www.combinatorics.ir

Transactions on Combinatorics
ISSN (print): 2251-8657, ISSN (on-line): 2251-8665
Vol. 3 No. 2 (2014), pp. 35-44.

www.ui.ac.ir

GLOBAL MINUS DOMINATION IN GRAPHS

M. ATAPOUR, S. NOROUZIAN AND S. M. SHEIKHOLESLAMI*

Communicated by Manouchehr Zaker

Abstract. A function $f : V(G) \to \{-1, 0, 1\}$ is a *minus dominating function* if for every vertex $v \in V(G)$, $\sum_{u\in N[v]} f(u) \geq 1$. A minus dominating function f of G is called a *global minus dominating function* if f is also a minus dominating function of the complement $\overline{G}$ of G. The *global minus domination number* $\gamma_g^-(G)$ of G is defined as $\gamma_g^-(G) = \min\{\sum_{v\in V(G)} f(v) \mid f$ is a global minus dominating function of $G\}$. In this paper we initiate the study of the global minus domination number in graphs and we establish lower and upper bounds for the global minus domination number.

1. Introduction

Throughout this paper, we only consider finite undirected graphs with neither loops nor multiple edges. Let $G = (V, E)$ be a graph with vertex set $V(G)$ and edge set $E(G)$ (briefly V and E). For every vertex $v \in V$, the open neighborhood $N(v)$ is the set $\{u \in V \mid uv \in E\}$ and its *closed neighborhood* is the set $N[v] = N(v) \cup \{v\}$. The *open neighborhood* of a set $S \subseteq V$ is the set $N(S) = \cup_{v\in S} N(v)$ and its *closed neighborhood* is $N[S] = N(S) \cup S$. The *minimum* and *maximum degree* of G are respectively denoted by δ and Δ. For a real-valued function $f : V \to \mathbb{R}$ the *weight* of f is $\omega(f) = \sum_{v\in V} f(v)$, and for $S \subseteq V$ we define $f(S) = \sum_{v\in S} f(v)$, so $\omega(f) = f(V)$. For a vertex v in V, we denote $f(N[v])$ by $f[v]$. For a vertex v in a rooted tree T, we let $C(v)$ denote the set of children of v and let $D(v)$ denote the set of descendants of v and $D[v] = D(v) \cup \{v\}$. The *maximal subtree* at v is the subtree of T induced by $D[v]$, and is denoted by T_v. A leaf of T is a vertex of degree 1 and a support vertex is a vertex adjacent to a leaf. The set of leaves and the set of support vertices in T are denoted by $L(T)$ and $S(T)$, respectively. Consult [19] for terminology and notation which are not defined here.

MSC(2010): Primary: 05C69; Secondary: 05C05

Keywords: minus domination number, global minus dominating function, global minus domination number.

Received: 9 February 2014, Accepted: 5 April 2014.

*Corresponding author.

A subset S of vertices of G is a *dominating set* if $N[S] = V$. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G. A dominating set S of G is a *global dominating set* of G if S is also a dominating set of $\overline{G}$. The *global domination number* $\gamma_g(G)$ of G is the minimum cardinality of a global dominating set. The global domination number was introduced independently by Brigham and Dutton [4] (the term factor domination number was used) and Sampathkumar [17] and has been studied by several authors (see for example [1, 20]). Since then some variants of the global domination parameter, such as connected (total) global domination, Global Roman domination, and global signed (total) domination, have been studied [3, 2, 7, 15, 16].

Let $f : V \to \{-1, 1\}$ be a function which assigns to each vertex of G an element of the set $\{-1, 1\}$. The function f is said to be a *signed dominating function* (SDF) of G (see [10]) if $f[v] \geq 1$ for every $v \in V$. The *signed domination number* of G, denoted by $\gamma_s(G)$, is the minimum weight of a signed dominating function on G. A $\gamma_s(G)$-function is a SDF f of G with $\omega(f) = \gamma_s(G)$. The signed domination numbers have been studied by several authors [5, 11, 12, 13, 14]. A signed dominating function $f : V(G) \to \{-1, 1\}$ is called a *global signed dominating function* (GSDF) if f is also a SDF of the complement $\overline{G}$ of G. The global signed domination number of G, denoted by $\gamma_{gs}(G)$, is the minimum weight of a GSDF on G. The global signed dominating function was introduced by Karami et al. in [15].

Let $f : V \to \{-1, 0, 1\}$ be a function. The function f is said to be a *minus dominating function* (abbreviated, MDF) of G if $f[v] \geq 1$ for every $v \in V$. The *minus domination number* of G, denoted by $\gamma^-(G)$, is the minimum weight of a minus dominating function of G. A $\gamma^-(G)$-function is a MDF f of G with $\omega(f) = \gamma^-(G)$. The minus domination number was introduced by Dunbar et al. in [8] and has been studied by several authors (see for example [6, 9]).

A minus dominating function f of G is called a *global minus dominating function* (abbreviated, GMDF) if f is also a MDF of the complement $\overline{G}$ of G. This definition is parallel to the definitions of a global signed dominating function of a graph defined in [15] and global signed total dominating function defined in [3]. The *global minus domination number* of G, denoted by $\gamma_g^-(G)$, is the minimum weight of a GMDF of G. A $\gamma_g^-(G)$-function is defined similarly. For a (global) minus dominating function f of G we define $P_f = \{v \in V \mid f(v) = 1\}$, $Z_f = \{v \in V \mid f(v) = 0\}$ and $M_f = \{v \in V \mid f(v) = -1\}$. Since every GMDF of G is a MDF on both G and $\overline{G}$ and since every global dominating function is a global minus dominating function, we have

$$\max\{\gamma^-(G), \gamma^-(\overline{G})\} \leq \gamma_g^-(G) \leq \gamma_g(G). \tag{1.1}$$

On the other hand, every global signed dominating function is a global minus dominating function. Hence the global signed domination and global minus domination number of a graph are related as follows:

$$\gamma_g^-(G) \leq \gamma_{gs}(G). \tag{1.2}$$

Note that the global minus domination number can differ significantly from the global signed domination number. For example, for $n \geq 3$, $\gamma_g^-(K_{1,n}) = 2$ and $\gamma_{gs}(K_{1,n}) = n$.

Our purpose in this paper is to initiate the study of the global minus domination in graphs. First we establish bounds for the global minus domination number of a graph. Then we prove that for any tree T of order $n \geq 2$, $\gamma_g^-(T) \leq \gamma^-(T) + 1$ and characterize all trees satisfying the equality.

We make use of the following results in this paper.

Theorem A. ([17]) For any graph G of order n, $\gamma_g(G) = n$ if and only if $G = K_n$ or $G = \overline{K_n}$.

An immediate consequence of (1.1) and Theorem A now follows.

Corollary 1.1. Let G be a graph of order n. Then $\gamma_g^-(G) = n$ if and only if $G = K_n$ or $G = \overline{K_n}$.

Theorem B. ([8]) For every graph G, $\gamma^-(G) \leq \gamma(G)$.

Theorem C. ([6]) If $\Delta(G) \leq 3$, then $\gamma^-(G) \geq \frac{n}{5}$.

Next result is an immediate consequence of Theorem C.

Corollary 1.2. For every graph G of order $n \geq 16$ with $2 \leq \delta \leq \Delta \leq 3$, $\gamma_g^-(G) = \gamma^-(G)$.

Proof. Assume f is a $\gamma^-(G)$-function and $v \in V$. We show that f is also a MDF of $\overline{G}$. By Theorem C, we have $\gamma^-(G) \geq \frac{n}{5}$ and hence $\gamma^-(G) \geq 4$. Since $\Delta(G) \leq 3$, $f(N_G(v)) \leq 3$ implying that

$$f(N_{\overline{G}}[v]) \geq \gamma^-(G) - 3 \geq 1.$$

Now the result follows by (1.1). □

Theorem D. ([8]) For every tree T, $\gamma^-(T) \geq 1$ with equality if and only if T is a star.

Theorem E. ([8]) For any path P_n and cycle C_n on n vertices, $\gamma^-(P_n) = \gamma^-(C_n) = \lceil \frac{n}{3} \rceil$.

Theorem F. ([4]) For $n \geq 3$, $\gamma_g(C_n) = \begin{cases} 3 & \text{if} \quad n = 3, 5 \\ \lceil \frac{n}{3} \rceil & \text{otherwise.} \end{cases}$

Next result is an immediate consequence of (1.1), Corollary 1.1 and Theorems E and F.

Corollary 1.3. For $n \geq 3$, $\gamma_g^-(C_n) = \begin{cases} 3 & \text{if} \quad n = 3, 5 \\ \lceil \frac{n}{3} \rceil & \text{otherwise.} \end{cases}$

2. Bounds on the global minus domination numbers

In this section, we give some bounds on the global minus domination numbers of general graphs. The first theorem shows that the global minus domination number of a graph is a positive integer.

Theorem 2.1. Let G be a graph of order $n \geq 2$. Then

$$\gamma_g^-(G) \geq \max\{2, \gamma^-(G), \gamma^-(\overline{G})\}.$$

Furthermore, this bound is sharp.

Proof. By (1.1), it suffices to prove that $\gamma_g^-(G) \geq 2$. Let f be a $\gamma_g^-(G)$-function. If $M, Z = \emptyset$, then we are done. First let $M \neq \emptyset$ and $x \in M$. Then

$$|N_G(x) \cap P| \geq |N_G(x) \cap M| + 2 \tag{2.1}$$

and

$$|N_{\overline{G}}(x) \cap P| \geq |N_{\overline{G}}(x) \cap M| + 2. \tag{2.2}$$

By (2.1) and (2.2),

$$|N_G(x) \cap P| + |N_{\overline{G}}(x) \cap P| \geq |N_G(x) \cap M| + |N_{\overline{G}}(x) \cap M| + 4.$$

Since $x \in M$, we deduce that $|P| \geq |M| + 3$ and so $\gamma_g^-(G) = |P| - |M| \geq 3$.

Let now $M = \emptyset$ and $Z \neq \emptyset$. Assume $x \in Z$. Then $|N_G(x) \cap P| \geq 1$ and $|N_{\overline{G}}(x) \cap P| \geq 1$ implying that $\gamma_g^-(G) = |P| \geq 2$. Thus $\gamma_g^-(G) \geq \max\{2, \gamma^-(G), \gamma^-(\overline{G})\}$.

To prove sharpness, let $G = K_{1,n}$, v be the center $K_{1,n}$ and $N(v) = \{v_1, \ldots, v_n\}$. Then $\overline{G} = K_n \cup K_1$. By Theorems D and B, $\gamma^-(G) = 1$ and $\gamma^-(\overline{G}) = 2$. Hence $\max\{2, \gamma^-(G), \gamma^-(\overline{G})\} = 2$. Define $g : V(G) \to \{-1, 0, 1\}$ by $g(v) = g(v_1) = 1$ and $g(x) = 0$ for $x \in V(G) - \{v, v_1\}$. It is easy to see that g is a GMDF of G of weight 2 and so $\gamma_g^-(G) = 2$. This completes the proof. □

A 2-packing set of a graph G is a subset $S \subseteq V(G)$ such that for any pair of distinct vertices $u, v \in S$, $d(u, v) \geq 3$. The 2-packing number $\rho(G)$ is the maximum cardinality of a 2-packing set of G.

Proposition 2.2. For any graph G of order n with $\delta(G) \geq 2$,

$$\gamma_g^-(G) \leq n - \rho(G) + 1.$$

Furthermore, the bound is sharp for complete graphs.

Proof. If $\rho(G) = 1$, then the result is immediate. Let $\rho(G) \geq 2$ and let $S = \{x_1, \ldots, x_{\rho(G)}\}$ be a 2-packing set of G.

First let $\rho(G)$ be odd. Define $g : V(G) \to \{-1, 0, 1\}$ by $g(x_i) = -1$ for $1 \leq i \leq \lfloor \frac{\rho(G)}{2} \rfloor$ and $g(x) = 1$ otherwise. Since S is a 2-packing, $|N_G[x] \cap S| \leq 1$ for each $x \in V(G)$. It follows from $\delta(G) \geq 2$ that g is a MDF of G. Now we show that g is a MDF of $\overline{G}$. Since the subgraph $\overline{G}[S]$ induced by S in $\overline{G}$ is complete, we have $\sum_{x \in N_{\overline{G}[x_i]}} g(x) \geq \sum_{i=1}^{\rho(G)} g(x_i) = 1$ for each i. On the other hand, since $|N_G[v] \cap S| \leq 1$ for each $v \in V(G) - S$, we can see that $\sum_{x \in N_{\overline{G}}[v]} g(x) \geq 1$. It follows that g is a MDF of $\overline{G}$. Thus g is a GMDF of G and hence $\gamma_g^-(G) \leq n - \rho(G) + 1$ when $\rho(G)$ is odd.

Now let $\rho(G)$ be even. Define $g : V(G) \to \{-1, 0, 1\}$ by $g(x_i) = -1$ for $1 \leq i \leq \lfloor \frac{\rho(G)}{2} \rfloor - 1$, $g(x_{\frac{\rho(G)}{2}}) = 0$ and $g(x) = 1$ otherwise. As above we can see that f is a GMDF of G and hence $\gamma_g^-(G) \leq n - \rho(G) + 1$ when $\rho(G)$ is even. This completes the proof. □

Moo Young Sohn et al. (Theorem 7 in [18]), proved that every connected cubic graph G whose every vertex is contained in at least one triangle satisfies $\rho(G) \geq \frac{n}{6}$. Using this and Proposition 2.2 we obtain the next result.

Corollary 2.3. Let G be connected cubic graph in which every vertex is contained in at least one triangle. Then $\gamma_g^-(G) \leq \frac{5n}{6} + 1$.

Next we show that the differences $\gamma_g(G) - \gamma_g^-(G)$ and $\gamma_g^-(G) - \max\{\gamma^-(G), \gamma^-(\overline{G})\}$ can be arbitrarily large.

Theorem 2.4. For every integer $k \geq 1$, there exists a graph G such that both of G and $\overline{G}$ are connected and

$$\gamma_g(G) - \gamma_g^-(G) \geq k.$$

Proof. Let G be the graph with vertex set $V(G) = \{u_i, v_i \mid 0 \leq i \leq k+2\} \cup \{w_1, \ldots, w_{k+2}\}$ and edge set $E(G) = \{v_i v_{i+1} \mid 0 \leq i \leq k+1\} \cup \{u_i v_i \mid 0 \leq i \leq k+2\} \cup \{w_i v_{i-1}, w_i v_i \mid 1 \leq i \leq k+2\}$. Obviously $\{u_0, v_1, \ldots, v_{k+2}\}$ is a global dominating set of G and hence $\gamma_g(G) \leq k+3$. On the other hand, every dominating set S of G satisfies $S \cap \{u_i, v_i\} \neq \emptyset$ for each i and so $\gamma_g(G) \geq \gamma(G) = k+3$. Thus $\gamma_g(G) = k+3$. Now define $f : V(G) \to \{-1, 0, 1\}$ by $f(u_0) = f(u_1) = 1, f(v_i) = 1$ for $0 \leq i \leq k+2$, $f(u_i) = 0$ for $2 \leq i \leq k+2$ and $f(x) = -1$ otherwise. It is easy to see that f is a GMDF of G implying that $\gamma_g^-(G) \leq 3$. Therefore $\gamma_g(G) - \gamma_g^-(G) \geq k$. □

Theorem 2.5. For every positive integer k, there exists a graph G such that both of G and $\overline{G}$ are connected and

$$\gamma_g^-(G) - \max\{\gamma^-(G), \gamma^-(\overline{G})\} \geq 2k.$$

Proof. The following graph was introduced by Karami et al. in [15]. Let G be the graph with vertex set $V(G) = \{u_i, v_i \mid 0 \leq i \leq 4k-1\}$ and edge set $E(G) = \{v_i v_j \mid 0 \leq i \neq j \leq 4k-1\} \cup \{u_i v_i, u_i v_{i+1}, \ldots, u_i v_{i+2k-1} \mid 0 \leq i \leq 4k-1\}$, where the sum is taken modulo $4k$. Clearly $G \simeq \overline{G}$ and so $\gamma^-(G) = \gamma^-(\overline{G})$. It is easy to see that the function $f : V(G) \to \{-1, 0, 1\}$ defined by $f(v_i) = 1$ if $i \in \{0, 1, \ldots, 3k\}$ and $f(x) = -1$ otherwise, is a MDF of G implying that $\gamma^-(G) \leq \omega(f) = 2 - 2k$. Therefore $\max\{\gamma^-(G), \gamma^-(\overline{G})\} \leq 2 - 2k$. It follows from Theorem 2.1 that $\gamma_g^-(G) - \max\{\gamma^-(G), \gamma^-(\overline{G})\} \geq 2k$. □

3. **Trees**

In this section we study the global minus domination numbers in trees. We begin with the following lemma.

Lemma 3.1. Let T be a tree of order $n \geq 2$ and f a $\gamma^-(T)$- function. Then $\sum_{u \in N_{\overline{T}}[v]} f(u) \geq 0$ for every $v \in V(T)$.

Proof. If $\text{diam}(T) \leq 2$, then $T = K_2$ or T is a star and clearly the theorem is true. Suppose $\text{diam}(T) \geq 3$. Then $\gamma^-(T) \geq 2$ by Theorem D. Let $v \in V(T)$ and root T at v. If v is a leaf adjacent to w, then

$$\sum_{u \in N_{\overline{T}}[v]} f(u) = f(V(T)) - f(w) \geq 1.$$

Assume v is not a leaf and let $N(v) = \{v_1, v_2, \ldots, v_t\}$. Suppose without loss of generality that $v_1, \ldots, v_s$ are the neighbors of v with degree at least 2. Consider three cases.

Case 1. $f(v) = -1$.

Then v is not a support vertex and the function f, restricted to T_{v_i}, is a MDF of T_{v_i} for each i with $1 \le i \le t$. By Theorem D, $f(V(T_{v_i})) \ge 1$. If $f(V(T_{v_i})) = 1$ for some i, then T_{v_i} is a star with center v_i implying that $f(N_T[v_i]) = 0$ which is a contradiction. Hence $f(V(T_{v_i})) \ge 2$ for each i with $1 \le i \le t$. It follows that $f(N_{\overline{T}}(v) \cap V(T_{v_i})) = f(V(T_{v_i})) - f(v_i) \ge 2 - f(v_i) \ge 1$. Since $t \ge 2$, we obtain

$$\sum_{u\in N_{\overline{T}}[v]} f(u) = \sum_{i=1}^{t} f(N_{\overline{T}}(v) \cap V(T_{v_i})) + f(v) \ge 1.$$

Case 2. $f(v) = 0$.

Then the function f, restricted to T_{v_i} is a MDF of T_{v_i} and so by Theorem D, $f(V(T_{v_i})) \ge 1$ for each i with $1 \le i \le t$. If $f(V(T_{v_i})) \ge 2$ for some i, then

$$\sum_{u\in N_{\overline{T}}[v]} f(u) = \sum_{i=1}^{t} f(N_{\overline{T}}(v) \cap V(T_{v_i})) = \sum_{i=1}^{t} (f(V(T_{v_i})) - f(v_i)) \ge 1.$$

Let now $f(V(T_{v_i})) = 1$ for each i with $1 \le i \le t$. Then by Theorem D, T_{v_i} is a star with center v_i such that $f(v_i) = 1$ and $f(u) = 0$ for $u \in C(v_i)$. This implies that

$$\sum_{u\in N_{\overline{T}}[v]} f(u) = \sum_{i=1}^{t} f(N_{\overline{T}}(v) \cap V(T_{v_i})) = \sum_{i=1}^{t} (f(V(T_{v_i})) - f(v_i)) \ge 0.$$

Case 3. $f(v) = 1$.

For each i with $1 \le i \le s$, suppose T_i is the subtree induced by $V(T_{v_i}) \cup \{v\}$. It is easy to see that $f(V(T_i)) \ge 1$ for each i. If $f(V(T_i)) = 1$ for some $1 \le i \le s$, then T_i is a star with center v_i by Theorem D and we must have $f(v_i) = 1$ and $f(u) = 0$ for each $u \in N(v_i)$ which is a contradiction. Thus we may assume $f(V(T_i)) \ge 2$ for each i with $1 \le i \le s$. It follows that

$$\sum_{u\in N_{\overline{T}}[v]} f(u) = \sum_{i=1}^{s} f(N_{\overline{T}}(v) \cap V(T_i)) + f(v) = \sum_{i=1}^{s} f(V(T_i) - f(v_i) - f(v)) + 1 \ge 1.$$

□

By a closer look at the proof of Lemma 3.1 we can see that:

Corollary 3.2. Let T be a tree of order $n \ge 2$ that is rooted at $v \in V(T)$, and let f be a $\gamma^-(T)$-function. If $\sum_{u\in N_{\overline{T}}[v]} f(u) = 0$, then either T is a star and v is a leaf of T or $\deg(v) \ge 2$, $f(v) = 0$ and for each $u \in N(v)$, T_u is a star such that f assigns 1 to u and 0 to its children.

Theorem 3.3. For any tree T of order $n \ge 2$,

$$\gamma_g^-(T) \le \gamma^-(T) + 1.$$

Proof. If T is a star, then $\gamma_g^-(T) = 2$ and $\gamma^-(T) = 1$ and so $\gamma_g^-(T) = \gamma^-(T) + 1$. Assume T is not a star and let f be a $\gamma^-(T)$-function. If f is a GMDF of T, then $\gamma_g^-(T) = \gamma^-(T)$ as desired. Assume f is not a GMDF of T and let v be a vertex of T for whcih $\sum_{u\in N_{\overline{T}}[v]} f(u) \le 0$. Then $\sum_{u\in N_{\overline{T}}[v]} f(u) = 0$ by

Lemma 3.1. Root T at v. By Corollary 3.2, $\deg(v) \geq 2$, $f(v) = 0$ and T_u is a star for each $u \in N(v)$, and f assigns 1 to every neighbor of v and 0 to the other vertices. Let $N(v) = \{v_1, \ldots, v_t\}$ and let $\deg(v_i) \geq 2$ for $1 \leq i \leq s$ where $s \leq t$. Since $f(v) = 0$, the function f, restricted to $V(T_{v_i})$ is a MDF of T_{v_i} for each i with $1 \leq i \leq t$. Hence by Theorem D, $f(V(T_{v_i})) \geq 1$ for each i. Let w be a leaf in T_{v_1} and define $g : V(T) \to \{-1, 0, 1\}$ by $g(w) = 1$ and $g(x) = f(x)$ for $x \in V(T) - \{w\}$. We claim that g is a GMDF of T. Obviously, g is a MDF of T. Let $x \in V(T)$. If $x = v$, then clearly $g(N_{\overline{T}}[x]) = f(N_{\overline{T}}[v]) + 1 = 1$. If $x \in \{v_2, \ldots, v_t\}$, then

$$g(N_{\overline{T}}[x]) = g(V(T_{v_1})) + \sum_{i\geq 2; v_i \neq x} f(V(T_{v_i})) + f(x) \geq t + 1 \geq 1.$$

If $x = v_1$, then

$$g(N_{\overline{T}}[x]) = f(N_{\overline{T}}[v_1]) = \sum_{i=2}^{t} f(V(T_{v_i})) + f(v_1) \geq t \geq 1.$$

Let $x \notin N_T[v]$. Then x is a leaf whose support vertex is v_i for some i. Then $g(N_{\overline{T}}[x]) = g(V(T)) - g(v_i) \geq 1$. Thus g is a GMDF of T and hence

$$\gamma_g^-(T) \leq \omega(g) = \omega(f) + 1 = \gamma^-(T) + 1.$$

□

In what follows, we characterize all extremal trees which attain the bound in Theorem 3.3. Let $\mathcal{F}$ be the family of all trees obtained from disjoint union of stars $S_1, \ldots, S_t$ either all of order 1 or all of order at least three, by adding a new vertex v and joining v to the centers of all stars.

Observation 3.4. If T is a tree and f is a $\gamma^-(T)$-function, then f assigns 0 or 1 to every leaf and every support vertex.

Observation 3.5. If T is a double $S(r, s)$ with $s \geq r \geq 1$, then $\gamma_g^-(T) = \gamma^-(T)$.

Proof. Let x and y be the centers of $S(r, s)$. Define f by $f(x) = f(y) = 1$ and $f(u) = 0$ otherwise. It is easy to see that f is a GMDF of T of weight 2. Hence $\gamma_g^-(T) \leq 2$ and it follows from Theorem 2.1 that $\gamma_g^-(T) = 2$.

Since T is not a star, by Theorem D we have $\gamma^-(T) \geq 2$ and by (1.1) we obtain $2 = \gamma_g^-(T) \geq \gamma^-(T) \geq 2$. Hence $\gamma_g^-(T) = \gamma^-(T)$. □

Lemma 3.6. If $T \in \mathcal{F}$, then $\gamma_g^-(T) = \gamma^-(T) + 1$.

Proof. Let $T \in \mathcal{F}$. Then T is obtained from disjoint union of stars $S_1, \ldots, S_t$ by adding a new vertex v and joining v to the centers of all stars. If $S_1 = \cdots = S_t = K_1$ or $t = 1$, then T is a star implying that $\gamma_g^-(T) = 2$ and $\gamma^-(T) = 1$, and so $\gamma_g^-(T) = \gamma^-(T)+1$. Let $t \geq 2$ and $S_1, \cdots, S_t$ be stars of order at least three. Suppose $v_1, \ldots, v_t$ be the centers of $S_1, \ldots, S_t$, respectively, and let $N(v_i) - \{v\} = \{v_{i_1}, \ldots, v_{i_\ell}\}$ for each i with $1 \leq i \leq t$. Define $h : V(T) \to \{-1, 0, 1\}$ by $h(v_1) = \cdots = h(v_t) = 1$ and $h(x) = 0$ otherwise. Obviously h is a MDF of T of weight t and so $\gamma^-(G) \leq t$. Root T at v.

Assume f is a $\gamma^-(T)$-function. By Observation 3.4, $\sum_{x\in N[v_i]-\{v\}} f(x) \geq 1$ for each i with $1 \leq i \leq t$. If $f(v) = 1$, then $\omega(f) \geq t+1$ which is a contradiction.

If $f(v) = -1$ then for each i with $1 \leq i \leq t$, we must have $\sum_{x\in N[v_i]-\{v\}} f(x) \geq 2$ because $\sum_{x\in N[v_i]} f(x) \geq 1$. It follows that $\gamma^-(T) = \omega(f) = f(v) + \sum_{i=1}^{t}(\sum_{x\in N[v_i]-\{v\}} f(x)) \geq 2t-1 \geq t+1$, a contradiction.

Assume $f(v) = 0$. Then the function f, restricted to $V(T_{v_i})$ is a MDF of T_{v_i} with $f(V(T_{v_i})) = 1$, otherwise we get a contradiction as above. Hence $f(v_1) = \cdots f(v_t) = 1$ and $f(x) = 0$ for the other vertices. In fact f is the unique $\gamma^-(T)$-function which is not a GMDF of T. This implies that $\gamma_g^-(T) \geq \gamma^-(T) + 1$ and the result follows by Theorem 3.3. □

Lemma 3.7. Let T be a tree with $\gamma_g^-(T) = \gamma^-(T) + 1$. Then $T \in \mathcal{F}$.

Proof. Let $\gamma_g^-(T) = \gamma^-(T)+1$. If $\mathrm{diam}(T) \leq 2$, then T is a star and clearly $T \in \mathcal{F}$. Let $\mathrm{diam}(T) \geq 3$. It follows from Observation 3.5 that $\mathrm{diam}(T) \geq 4$. Let f be a $\gamma^-(T)$- function. Since f is not a GMDF of T, $f(N_{\overline{T}}[v]) \leq 0$ for some vertex $v \in V(T)$. It follows from Lemma 3.1 that $f(N_{\overline{T}}[v]) = 0$. Root T at v. By Corollary 3.2 and the fact $\mathrm{diam}(T) \geq 4$, we deduce that $\deg(v) \geq 2$, $f(v) = 0$ and for each $u \in N(v)$, T_u is a star such that f assigns 1 to u and 0 to its children. Assume $N(v) = \{v_1, \ldots, v_t\}$ where $t \geq 2$. Since $\mathrm{diam}(T) \geq 4$, we may assume T_{v_1} and T_{v_2} are stars of order at least two. It will now be shown that v is not a support vertex. Let v be a support vertex and let without loss of generality that v_t be a leaf. Since $f(v) + f(v_t) \geq 1$, we must have $f(v_t) = 1$ because $f(v) = 0$. Define $g : V(T) \to \{-1, 0, 1\}$ by $g(v) = 1$, $g(v_t) = 0$ and $g(x) = f(x)$ otherwise. Clearly g is a MDF of T. We claim that g is a GMDF of T. If $x = v$, then $g(N_{\overline{T}}[x]) = f(N_{\overline{T}}[v]) + 1 = 1$. If $x = v_t$, then $g(N_{\overline{T}}[x]) = g(V(T)) - 1 = f(V(T)) - 1 \geq 1$. If $x = v_j$ for some $v_j \in \{v_1, \ldots, v_{t-1}\}$, then

$$g(N_{\overline{T}}[x]) = \sum_{i\neq t,j} f(V(T_{v_i})) + f(x) \geq t-1 \geq 1.$$

Finally, let $x \notin N_T[v]$. Then x is a leaf of T. Assume v_i is the support vertex of x. Then $g(v_i) = 1$ and we have $g(N_{\overline{T}}[x]) = g(V(T)) - g(v_i) \geq 1$. Therefore g is a GMDF of T implying that $\omega(g) = \omega(f) = \gamma^-(T)$ which is a contradiction. Hence, v is not a support vertex and T_{v_i} is a star of order at two for each i. Now we show that T_{v_i} is a star of order at least three for each i which implies that $T \in \mathcal{F}$. Assume to the contrary that T_{v_t} is a tree of order two and let v_t' is a leaf adjacent to v_t. It is easy to see that $h : V(T) \to \{-1, 0, 1\}$ defined by $g(v_t') = 1$, $g(v_t) = 0$ and $g(x) = f(x)$ otherwise, is a GMDF of T of weight $\omega(f)$ which is a contradiction and the proof is completed. □

Next result is an immediate consequence of Lemmas 3.6 and 3.7.

Theorem 3.8. Let T be a tree. Then $\gamma_g^-(T) = \gamma^-(T) + 1$ if and only if $T \in \mathcal{F}$.

An immediate consequence of Theorems E and 3.8 now follows.

Corollary 3.9. For $n \geq 4$, $\gamma_g^-(P_n) = \lceil \frac{n}{3} \rceil$.

Acknowledgments

This work has been supported by the Research Office of Azarbaijan Shahid Madani University.

References

[1] S. Arumugam, I. S. Hamid and K. Karuppasamy, Fractional global domination in graphs, *Discuss. Math. Graph Theory*, **30** (2010) 33–44.

[2] M. Atapour, S. M. Sheikholeslami and L. Volkmann, Global Roman domination in graphs, *Graphs Combin.*, (to appear).

[3] M. Atapour, S. M. Sheikholeslami and A. Khodkar, Global signed total domination in graphs, *Publ. Math. Debrecen*, **79** (2011) 7–22.

[4] R. C. Brigham and R. D. Dutton, Factor domination in graphs, *Discrete Math.*, **86** (1990) 127–136.

[5] E. J. Cockayne and C. M. Mynhardt, On a generalisation of signed dominating functions of a graph, *Ars Combin.*, **43** (1996) 235-245.

[6] P. Damaschke, Minus domination in small-degree graphs, *Discrete Appl. Math.*, **108** (2001) 53–64.

[7] D. Delić and C. P. Wang, The global connected domination in graphs, *Ars Combin.*, (to appear).

[8] J. Dunbar, S. Hedetniemi, M. A. Henning and A. McRae, Minus domination in graphs, *Discrete Math.*, **199** (1999) 35–47.

[9] J. Dunbar, S. Hedetniemi, M. A. Henning and A. McRae, Minus domination in regular graphs, *Discrete Math.*, **149** (1996) 311–312.

[10] J. Dunbar, S. T. Hedetniemi, M. A. Henning and P. J. Slater, *Signed domination in graphs*, Graph Theory, Combinatorics, and algorithms, **1,2**, Wiley, New York, 1995, 311-321.

[11] O. Favaron, Signed domination in regular graphs, *Discrete Math.*, **158** (1996) 287-293.

[12] Z. Füredi and D. Mubayi, Signed domination in regular graphs and set-systems, *J. Combin. Theory Ser. B*, **76** (1999) 223-239.

[13] J. H. Hattingh, M. A. Henning and P. J. Slater, The algorithmic complexity of signed domination in graphs, *Australas. J. Combin.*, **12** (1995) 101-112.

[14] T. W. Haynes, S. T. Hedetniemi, P. J. Slater and (Eds.), *Domination in Graphs: Advanced Topics*, Marcel Dekker, New York, 1998.

[15] H. Karami, R. Khoeilar, S. M. Sheikholeslami and A. Khodkar, Global signed domination in graphs, *Bull. Malays. Math. Sci. Soc.(2)*, **36** (2013) 363-372.

[16] V. R. Kulli and B. Janakiram, The total global domination number of a graph, *India J. Pure Appl. Math.*, **27** (1996) 537–542.

[17] E. Sampathkumar, The global domination number of a graph, *J. Math. Phy. Sci.*, **23** (1989) 377–385.

[18] M. Y. Sohn, J. Leey and Y. S. Kwon, Lower bounds of signed domination number of a graph, *Bull. Korean Math. Soc.*, **41** (2004) 181–188.

[19] D. B. West, *Introduction to Graph Theory*, Prentice-Hall, Inc., 1996.

[20] V. Zverovich and A. Poghosyan, On Roman, global and restrained domination in graphs, *Graphs Combin.*, **27** (2011) 755–768.

Maryam Atapour

Department of Mathematics, University of Bonab, Bonab, I. R. Iran

Email: m.atapour@bonabu.ac.ir

Sepideh Norouzian
Department of Mathematics, Azarbaijan Shahid Madani University, Tabriz, I. R. Iran

Seyed Mahmoud Sheikholeslami
Department of Mathematics, Azarbaijan Shahid Madani University, Tabriz, I. R. Iran
Email: `s.m.sheikholeslami@azaruniv.edu`