinality of a matching in G . A *clique* of G is a complete graph which is a subgraph of G . A graph is a *split graph* if its vertices can be partitioned into a clique and an independent set. We refer to [1] for graph theory notions and terminology not described in this work.

A dominating set D is a *split dominating set* if the induced subgraph $\langle V \setminus D \rangle$ is disconnected. The *split domination number* $\gamma_s(G)$ of a graph G is the minimum cardinality of a split dominating set of G . A γ_s -set is a minimum split dominating set. These concepts were introduced by Kulli and Janakiram in [3]. A dominating set D is a *strong split dominating set* if the induced subgraph $\langle V \setminus D \rangle$ is totally disconnected. The *strong split domination number* $\gamma_{ss}(G)$ is the minimum cardinality of a strong split dominating set of G . A γ_{ss} -set is a minimum strong split dominating set. These concepts were introduced by Kulli and Janakiram in [4]. Observe that both a γ_s -set and a γ_{ss} -set exist if, and only if, the graph has either one component which is not complete or two non-trivial components.

In this paper, we study a variant of domination. A dominating set $D \subseteq V$ is called a *semi-strong split dominating set* (SSSDS) if $|V \setminus D| \geq 1$ and the maximum degree of the induced subgraph $\langle V \setminus D \rangle$ is 1. Thus the induced subgraph $\langle V \setminus D \rangle$ is isomorphic to $sK_1 \cup tK_2$, where s and t are non-negative integers with $s > 0$ or $t > 0$. The minimum cardinality of a semi-strong split dominating set of G is the *semi-strong split domination number* of G , denoted $\gamma_{sss}(G)$, and an SSSDS of minimum cardinality in G is a $\gamma_{sss}(G)$ -set. Since every strong split dominating set is an SSSDS, it is clear that $\gamma_{sss}(G)$ exists if, and only if, G is not totally disconnected. Therefore, whenever $\gamma_{sss}(G)$ exists we have $\gamma_{sss}(G) \leq n - 1$.

The concept of semi-strong split domination has interesting applications. As an example, consider a case of epidemics in a given area. We can model the situation as a graph, where villages are vertices and two villages are adjacent if there is regular traffic between them. Of course, a desirable situation is to have a disease-control post in each village of the area, but sometimes it is not possible. A much weaker option is to make every village lacking disease-control post (unprotected village) adjacent to at least one village having a post (protected village), that is, to make the set of protected villages dominating in the graph.

However, there are some intermediate solutions: We may arrange the posts in such a way that every unprotected village is adjacent only to protected villages, which corresponds to a strong split dominating set in the graph. If we do not have enough resources for that, we may distribute the posts so that every unprotected village is adjacent to at most another unprotected village, that is, we have a post in each vertex of a semi-strong split dominating set. The concept is useful because there are connected graphs such that $\gamma_{ss} - \gamma_{sss}$ is as large as desired (see Proposition 2.7). Of course, if we do not have enough resources even for that, we may place the posts in the vertices of a split dominating set, so at least the unprotected villages are split into two non-communicating sets. Again, this may prove useful since $\gamma_{sss} - \gamma_s$ may be as large as desired even for connected graphs, as shown in Proposition 2.7, and is still better than taking just a dominating set.

Instead of disease-control posts in case of epidemics, we may consider fire-extinguishing teams in case of a huge forest fire, where the vertices of our graph represent vital points for extinguishing it, or police stations in a city, where the vertices may represent neighborhoods.

The graph G given in Figure 1 is an example where $\gamma(G)$, $\gamma_s(G)$, $\gamma_{sss}(G)$ and $\gamma_{ss}(G)$ are all different, since G has $\{v_4, v_5\}$ as a γ -set, $\{v_1, v_4, v_8\}$ as a γ_s -set, $\{v_1, v_4, v_5, v_8\}$ as a γ_{sss} -set, and $\{v_1, v_2, v_4, v_5, v_7, v_8\}$ as a γ_{ss} -set. Hence $\gamma(G) = 2$, $\gamma_s(G) = 3$, $\gamma_{sss}(G) = 4$, and $\gamma_{ss}(G) = 6$.

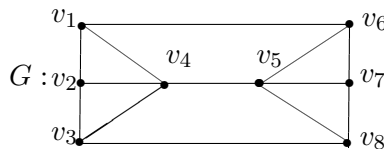


Figure 1:

Take the Petersen graph $G = (V, E)$ as given in Figure 2. Then the set $\{v_1, v'_2, v'_3, v_4\}$ is an SSSDS of G , so $\gamma_{sss}(G) \leq 4$. Now suppose $D \subseteq V(G)$ is an SSSDS of G with $|D| = 3$. Since $\gamma(G) = 3$, D is a $\gamma(G)$ -set of G , so $D = N(v)$ for some $v \in V(G)$. Since for every two vertices $\{u, w\} \subseteq V$ there is an automorphism f of G such that $f(u) = w$, we may assume $D = N(v_1)$, but then $\deg_{(V \setminus D)}(v_4) = 2$, so D is not an SSSDS. Therefore, $\gamma_{sss}(G) = 4$.

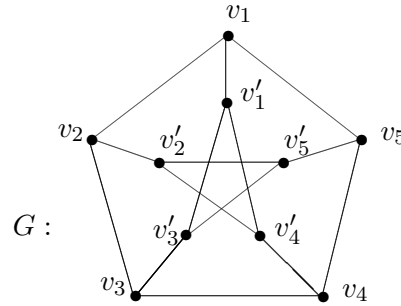


Figure 2:

Theorem 1.1. [4] For any graph G such that $\gamma_s(G)$ exists, $\gamma(G) \leq \gamma_s(G) \leq \gamma_{ss}(G)$.

Theorem 1.2. [5] For any graph G , $\lceil \frac{n}{\Delta(G)+1} \rceil \leq \gamma(G) \leq n - \Delta(G)$.

Theorem 1.3. [2], [7] Let G be a connected graph. Then $\gamma(G) = n(G)/2$ if, and only if, G is the corona graph of any connected graph J or G is isomorphic to the cycle C_4 .

2. Main results

In the following proposition, we establish some basic results. The proofs are straightforward and are therefore omitted.

Proposition 2.1.

- (1) For the complete graph K_n , $\gamma_{sss}(K_n) = n - 2$.
- (2) For the path P_n , $\gamma_{sss}(P_n) = \lceil \frac{n}{3} \rceil$.
- (3) For the cycle C_n , $\gamma_{sss}(C_n) = \lceil \frac{n}{3} \rceil$.
- (4) For the wheel $W_n = C_{n-1} + K_1$ of order $n \geq 4$, $\gamma_{sss}(W_n) = \lceil \frac{n}{3} \rceil + 1$.
- (5) For the complete multipartite graph K_{n_1, n_2, \dots, n_t} with $2 \leq n_1 \leq n_2 \leq \dots \leq n_t$, $\gamma_{sss}(K_{n_1, n_2, \dots, n_t}) = n_1 + n_2 + \dots + n_{t-1}$.
- (6) Let G be any bipartite graph with bipartition $V(G) = V_1 \cup V_2$. Then $\gamma_{sss}(G) \leq \min\{|V_1|, |V_2|\}$.
- (7) Let G be any connected graph of order $n \geq 3$. Then $\gamma_{sss}(G) = 1$ if, and only if, G is isomorphic to $K_1 + (sK_1 \cup tK_2)$ for some integers $s, t \geq 0$.
- (8) Let $G = (V, E)$ be a non-trivial connected graph. If there exists $v \in V(G)$ with $\deg(v) = \Delta(G)$ such that the induced subgraph $\langle N(v) \rangle$ is isomorphic to $sK_1 \cup tK_2$ for some integers $s, t \geq 0$, then $\gamma_{sss}(G) \leq n - \Delta(G)$.

Proposition 2.2. Let G be any connected graph of order $n \geq 3$, let p be the maximum length of a path in G , and let c be the maximum length of a cycle in G . Then

- (1) $\lceil \frac{p+1}{3} \rceil \leq \gamma_{sss}(G) \leq n - \lfloor \frac{2p+2}{3} \rfloor$, and the bounds are sharp.
- (2) $\lceil \frac{c}{3} \rceil \leq \gamma_{sss}(G) \leq n - \lfloor \frac{2c}{3} \rfloor$, and the bounds are sharp.

Proof. (i) Let $P = (v_1, v_2, \dots, v_{p+1})$ be a path of maximum length in G , and let D be a γ_{sss} -set of P . By (ii) of Proposition 2.1, we have $\gamma_{sss}(P) = |D| = \lceil \frac{p+1}{3} \rceil$. We define $S = D \cup (V(G) \setminus V(P))$, then it is clear that S is an SSSDS of G , which implies that $\gamma_{sss}(G) \leq |D| + (n - |V(P)|) = n - \lfloor \frac{2p+2}{3} \rfloor$.

Now let S_1 be any SSSDS of G . Then S_1 has a non-empty intersection with each set of three consecutive vertices in $V(P)$, which implies that $|S_1 \cap V(P)| \geq \lceil \frac{p+1}{3} \rceil$, and hence the lower bound holds. Both equalities are trivially satisfied by any path P_n .

(ii) If G has no cycles, the bounds are trivial. Otherwise, the proof is similar to that of (i). Both equalities are satisfied by any cycle C_n . □

Notice that for every graph G , $c \leq p + 1$. Therefore, in Proposition 2.2 the bounds established in (i) are better than those of (ii).

Theorem 2.3. *Let G be a graph. Then $\gamma_{sss}(G) \leq n - 2$ if, and only if, G has either two components isomorphic to K_2 or a component with at least three vertices.*

Proof. If $G = (V, E)$ has two components isomorphic to K_2 , take two vertices u and v , one from each of such components. If G has a component H with at least three vertices, let u and v be pendant vertices of a spanning tree T of H . In both cases, $V \setminus \{u, v\}$ is an SSSDS of G .

Conversely, take a graph G with neither two components isomorphic to K_2 nor a component with at least three vertices. If G is totally disconnected, $\gamma_{sss}(G)$ does not exist. If G is of the form $K_2 \cup sK_1$ with $s \geq 0$, then every SSSDS of G has exactly one vertex from the non-trivial component, plus all isolated vertices, so $\gamma_{sss}(G) = n - 1$. □

Corollary 2.4. *For any graph G , $\gamma_{sss}(G) = n - 1$ if, and only if, G is of the form $K_2 \cup sK_1$ with $s \geq 0$.*

Next corollary states two Nordhaus-Gaddum type results.

Corollary 2.5. *Let G be a graph of order $n \geq 3$ which is not complete nor totally disconnected. Then*

- (1) $\gamma_{sss}(G) + \gamma_{sss}(\bar{G}) \leq 2n - 3$.
- (2) $\gamma_{sss}(G) \cdot \gamma_{sss}(\bar{G}) \leq n^2 - 3n + 2$.

Furthermore, equality holds only for $K_n - e$, where e is any edge of K_n and $n \geq 3$ (and for its complement, of course).

Proof. The inequalities follow straightforwardly from Theorem 2.3, since the complement of a disconnected graph is always connected. It is clear that equality holds for $K_n - e$, $n \geq 3$ and its complement. Now let G be a graph such that equality holds. Without loss of generality we may assume $\gamma_{sss}(\bar{G}) = n - 1$, so Corollary 2.4 implies that $\bar{G} = K_2 \cup sK_1$ for $s \geq 0$. If $s = 0$ then $\bar{G} = K_2$, which is complete. If $s > 0$, $G = K_n - e$ for an edge e of K_n , $n \geq 3$. □

Theorem 2.6. *Let G be any graph. Then $\gamma_{sss}(G) = n - 2$ if, and only if,*

- (1) G is of the form $2K_2 \cup sK_1$ with $s \geq 0$,
- (2) G is of the form $H \cup sK_1$ with $s \geq 0$, where H is isomorphic to P_3, C_3, P_4, C_4, K_4 , or $K_4 - e$ for an edge e of K_4 ,
- (3) G is of the form $H \cup sK_1$ with $s \geq 0$, $|H| = p \geq 5$, and $\delta(H) \geq p - 2$.

Proof. Let $G = (V, E)$ be a graph with more than one non-trivial component, and such that $\gamma_{sss}(G) = n - 2$. If G has more than two non-trivial components, then V minus one vertex from each of such components is an SSSDS of G of cardinality less than $n - 2$. If G has a component H of order 3 or more, then V minus two vertices from H minus one vertex from all other non-trivial components, is an SSSDS of G of cardinality less than $n - 2$. Then G is of the form $2K_2 \cup sK_1$ with $s \geq 0$. It is straightforward that equality holds for all such graphs. It follows that every other graph satisfying equality has exactly one non-trivial component.

Let G be a totally disconnected graph. Then no SSSDS exists.

Let G be a graph with only one non-trivial component H of order 2. Then Corollary 2.4 implies that $\gamma_{sss}(G) = n - 1$.

Let G be a graph with only one non-trivial component H of order 3. Then H is isomorphic to P_3 or C_3 , and clearly $\gamma_{sss}(G) = n - 2$.

Let G be a graph with only one non-trivial component H of order 4. If H is a star or a C_3 plus one pendant vertex, $\gamma_{sss} = 1 = n - 3$. Otherwise H is isomorphic to P_4 , C_4 , K_4 , or $K_4 - e$ for an edge e of K_4 . It is easy to verify that in those cases we have $\gamma_{sss}(G) = n - 2$.

Let G be a graph with only one non-trivial component $H = (V', E')$ of order $p \geq 5$. If $\delta(H) \geq p - 2$, any three vertices form a P_3 or a C_3 , so Theorem 2.3 implies $\gamma_{sss}(G) = n - 2$. Assume $\delta(H) < p - 2$; take $v \in V'$ of minimum degree and $\{u, w\} \subseteq V' \setminus N(v)$. If $D = V' \setminus \{u, v, w\}$ is a dominating set, then D is an SSSDS of G . Now suppose D is not a dominating set. Since H is connected, both v and (without loss of generality) u are adjacent to at least one vertex of D , and w is adjacent to u . Take $z \in D \cap N(u)$. If $D \cap N(v) \neq \{z\}$, then $(D \setminus \{z\}) \cup \{u\}$ is an SSSDS of G . If $D \cap N(v) = \{z\}$, notice that since $|H| \geq 5$ there is a vertex $x \in D \setminus \{z\}$, and H connected implies $x \in N(u)$ or $x \in N(z)$. It follows that $V' \setminus \{v, w, x\}$ is an SSSDS of G . \square

Theorem 2.6 implies that the only trees satisfying equality are P_3 and P_4 , and the only cycles satisfying equality are C_3 and C_4 . Equality holds as well for K_n and $K_n - M$, where M is any match of K_n , provided $n \geq 3$.

Proposition 2.7. *For every positive integer k , there is a connected graph G such that $\gamma_{ss} - \gamma_{sss} \geq k$ and $\gamma_{sss} - \gamma_s \geq k$.*

Proof. For $p \geq 3$, let G be the graph consisting on k cliques of order p sharing a common vertex. Then $\gamma_s(G) = 1$, $\gamma_{sss}(G) = k(p - 3) + 1$, and $\gamma_{ss}(G) = k(p - 2) + 1$. So for $p = 4$ we have $\gamma_s(G) = 1$, $\gamma_{sss}(G) = k + 1$, and $\gamma_{ss}(G) = 2k + 1$. \square

Theorem 2.8. *Let G be a graph such that $\gamma_s(G)$ exists. Then $\gamma_s(G) \leq \gamma_{sss}(G)$.*

Proof. Since $\gamma_s(G)$ exists, G has either a component which is not complete or two components which are non-trivial. Then $\gamma_{sss}(G)$ exists and Theorem 2.3 implies $\gamma_{sss}(G) \leq n - 2$. We consider two cases:

Case 1. $\gamma_{sss}(G) \leq n - 3$.

Let D be any $\gamma_{sss}(G)$ -set. Since $|V \setminus D| \geq 3$ and every vertex in the induced subgraph $\langle V \setminus D \rangle$ has degree at most 1, then $\langle V \setminus D \rangle$ is a disconnected graph. Hence D is a split dominating set of G .

Case 2. $\gamma_{sss}(G) = n - 2$.

If G has more than one non-trivial component then every $\gamma_{sss}(G)$ -set is a $\gamma_s(G)$ -set, as shown in the proof of Theorem 2.6. If G has only one non-trivial component H , then H is not complete, so $|V(H)| \geq 3$. This implies that there exist two non-adjacent vertices $u, v \in V(H)$ such that $V(G) \setminus \{u, v\}$ is a $\gamma_{sss}(G)$ -set. Since $\{u, v\}$ is an independent set, we have $\gamma_s(G) \leq n - 2$. \square

Corollary 2.9. *Let G be any graph such that $\gamma_s(G)$ exists. Then*

$$\gamma(G) \leq \gamma_s(G) \leq \gamma_{sss}(G) \leq \gamma_{ss}(G).$$

Proof. The inequality $\gamma(G) \leq \gamma_s(G)$ holds by Theorem 1.1. Since every strong split dominating set is an SSSDS, it follows that $\gamma_{sss}(G) \leq \gamma_{ss}(G)$. Theorem 2.8 implies $\gamma_s(G) \leq \gamma_{sss}(G)$. \square

Theorem 2.10. *Let T be any tree of order $n \geq 3$. Then $\gamma_{sss}(T) \leq n - \varepsilon(T)$, where $\varepsilon(T)$ denotes the number of pendant vertices of T . Furthermore, equality holds if, and only if, every non-pendant vertex is adjacent to at least one pendant vertex.*

Proof. Let $T = (V, E)$ be a tree, and let S be the set of all pendant vertices of T . Then clearly $\langle S \rangle$ is an independent set, which implies that $V \setminus S$ is an SSSDS, and hence $\gamma_{sss}(T) \leq n - \varepsilon(T)$. If every vertex in $V \setminus S$ is adjacent to a vertex in S , it is clear that $V \setminus S$ is a $\gamma_{sss}(T)$ -set, so $\gamma_{sss}(T) = n - \varepsilon(T)$. If there is a vertex $v \in V \setminus S$ such that $N(v) \cap S = \emptyset$, then $V \setminus (S \cup \{v\})$ is an SSSDS of T , which implies $\gamma_{sss}(T) < n - \varepsilon(T)$. \square

Theorem 2.11. *For any tree T , $\gamma_{sss}(T) \leq \frac{n}{2}$. Equality holds if, and only if, every non-pendant vertex is adjacent to exactly one pendant vertex.*

Proof. Let $T = (V, E)$ be a tree with stem S_0 , and let L_0 be the set of pendant vertices of T . Now take $T_1 = \langle V \setminus (S_0 \cup L_0) \rangle$ with stem S_1 and set of pendant vertices L_1 . Repeat till there are no vertices left, after k steps. Consider the set $D = \bigcup_{i=0}^k S_i$. It is clear that D is a dominating set in T , and $V \setminus D$ is an independent set, so D is an SSSDS of T . Since trivially $|S_i| \leq |L_i|$ for $0 \leq i \leq k$, we have $|D| \leq \frac{n}{2}$. The condition for equality then follows from Theorem 1.3 and Corollary 2.9. \square

Theorem 2.12. *Let $G = (V, E)$ be a graph. An SSSDS D of G is minimal if, and only if, for every vertex $v \in D$ at least one of the following conditions hold:*

- (1) v is an isolated vertex in $\langle D \rangle$.
- (2) There exists a vertex $u \in V \setminus D$, for which $N(u) \cap D = \{v\}$.
- (3) $|N(v) \cap (V \setminus D)| \geq 2$.
- (4) v is adjacent to some vertex $u \in V \setminus D$ which belongs to a K_2 component of the induced subgraph $\langle V \setminus D \rangle$.

Proof. Let D be any minimal SSSDS of G and let $v \in D$. Then $D \setminus \{v\}$ is not an SSSDS, which implies that either $D \setminus \{v\}$ is not a dominating set or the induced subgraph $\langle (V \setminus D) \cup \{v\} \rangle$ contains a vertex of degree at least two. If $D \setminus \{v\}$ is not a dominating set, then either v dominates itself or v has a private neighbor $u \in V \setminus D$, so either (i) or (ii) hold. If the induced subgraph $\langle (V \setminus D) \cup \{v\} \rangle$

contains a vertex of degree at least two, then either v is adjacent to at least two vertices in $V \setminus D$, or v is adjacent to some vertex $u \in V \setminus D$ which belongs to a K_2 component of the induced subgraph $\langle V \setminus D \rangle$, so either (iii) or (iv) hold. The converse is straightforward. \square

Theorem 2.13. *Let G be a graph without isolated vertices. If D is a minimal SSSDS, then $V \setminus D$ is a dominating set.*

Proof. Let D be any minimal SSSDS of G . Suppose $V \setminus D$ is not a dominating set. Then there exists a vertex u in D such that $N(u) \cap (V \setminus D) = \emptyset$. Since G has no isolates, u is dominated by $D' = D \setminus \{u\}$, and u is an isolated vertex of $\langle V \setminus D' \rangle$. This implies that D' is an SSSDS of G , contradicting the minimality of D . Therefore, $V \setminus D$ is a dominating set of G . \square

Corollary 2.14. *For any graph G without isolated vertices, $\gamma_{sss}(G) + \gamma(G) \leq n$. For a tree T , equality holds if, and only if, every non-pendant vertex is adjacent to exactly one pendant vertex.*

Proof. The inequality follows directly from Theorem 2.13. Now take a tree T such that $\gamma_{sss}(T) + \gamma(T) = n$. Then Theorem 2.11 implies that $\gamma(T) = \gamma_{sss}(T) = \frac{n}{2}$. Conversely, if $\gamma(T) = \frac{n}{2}$ then from Corollary 2.9 and Theorem 2.11 it follows that $\gamma_{sss}(T) = \frac{n}{2}$. Therefore, Theorem 1.3 implies the result. \square

Regarding Corollary 2.14, notice that given any graph G without isolates, equality holds for G if, and only if, it holds for each of its components, so we may restrict ourselves to connected graphs. Moreover, equality holds whenever $\gamma(G) = \frac{n}{2}$, so Theorem 1.3 implies that the corona graph of every graph without isolates satisfies equality. However, there are other connected graphs for which equality holds, like $K_p^- + 2K_1$ for p even, $p \geq 2$, where K_p^- is the graph resulting from deleting a maximum matching from K_p . No graph of order 3 satisfies equality, and the only connected graphs of order 4 for which equality holds are C_4 and P_4 .

Corollary 2.15. *For any graph G without isolated vertices, $\gamma_{sss}(G) \leq \lfloor \frac{n\Delta(G)}{\Delta(G)+1} \rfloor$. The bound is sharp.*

Proof. By Theorem 1.2 and Corollary 2.14, $\lceil \frac{n}{\Delta(G)+1} \rceil \leq \gamma(G) \leq n - \gamma_{sss}(G)$. Hence $\gamma_{sss}(G) \leq \lfloor \frac{n\Delta(G)}{\Delta(G)+1} \rfloor$. Equality holds for C_4 and P_4 . \square

Theorem 2.16. *Let T be any tree. Then $\gamma_{sss}(T) = n - \Delta(T)$ if, and only if, T is a wounded spider.*

Proof. Suppose T is a wounded spider. Then it is easy to check that $\gamma_{sss}(T) = n - \Delta(T)$. Conversely, suppose T is a tree with $\gamma_{sss}(T) = n - \Delta(T)$. Let $v \in V(T)$ be a vertex of maximum degree. If $V(T) \setminus N[v] = \emptyset$, then T is a star $K_{1,t}$, $t \geq 1$, which is a wounded spider. We assume that there is at least one vertex in $V(T) \setminus N[v]$. Since $\langle V(T) \setminus N[v] \rangle$ is a bipartite graph, we may take a partition (X, Y) of $V(T) \setminus N[v]$ such that $|X|$ is maximum. Then X is a maximal independent set of the induced subgraph $\langle V(T) \setminus N[v] \rangle$, and $X \cup \{v\}$ is an independent SSSDS of T . Thus $n = \gamma_{sss}(T) + \Delta(T) \leq |X| + 1 + \Delta(T)$, which implies that $Y = \emptyset$, and so $V(T) \setminus N(v)$ is an independent set. The connectivity of T implies that each vertex in $V(T) \setminus N[v]$ is adjacent to exactly one vertex in $N(v)$. Therefore, T is a spider. If it is healthy, then $n = 2\Delta(T) + 1$ but $N(v)$ is an SSSDS. It follows that at least one vertex in $N(v)$ is not adjacent to any vertex in $V(T) \setminus N[v]$, that is, T is a wounded spider. \square

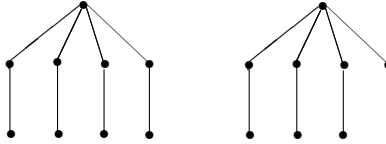


Figure 3: A healthy spider and a wounded spider.

Theorem 2.17. *Let G be any graph without isolates. Then $\gamma_{sss}(G) \leq \min\{n - \beta_0(G), \alpha_0(G), 2\beta_1(G)\}$.*

Proof. Let S be an independent set of G with $\beta_0(G)$ vertices. Then the induced subgraph $\langle S \rangle$ contains no edges, and each vertex in S has a neighbor in $V \setminus S$, since $\delta(G) \geq 1$. This implies that $V \setminus S$ is an SSSDS of G , and hence $\gamma_{sss}(G) \leq n - \beta_0(G)$.

Now let S be a minimum vertex cover of G . Then the induced subgraph $\langle V \setminus S \rangle$ is totally disconnected, and each vertex in $V \setminus S$ has a neighbor in S , since $\delta(G) \geq 1$. This implies that S is an SSSDS of G , and hence $\gamma_{sss}(G) \leq \alpha_0(G)$.

Let $M = \{e_i = u_i v_i : i = 1, 2, \dots, \beta_1\}$ be a maximum matching of G , and let $D = \{u_i, v_i : i = 1, 2, \dots, \beta_1\}$. Since G has no isolates, M has at least one edge in each component of G . Suppose D is not a dominating set of G . Then $d(x, D) = \min\{d(x, w) : w \in D\} \geq 2$ for some $x \in V \setminus D$, which implies (without loss of generality) that there exists a path (x, y, \dots, u_i) , where $u_i \in D$ with $d(x, u_i) = \min\{d(x, w) : w \in D\}$. Then $M \cup \{xy\}$ is a matching of G , which is a contradiction to the maximality of M . Also, M maximal implies that the set $V \setminus D$ is independent. It follows that D is an SSSDS of G , and hence $\gamma_{sss}(G) \leq 2\beta_1(G)$.

Therefore $\gamma_{sss}(G) \leq \min\{n - \beta_0(G), \alpha_0(G), 2\beta_1(G)\}$. □

Corollary 2.18. *For any tree T , $\gamma_{sss}(T) \leq \beta_0(T)$.*

Corollary 2.19. *Let G be any connected non-trivial graph. Then $\gamma_{sss}(G) + i(G) \leq n$.*

Proof. Since $i(G) \leq \beta_0(G)$ and $\gamma_{sss}(G) \leq n - \beta_0(G)$, it follows that $\gamma_{sss}(G) + i(G) \leq n$. □

3. Graph operations

This section is mostly focused on determining the semi-strong split domination number of a graph obtained by applying graph operations on two graphs.

Theorem 3.1. *Let G_1 and G_2 be any two graphs of order n_1 and n_2 respectively, with $\min\{n_1, n_2\} \geq 1$, and such that $V(G_1) \cap V(G_2) = \emptyset$. Then $\gamma_{sss}(G_1 + G_2) = \min\{n_1 + k_2, n_2 + k_1\}$, where k_i is the minimum cardinality of a set $D_i \subset V(G_i)$ such that $\langle V(G_i) \setminus D_i \rangle$ is isomorphic to $sK_1 \cup tK_2$ for some integers $s, t \geq 0$.*

Proof. Let D be any SSSDS of $G_1 + G_2$, and let $V' = V(G_1 + G_2) \setminus D$. If $|V' \cap V(G_i)| \geq 1$ and $|V' \cap V(G_j)| \geq 2$, where $\{i, j\} = \{1, 2\}$, then $\deg_{\langle V' \rangle}(x) \geq 2$ for every $x \in V' \cap V(G_i)$, which is a contradiction. If $|V' \cap V(G_1)| = |V' \cap V(G_2)| = 1$, then D is an SSSDS of cardinality $n_1 + n_2 - 2 \geq \min\{n_1 + k_2, n_2 + k_1\}$. We now assume that $V' \cap V(G_i) = \emptyset$, for $i = 1$ or $i = 2$. Then $|D \cap V(G_j)| \geq k_j$, according to the definition of k_j , which implies that $\gamma_{sss}(G_1 + G_2) \geq \min\{n_1 + k_2, n_2 + k_1\}$.

Let D_i be a minimum subset of $V(G_i)$ such that the graph $\langle V(G_i) \setminus D_i \rangle$ contains no vertex of degree 2 or more, for $i = 1, 2$. Then $D_1 \cup V(G_2)$ and $D_2 \cup V(G_1)$ are SSSDSs of $G_1 + G_2$. Hence $\gamma_{sss}(G_1 + G_2) \leq \min\{n_1 + k_2, n_2 + k_1\}$. Therefore, $\gamma_{sss}(G_1 + G_2) = \min\{n_1 + k_2, n_2 + k_1\}$. \square

Theorem 3.2. *Let G_1 and G_2 be any two graphs of order n_1 and n_2 respectively, such that $V(G_1) \cap V(G_2) = \emptyset$. Then $\gamma_{sss}(G_1 \circ G_2) = n_1(k + 1)$, where k is the minimum cardinality of a set $S \subset V(G_2)$ such that $\langle V(G_2) \setminus S \rangle$ is isomorphic to $sK_1 \cup tK_2$ for some integers $s, t \geq 0$.*

Proof. Let $V = V(G_1 \circ G_2) = \left(\bigcup_{i=1}^{n_1} V_i\right) \cup V(G_1)$, where $V_i = \{u_{i_1}, \dots, u_{i_{n_2}}\}$ and $\langle V_i \rangle \cong G_2$ for $i = 1, \dots, n_1$, and $V(G_1) = \{v_1, \dots, v_{n_1}\}$. Let $D = V(G_1) \cup \left(\bigcup_{i=1}^{n_1} S_i\right)$, where $S_i \subset V_i$ is a set of minimum cardinality such that the induced subgraph $\langle V_i \setminus S_i \rangle$ is isomorphic to $sK_1 \cup tK_2$ for some integers $s, t \geq 0$, for each $i = 1, \dots, n_1$. Then clearly D dominates the set $V(G_1 \circ G_2)$, and the induced subgraph $\langle V(G_1 \circ G_2) \setminus D \rangle \cong n_1 sK_1 \cup n_1 tK_2$. This implies that D is an SSSDS of $G_1 \circ G_2$, and hence $\gamma_{sss}(G_1 \circ G_2) \leq n_1 k + n_1$.

Conversely, let D' be any SSSDS of $G_1 \circ G_2$. If G_2 has no components of order 3 or greater, then $k = 0$ and $V(G_1)$ is a $\gamma_{sss}(G_1 \circ G_2)$ -set, since for every vertex $v_i \in V(G_1) \setminus D'$ there must be at least one vertex of D' in the i^{th} copy of G_2 .

Assume G_2 has a component of order 3 or greater, then Theorem 2.3 implies that $k \leq n_2 - 2$. If $v_i \notin D'$ for some i , $1 \leq i \leq n_1$, then $|D' \cap V_i| \geq n_2 - 1$, else $d_{\langle V \setminus D' \rangle}(v_i) \geq 2$. If $v_i \in D'$ for some i , $1 \leq i \leq n_1$, then $|D' \cap V_i| \geq k$, according to the definition of k . Therefore, $|D'| \geq a(n_2 - 1) + (n_1 - a)k + (n_1 - a) = n_1 k + n_1 + at$, where $a = |V(G_1) \setminus D'|$ and $t = (n_2 - k - 2)$. Since $k \leq n_2 - 2$, we have $t \geq 0$; also clearly $a \geq 0$, so $|D'| \geq n_1 k + n_1$. Therefore, in any case $\gamma_{sss}(G_1 \circ G_2) = n_1(k + 1)$. \square

Theorem 3.3. *Let G be any graph of order n which is not totally disconnected. Then $\gamma_{sss}(T_k(G)) = n$ for every $k \geq 2$.*

Proof. Take $V = V(T_k(G))$ and $D = V(G) = \{v_1, v_2, \dots, v_n\}$. Let $e_i^1, e_i^2, \dots, e_i^k$ be the new edges of $T_k(G)$ corresponding to the edge $e_i \in E(G)$, for $i = 1, 2, \dots, m$. Then the induced subgraph $\langle V \setminus D \rangle$ is isomorphic to $(mk)K_2$. Since D is a dominating set of $V(T_k(G))$, then D is an SSSDS of $T_k(G)$, so $\gamma_{sss}(T_k(G)) \leq n$.

Now let D' be a minimum SSSDS of $(T_k(G))$. It is clear that every isolated vertex belongs to D' . If $v_i \notin D'$ for some i , $1 \leq i \leq n$, where $v_i v_j = e_t \in E(G)$, then to dominate the ends of $e_t^1, e_t^2, \dots, e_t^k$ which are adjacent to v_i we need at least k vertices which are incident to $e_t^1, e_t^2, \dots, e_t^k$. Hence $|D'| \geq a + (n - a)k$, where $a = |V(G) \cap D'|$. Since $k \geq 2$ and $a \leq n$, then $|D'| \geq 2n - a \geq n$, which implies that $\gamma_{sss}(G) \geq n$. Therefore, $\gamma_{sss}(G) = n$. \square

Lemma 3.4. *Let G be a graph, and let $K_{n_1}, K_{n_2}, \dots, K_{n_t}$ be (not necessarily maximal) disjoint cliques of G , where $\min\{n_1, n_2, \dots, n_t\} \geq 2$. Then $\gamma_{sss}(G) \geq \sum_{i=1}^t (n_i - 2)$.*

Proof. Let D be any SSSDS of G . If $|V(K_{n_i}) \cap D| \leq n_i - 3$ for some i , $1 \leq i \leq t$, then the induced subgraph $\langle V(K_{n_i}) \setminus D \rangle$ is a clique of $\langle V \setminus D \rangle$ of order at least 3, which is a contradiction. Therefore $|V(K_{n_i}) \cap D| \geq n_i - 2$ for every $i = 1, 2, \dots, t$, so $\gamma_{sss}(G) \geq \sum_{i=1}^t (n_i - 2)$. \square

Theorem 3.5. *Let G be a connected split graph with split partition $S \cup I$, where $\langle S \rangle$ is a clique and I is an independent set of G . Then $|S| - 2 \leq \gamma_{sss}(G) \leq |S|$. Furthermore,*

- (1) $\gamma_{sss}(G) = |S|$ if, and only if, for every $x \in S$ it holds that either x has a private neighbor or $|N(x) \cap I| \geq 2$.
- (2) $\gamma_{sss}(G) = |S| - 1$ if, and only if, there is at least one vertex x having no private neighbors and such that $|N(x) \cap I| \leq 1$, and every $y \in S$, $y \neq x$ has at least one neighbor in I .
- (3) $\gamma_{sss}(G) = |S| - 2$ if, and only if, there are at least two vertices in S having no neighbors in I .

Proof. The inequality $\gamma_{sss}(G) \geq |S| - 2$ follows from Lemma 3.4, since it holds trivially if $|S| = 1$. On the other hand, since G is connected then S is a dominating set, and $V \setminus S = I$ is an independent set. Therefore, S is always an SSSDS of G , hence $\gamma_{sss}(G) \leq |S|$.

(i) Assume that $\gamma_{sss}(G) = |S|$. If there exists $x \in S$ such that x has no private neighbor and $|N(x) \cap I| \leq 1$, then there exists $y \in S$, $y \neq x$, such that $N(x) \cap I \subseteq N(y)$. This implies that $S \setminus \{x\}$ is an SSSDS of G , which is a contradiction. Conversely, assume that each $x \in S$ has either a private neighbor or $|N(x) \cap I| \geq 2$. Suppose there exists $x \in S$ such that $D = S \setminus \{x\}$ is an SSSDS of G . Then clearly x has no private neighbors, because D is a dominating set of G . Since $|N(x) \cap I| \geq 2$, we have $\deg_{\langle V \setminus D \rangle}(x) \geq 2$, and so D is not an SSSDS of G . Therefore, $\gamma_{sss}(G) = |S|$.

(ii) Assume that $\gamma_{sss}(G) = |S| - 1$. Then there must be at least one vertex x having no private neighbors and such that $|N(x) \cap I| \leq 1$. Suppose there are two vertices x, y in S such that $|N(x) \cap I| = |N(y) \cap I| = 0$. Then $S \setminus \{x, y\}$ is an SSSDS of G , which is a contradiction. Conversely, if there is one vertex x having no private neighbors and such that $|N(x) \cap I| \leq 1$, and every $y \in S$, $y \neq x$ has at least one neighbor in I , then $S \setminus \{x\}$ is an SSSDS of G , but $S \setminus \{u, v\}$ is not an SSSDS of G for any $\{u, v\} \subset S$.

(iii) Assume that $\gamma_{sss}(G) = |S| - 2$, and let $D = S \setminus \{x, y\}$ be a minimum SSSDS of G . Suppose $|N(x) \cap I| \geq 1$. Then $\deg_{\langle V \setminus D \rangle}(x) \geq 2$, which is a contradiction. Conversely, if there are two vertices x, y in S having no neighbors in I , then $S \setminus \{x, y\}$ is an SSSDS of G . \square

4. Conclusions and scope

In this paper we introduced the concept of semi-strong split domination and began the study of γ_{sss} . Of course, these are only introductory steps, and there is still much to be discovered in this topic. Among the questions raised by this research, the following are of particular interest to the authors:

(1) Characterize graphs G for which $\gamma(G) = \gamma_{sss}(G)$.

(2) Characterize graphs G for which $\gamma_s(G) = \gamma_{sss}(G)$.

Corollary 2.9 implies that if $\gamma(G) = \gamma_{sss}(G)$, then $\gamma_s(G) = \gamma_{sss}(G)$. The converse is not true.

(3) Characterize graphs G for which $\gamma_{sss}(G) = \gamma_{ss}$.

(4) Characterize graphs G without isolated vertices such that $\gamma_{sss}(G) + \gamma(G) = n$.

As noticed earlier, it is enough to solve the problem for connected graphs.

(5) Given positive integers a , b , c , and d with $a \leq b \leq c \leq d$, under which conditions there exists a graph G such that $\gamma(G) = a$, $\gamma_s(G) = b$, $\gamma_{sss}(G) = c$ and $\gamma_{ss}(G) = d$?

It is also interesting to consider the problem in general: Given a graph $G = (V, E)$ and a positive integer k , a set $S \subseteq V$ is *k-split dominating* if it is dominating and $\langle V \setminus S \rangle$ has no vertices of degree greater than k . Then a strong split dominating set is a 0-split dominating set, a semi-strong split dominating set is a 1-split dominating set, and so on. The *k-split dominating number* γ_s^k is defined accordingly. Following the idea of Proposition 2.7, it is easily shown that for every positive integer t , there is a connected graph such that $\gamma_s^k - \gamma_s^{k-1} \geq t$: The graph consisting on t cliques of order p sharing a common vertex, where $p \geq k + 2$. This suggests several questions, as well as useful applications, regarding this generalized concept.

Another possible variant is the following: Given a graph $G = (V, E)$ and a positive integer k , a set $S \subseteq V$ is *k-clique dominating* if it is dominating and $\langle V \setminus S \rangle$ has no cliques of order greater than k . Then a strong split dominating set is a 1-clique dominating set, a semi-strong split dominating set is a 2-clique dominating set, etc. Again, we define the *k-clique dominating number* γ_c^k , and for every positive integer t there is a connected graph such that $\gamma_c^k - \gamma_c^{k-1} \geq t$ (same example as in the previous paragraph, but asking only $p \geq k + 1$, because of the "shift" in notation). It is straightforward that for every positive integer $k \geq 2$, $\gamma_c^{k+1} \leq \gamma_s^k$ (as mentioned earlier, equality holds in every graph for $k = 1, 2$).

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