

## CONNECTED GRAPHS COSPECTRAL WITH A FRIENDSHIP GRAPH

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ABSTRACT. Let  $n$  be any positive integer, the friendship graph  $F_n$  consists of  $n$  edge-disjoint triangles that all of them meeting in one vertex. A graph  $G$  is called cospectral with a graph  $H$  if their adjacency matrices have the same eigenvalues. Recently in <http://arxiv.org/pdf/1310.6529v1.pdf> it is proved that if  $G$  is any graph cospectral with  $F_n$  ( $n \neq 16$ ), then  $G \cong F_n$ . Here we give a proof of a special case of the latter: Any connected graph cospectral with  $F_n$  is isomorphic to  $F_n$ . Our proof is independent of ones given in <http://arxiv.org/pdf/1310.6529v1.pdf> and the proofs are based on our recent results given in [*Trans. Comb.*, **2** no. 4 (2013) 37-52.] using an upper bound for the largest eigenvalue of a connected graph given in [*J. Combinatorial Theory Ser. B* **81** (2001) 177-183.].

### 1. Introduction

The friendship graph  $F_n$  is the graph consisting of  $n$  edge-disjoint cycles of length 3, all meeting in a common vertex (see Figure 1). The adjacency spectrum  $Spec(G)$  of a graph  $G$  is the multiset of eigenvalues of its adjacency matrix. If distinct eigenvalues of the graph  $G$  are  $\lambda_1, \dots, \lambda_k$  with multiplicities  $m_1, \dots, m_k$  respectively, then we denote  $Spec(G)$  by  $\{\lambda_1^{m_1}, \dots, \lambda_k^{m_k}\}$ . A graph  $G$  is called determined by its adjacency spectrum (for short *DS*), if  $Spec(G) = Spec(H)$  for some graph  $H$ , then  $G \cong H$ . In [5, 6], it is conjectured that the friendship graph is *DS*. In [3], it is claimed that the conjecture is valid, but it is noted in [1] that the proof has a flaw. In [1] some results on graphs cospectral with  $F_n$  has been obtained. Finally in [2] a more general result about graphs with all but two eigenvalues equal to  $\pm 1$  is obtained and as a corollary it is shown that  $F_n$  is *DS* if  $n \neq 16$  and  $F_{16}$  is cospectral with the disjoin union of 10 complete graph  $K_2$  with the graph given in Figure 2. The spectrum of the graph of Figure 2 is  $\{(1 \pm \sqrt{129})/2, 1^5, -1^6\}$ . Our main result is the following:

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**Theorem 1.1.** Any connected graph cospectral with  $F_n$  is isomorphic to  $F_n$ .

The proof is based on our previous recent results in [1] by using an upper bound given in [4] for the spectral radius of a connected graph in terms of the minimum degree of the graph and its number of vertices and edges. The proof is independent of that of given in [2].

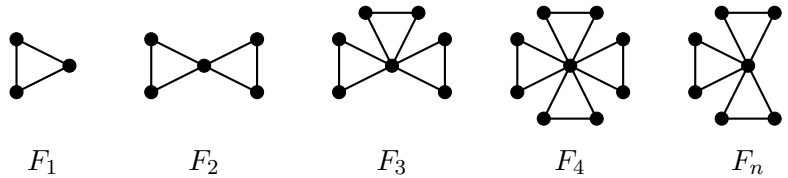


FIGURE 1. Friendship graphs  $F_1, F_2, F_3, F_4$  and  $F_n$

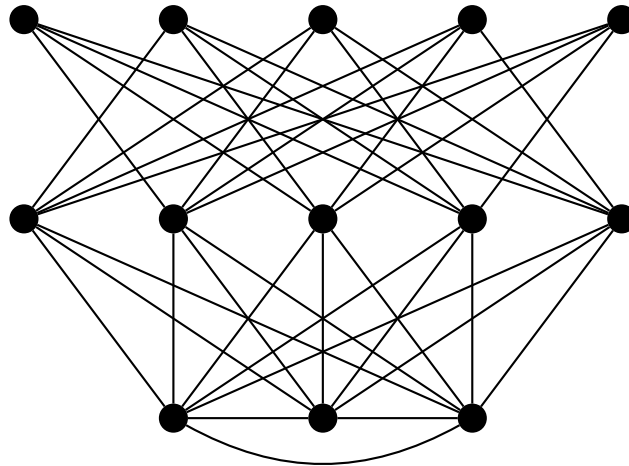


FIGURE 2. The disjoint union of the above graph with  $10K_2$  is cospectral with  $F_{16}$

### 2. Proof of Theorem 1.1

In the following, we give results that we will need in the proof of Theorem 1.1.

**Proposition 2.1.** [1, Proposition 2.3] Let  $F_n$  denote the friendship graph with  $2n + 1$  vertices. Then

$$Spec(F_n) = \left\{ \left( \frac{1}{2} - \frac{1}{2}\sqrt{1 + 8n} \right)^1, (-1)^n, (1)^{n-1}, \left( \frac{1}{2} + \frac{1}{2}\sqrt{1 + 8n} \right)^1 \right\}.$$

The maximum eigenvalue of a graph  $G$  is called spectral radius and it is denoted by  $\varrho(G)$ .

**Lemma 2.2.** [1, Lemma 3.2] Let  $G$  be a connected graph that is cospectral with  $F_n$  and  $\delta(G)$  be the minimum degree of  $G$ . Then,  $\delta(G) = 2$  and  $G$  has at least  $1 + \varrho(F_n)$  vertices with this minimum degree.

**Theorem 2.3.** [1, Theorem 3.11] *Let  $G$  be a graph cospectral with  $F_n$  and  $G$  has two adjacent vertices of degree 2. Then  $G$  is isomorphic to  $F_n$ .*

A graph is called bidegreed if the set of degrees of vertices consists of exactly two distinct elements.

**Theorem 2.4.** [4, Theorem 2.3] *Let  $G$  be a simple graph with  $n$  vertices and  $m$  edges. Let  $\delta = \delta(G)$  be the minimum degree of vertices of  $G$  and  $\varrho(G)$  be the spectral radius of the adjacency matrix of  $G$ . Then*

$$\varrho(G) \leq \frac{\delta - 1}{2} + \sqrt{2m - n\delta + \frac{(\delta + 1)^2}{4}}.$$

*Equality holds if and only if  $G$  is either a regular graph or a bidegreed graph in which each vertex is of degree either  $\delta$  or  $n - 1$ .*

Now we are ready to prove the main result.

**Proof of Theorem 1.1.** Let  $G$  be a connected graph and cospectral with the friendship graph  $F_n$ . By Proposition 2.1,  $\varrho(G) = \frac{1}{2} + \frac{1}{2}\sqrt{1 + 8n}$  and the number of vertices and edges of  $G$  are the same as  $F_n$ . Also by Lemma 2.2,  $\delta = \delta(G) = 2$ . Therefore, the equality in Theorem 2.4 holds and so  $G$  is either a regular graph or a bidegreed graph in which each vertex is of degree either 2 or  $2n$ . If  $n = 1$ , then it is easy to see that  $G \cong F_1 = K_3$ . If  $n > 1$  then  $F_n$  is not regular and so  $G$  is not a regular graph, since regularity can be determined by the adjacency spectrum of a graph. Hence every vertex of  $G$  has degree 2 or  $2n$ . Since  $G$  has  $2n + 1$  vertices and  $3n$  edges, it follows that  $G$  has only one vertex of degree  $2n$  and all other vertices has degree 2. Thus  $G$  has at least two adjacent vertices of degree 2. Now Theorem 2.3 completes the proof.  $\square$

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