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ON THE EIGENVALUES OF FIREFLY GRAPHS

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Abstract. The sharp upper bounds and the sharp lower bounds of the largest eigenvalues λ_1, the least eigenvalue λ_n, the second largest eigenvalue λ_2, the spread and the separator among all firefly graphs on n vertices are determined.

1. Introduction

Let $G = (V(G), E(G))$ be a simple and connected graph with n vertices and $A(G)$ be the (0,1) adjacency matrix of G. The eigenvalues λ_i $(1 \leq i \leq n)$ of $A(G)$ satisfy $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. The characteristic polynomial $P(G, \lambda)$ of G is defined as $P(G, \lambda) = det(\lambda I - A(G))$. We call $\lambda_i(G)$ the i–th largest eigenvalue. The largest eigenvalue $\lambda_1(G)$ is called the index (or spectral radius) of G. In particular, $A(G)$ is irreducible if G is connected and it is well known that $\lambda_1(G)$ has the multiplicity one and there exists a unique positive unit eigenvector corresponding to $\lambda_1(G)$ by the Perron–Frobenius theory of nonnegative matrices. The spread and the separator of G are defined as $S(G) = \lambda_1(G) - \lambda_n(G)$, $S_A(G) = \lambda_1(G) - \lambda_2(G)$, respectively.

Definition 1.1. ([9]) *A firefly graph $F_{s,t,n-2s-2t-1}$ ($s \geq 0$, $t \geq 0$ and $n - 2s - 2t - 1 \geq 0$) is a graph of order n that consists of s triangles, t pendent paths of length 2 and $n - 2s - 2t - 1$ pendent edges, sharing a common vertex.*

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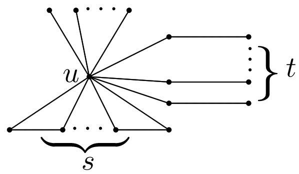

Fig.1. A firefly graph $F_{s,t,n-2s-2t-1}$

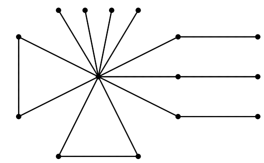

Fig.2. A firefly graph $F_{2,3,4}$

The graph $F_{2,3,4}$ illustrated in Fig.2 is an example of a firefly graph. Let $\mathscr{F}_n$ be the set of all firefly graphs $F_{s,t,n-2s-2t-1}$. Note that $\mathscr{F}_n$ contains the stars S_n ($\cong F_{0,0,n-1}$), stretched stars ($\cong F_{0,t,n-2t-1}$), friendship graphs ($\cong F_{\frac{n-1}{2},0,0}$) and butterfly graphs ($\cong F_{s,0,n-2s-1}$).

Many extremal graphs belong to $\mathscr{F}_n$. For trees, the stars S_n have the maximum spread. For unicyclic graphs, Hong [8] determined the unique graph $F_{1,0,n-3}$ with maximum largest eigenvalue. Fan et al. [7] determined the unique graph $F_{1,0,n-3}$ with minimum least eigenvalue and maximum spread among all unicyclic graphs of order n when $n \geq 12$. In [12], Petrović et al. determined the unique graph $F_{1,0,n-3}$ with minimum least eigenvalue among the cacti with n vertices ($n \geq 12$) and k cycles, where $0 \leq k \leq \lfloor \frac{n-1}{2} \rfloor$. Moreover, Li et al. [10] characterized graphs $F_{\lfloor \frac{n-1}{2} \rfloor,0,n-2\lfloor \frac{n-1}{2} \rfloor -1}$ with the largest signless Laplacian spectral radius among all the cacti with n vertices.

There are many results in the literatue about the largest eigenvalue of simple graphs [5, 11, 13]. The study of the second largest eigenvalue of graphs also has gotten much attention [4, 14]. The interest in studying the least eigenvalue of graphs has increased [2, 3, 7, 12] recently.

In this paper, we study the largest, the second largest and the least eigenvalue, $\lambda_1, \lambda_2, \lambda_n$, of the graphs in $\mathscr{F}_n$, obtain the sharp upper bounds and the sharp lower bounds of the largest eigenvalues λ_1, the least eigenvalue λ_n, and the second largest eigenvalue λ_2 in Section 2, investigate the minimum spread, and the maximum (minimum) separator among all firefly graphs on n vertices in Sections 3 and 4, respectively. We also propose a conjecture of the maximum spread of firefly graphs on n vertices.

2. **Eigenvalues among all firefly graphs**

In Subsection 2.1, we determine the unique graph with the maximum largest or the minimum largest eigenvalue. In Subsection 2.2, the unique graph with the maximum least or the minimum least eigenvalue is characterized. The sharp upper and lower bounds of the second largest eigenvalue are also determined in Subsection 2.3.

2.1. **The largest eigenvalue λ_1.**

In this subsection, we determine the unique graph with the maximum largest or the minimum largest eigenvalue.

For an edge subset $F \subseteq E(G)$, $G - F$ denotes the graph obtained from G by deleting the edges in F. For an edge subset $F' \cap E(G) = \emptyset$, $G + F'$ denotes the graph obtained from G by adding the edges in F'

Lemma 2.1. ([6]) *Let G be a connected graph with $e \notin E(G)$. Then $\lambda_1(G) < \lambda_1(G + e)$.*

Lemma 2.2. ([11]) *Let G be a connected graph with $u \in V(G)$. Let $G_{r,s}$ be the graph obtained from G by attaching two vertex-disjoint paths, one of r vertices and the other of s vertices, at one end vertex to u respectively, where $r \geq s \geq 1$. If $s \geq 2$, then $\lambda_1(G_{r+1,s-1}) < \lambda_1(G_{r,s})$.*

For $u \in V(G)$, let $G-u$ be the graph obtained from G by deleting the vertex u and its incident edges. We denote the set of all neighbors of the vertex u by $N(u)$.

Lemma 2.3. ([6]) *Let $u \in V(G)$ and $C(u)$ be the set of all cycles of G containing u. Then*

$$P(G,\lambda) = \lambda P(G-u,\lambda) - \sum_{uv\in E(G)} P(G-u-v,\lambda) - 2\sum_{Z\in C(u)} P(G-V(Z),\lambda).$$

By Lemma 2.3, we have the following lemma.

Lemma 2.4. *Let $G = F_{s,t,n-2s-2t-1} \in \mathscr{F}_n$. Then*

$$P(G,\lambda) = \lambda^{n-2s-2t-2}(\lambda+1)^{s+t-1}(\lambda-1)^{s+t-1}[\lambda^4-(n-t)\lambda^2-2s\lambda+n-2s-2t-1].$$

Proof. By applying Lemma 2.3 to the common vertex $u \in V(G)$, we obtained

$$\begin{aligned} P(G,\lambda) &= \lambda P(G-u,\lambda) - \sum_{uv\in E(G)} P(G-u-v,\lambda) - 2\sum_{Z\in C(u)} P(G-V(Z),\lambda) \\ &= \lambda^{n-2s-2t}(\lambda^2-1)^{s+t} - (2s+t)\lambda^{n-2s-2t}(\lambda^2-1)^{s+t-1} \\ &\quad -(n-2s-2t-1)\lambda^{n-2s-2t-2}(\lambda^2-1)^{s+t} - 2s\lambda^{n-2s-2t-1}(\lambda^2-1)^{s+t-1} \\ &= \lambda^{n-2s-2t-2}(\lambda+1)^{s+t-1}(\lambda-1)^{s+t-1}[\lambda^4-(n-t)\lambda^2-2s\lambda+n-2s-2t-1]. \end{aligned}$$ □

Lemma 2.5. *Let $0 \leq s,t \leq \lfloor \frac{n-1}{2} \rfloor$ and $f(x) = x^4-(n-t)x^2-2sx+n-2s-2t-1$. Then $f(x)$ has four roots x_1,x_2,x_3,x_4 satisfying*

$$x_1 > 1 > x_2 \geq 0 > x_3 \geq -1 > x_4 > -\sqrt{n-1}.$$

Proof. $f(0) = n-2s-2t-1 \geq 0$, $f(1) = -4s-t < 0$, $f(-1) = -t < 0$ and

$$f(-\sqrt{n-1}) = (n-1)(t-1)+2s\sqrt{n-1}+n-2s-2t-1 > 0.$$

Since $\lim_{x\to+\infty} f(x) = +\infty$, then the function $f(x)$ has exactly four roots:

$$x_1 > 1 > x_2 \geq 0 > x_3 > -1 > x_4 > -\sqrt{n-1}.$$ □

Theorem 2.6. *Let $G = F_{s,t,n-2s-2t-1} \in \mathscr{F}_n$. Then*

$$\lambda_1(F_{0,\lfloor\frac{n-1}{2}\rfloor,n-2\lfloor\frac{n-1}{2}\rfloor-1}) \leq \lambda_1(G) \leq \lambda_1(F_{\lfloor\frac{n-1}{2}\rfloor,0,n-2\lfloor\frac{n-1}{2}\rfloor-1}),$$

and

$$\lambda_1(G) = \lambda_1(F_{\lfloor\frac{n-1}{2}\rfloor,0,n-2\lfloor\frac{n-1}{2}\rfloor-1}) \text{ if and only if } G = F_{\lfloor\frac{n-1}{2}\rfloor,0,n-2\lfloor\frac{n-1}{2}\rfloor-1},$$

$$\lambda_1(G) = \lambda_1(F_{0,\lfloor\frac{n-1}{2}\rfloor,n-2\lfloor\frac{n-1}{2}\rfloor-1}) \text{ if and only if } G = F_{0,\lfloor\frac{n-1}{2}\rfloor,n-2\lfloor\frac{n-1}{2}\rfloor-1}.$$

Proof. First, we show that $\lambda_1(G) \leq \lambda_1(F_{\lfloor\frac{n-1}{2}\rfloor,0,n-2\lfloor\frac{n-1}{2}\rfloor-1})$.

Let u be the common vertex of G and $\{v_1,v_2,\ldots,v_t\}$ be the set of pendent vertices of the pendent paths of length 2. Let $G_1 = G+\{uv_i : 1 \leq i \leq t\}$. Then $G_1 \in \mathscr{F}_n$ and $\lambda_1(G) < \lambda_1(G_1)$ by Lemma 2.1. Let $\{z_1,z_2,\ldots,z_{n-2s-2t-1}\}$ be the set of pendent vertices which are the neighbors of u.

If $n \equiv 0 \pmod 2$, then let $G_2 = G_1 + \{z_1z_2, z_3z_4, \ldots, z_{n-2s-2t-3}z_{n-2s-2t-2}\}$. Obviously, $G_2 \cong F_{\frac{n-2}{2},0,1} \in \mathscr{F}_n$ and $\lambda_1(G) < \lambda_1(G_1) < \lambda_1(G_2)$ by Lemma 2.1.

If $n \equiv 1 \pmod 2$, then let $G_3 = G_1 + \{z_1z_2, z_3z_4, \ldots, z_{n-2s-2t-2}z_{n-2s-2t-1}\}$. Obviously, $G_3 \cong F_{\frac{n-1}{2},0,0} \in \mathscr{F}_n$ and $\lambda_1(G) < \lambda_1(G_1) < \lambda_1(G_3)$ by Lemma 2.1.

Then the result follows.

Second, we show that $\lambda_1(G) \geq \lambda_1(F_{0,\lfloor\frac{n-1}{2}\rfloor,n-2\lfloor\frac{n-1}{2}\rfloor-1})$.

By Lemma 2.1, for any $G \in \mathscr{F}_n$,

$\lambda_1(G) = \lambda_1(F_{s,t,n-2s-2t-1}) > \lambda_1(F_{s-1,t,n-2s-2t+1}) > \cdots > \lambda_1(F_{0,t,n-2t-1})$ for $0 \leq t \leq \lfloor\frac{n-1}{2}\rfloor$.

By Lemma 2.2,

$$\lambda_1(F_{0,t,n-2t-1}) > \lambda_1(F_{0,t+1,n-2t-3}) > \cdots > \lambda_1(F_{0,\lfloor\frac{n-1}{2}\rfloor,n-2\lfloor\frac{n-1}{2}\rfloor-1}).$$

Then the result follows. $\square$

By Lemma 2.4 and Theorem 2.6, we have the following corollary.

Corollary 2.7. *Let $G = F_{s,t,n-2s-2t-1} \in \mathscr{F}_n$. Then*

$$\begin{cases} \sqrt{\frac{n+1}{2}} \leq \lambda_1(G) \leq \frac{1+\sqrt{4n-3}}{2}, & \text{if } n \equiv 1 \pmod 2; \\ \frac{\sqrt{n+2+\sqrt{n^2+4n-12}}}{2} \leq \lambda_1(G) \leq \lambda_1(F_{\frac{n-2}{2},0,1}), & \text{if } n \equiv 0 \pmod 2. \end{cases}$$

$$\lambda_1(G) = \begin{cases} \frac{1+\sqrt{4n-3}}{2}, & \text{if } n \equiv 1 \pmod 2; \\ \lambda_1(F_{\frac{n-2}{2},0,1}), & \text{if } n \equiv 0 \pmod 2 \end{cases} \quad \text{if and only if } G = F_{\lfloor\frac{n-1}{2}\rfloor,0,n-2\lfloor\frac{n-1}{2}\rfloor-1},$$

$$\lambda_1(G) = \begin{cases} \sqrt{\frac{n+1}{2}}, & \text{if } n \equiv 1 \pmod 2; \\ \frac{\sqrt{n+2+\sqrt{n^2+4n-12}}}{2}, & \text{if } n \equiv 0 \pmod 2 \end{cases} \quad \text{if and only if } G = F_{0,\lfloor\frac{n-1}{2}\rfloor,n-2\lfloor\frac{n-1}{2}\rfloor-1}.$$

2.2. **The least eigenvalue λ_n.** In this subsection, we determine the unique graph with the maximum least or the minimum least eigenvalue.

Theorem 2.8. *Let $G = F_{s,t,n-2s-2t-1} \in \mathscr{F}_n$ $(n \geq 8)$. Then*

$$\lambda_n(S_n) \leq \lambda_n(G) \leq \lambda_n(F_{0,\lfloor\frac{n-1}{2}\rfloor,n-2\lfloor\frac{n-1}{2}\rfloor-1}),$$

and

$$\lambda_n(G) = \lambda_n(S_n) \text{ if and only if } G = F_{0,0,n-1} \cong S_n.$$

$$\lambda_n(G) = \lambda_n(F_{0,\lfloor\frac{n-1}{2}\rfloor,n-2\lfloor\frac{n-1}{2}\rfloor-1}) \text{ if and only if } G = F_{0,\lfloor\frac{n-1}{2}\rfloor,n-2\lfloor\frac{n-1}{2}\rfloor-1}.$$

Proof. First, we show that $\lambda_n(G) \geq \lambda_n(S_n)$ by the following two cases.

Case 1. $s = 0$ and $t = 0$.

By Lemma 2.4, $P(F_{0,0,n-1}, \lambda) = P(S_n, \lambda) = \lambda^{n-2}(\lambda^2 - n + 1)$. Then $\lambda_1(S_n) = \sqrt{n-1}$ and $\lambda_n(S_n) = -\sqrt{n-1}$.

Case 2. $s \neq 0$ or $t \neq 0$.

By Lemma 2.4, we have

$$P(F_{s,t,n-2s-2t-1},\lambda) = \lambda^{n-2s-2t-2}(\lambda+1)^{s+t-1}(\lambda-1)^{s+t-1}[\lambda^4-(n-t)\lambda^2-2s\lambda+n-2s-2t-1].$$

Let $f(s,t,x) = x^4-(n-t)x^2-2sx+n-2s-2t-1$. By Lemma 2.5, $f(s,t,x)$ has the same least root as $P(F_{s,t,n-2s-2t-1},\lambda)$. Therefore $\lambda_n(F_{s,t,n-2s-2t-1}) > \lambda_n(S_n)$.

Second, we show that $\lambda_n(G) \le \lambda_n(F_{0,\lfloor\frac{n-1}{2}\rfloor,n-2\lfloor\frac{n-1}{2}\rfloor-1})$. We only consider $G \in \mathscr{F}_n\backslash\{F_{0,0,n-1}\}$.

The difference between polynomials $f(s,t,x)$ and $f(s+1,t,x)$ does not depend on the value of parameter s:

$$f(s,t,x) - f(s+1,t,x) = 2x+2 = g(x).$$

The polynomial $g(x)$ has exactly one root $x_0 = -1$. Noting that $\lambda_n(F_{s,t,n-2s-2t-1}) < -1$, then $f(s,t,x) < f(s+1,t,x)$ while $x \in (-\infty,-1)$.

Since $\lim\limits_{x\to-\infty} f(s,t,x) = +\infty$, by the graph of the polynomial function, we conclude that for $0 \le t \le \lfloor\frac{n-1}{2}\rfloor$, $x_4(f(s,t,x)) < x_4(f(s+1,t,x))$ while $x_4(f(s,t,x))$ is the least root of $f(s,t,x)$.

Thus, $\lambda_n(F_{0,t,n-2t-1}) < \lambda_n(F_{1,t,n-2t-3}) < \cdots < \lambda_n(F_{\lfloor\frac{n-2t-1}{2}\rfloor,t,n-2t-2\lfloor\frac{n-2t-1}{2}\rfloor-1})$.

Let $h(t,x) = x^4-(n-t)x^2-2\lfloor\frac{n-2t-1}{2}\rfloor x+n-2\lfloor\frac{n-2t-1}{2}\rfloor-2t-1$. By Lemmas 2.4$\sim$2.5, $h(t,x)$ has the same least root as $P(F_{s,t,n-2s-2t-1},\lambda)$. The difference between polynomials $h(t,x)$ and $h(t+1,x)$ does not depend on the value of parameter t:

$$h(t,x) - h(t+1,x) = -(x+1)^2+1.$$

Since $h(t,-2) = \begin{cases} 14-2n, & \text{if } n \equiv 1 \pmod 2; \\ 13-2n, & \text{if } n \equiv 0 \pmod 2, \end{cases}$ then $h(t,-2) < 0$ for $n \ge 8$. Thus $h(t,x) < h(t+1,x)$ while $x \in (-\infty,-2)$.

Noting that $\lim\limits_{x\to-\infty} h(t,x) = +\infty$, by the graph of the polynomial function, we conclude that $x_4(h(t,x)) < x_4(h(t+1,x))$ while $x_4(h(t,x))$ is the least root of $h(t,x)$.

Thus, $\lambda_n(F_{\lfloor\frac{n-1}{2}\rfloor,0,n-2\lfloor\frac{n-1}{2}\rfloor-1}) < \lambda_n(F_{\lfloor\frac{n-3}{2}\rfloor,1,n-2\lfloor\frac{n-3}{2}\rfloor-3}) < \cdots < \lambda_n(F_{0,\lfloor\frac{n-1}{2}\rfloor,n-2\lfloor\frac{n-1}{2}\rfloor-1})$.

Then the result follows. $\square$

By Lemma 2.4 and Theorem 2.8, we have the following corollary.

Corollary 2.9. *Let $G = F_{s,t,n-2s-2t-1} \in \mathscr{F}_n$ $(n \ge 8)$. Then*

$$\begin{cases} -\sqrt{n-1} \le \lambda_n(G) \le -\sqrt{\frac{n+1}{2}}, & \textit{if } n \equiv 1 \pmod 2; \\ -\sqrt{n-1} \le \lambda_n(G) \le -\frac{\sqrt{n+2+\sqrt{n^2+4n-12}}}{2}, & \textit{if } n \equiv 0 \pmod 2. \end{cases}$$

$$\lambda_n(G) = \begin{cases} -\sqrt{\frac{n+1}{2}}, & \textit{if } n \equiv 1 \pmod 2; \\ -\frac{\sqrt{n+2+\sqrt{n^2+4n-12}}}{2}, & \textit{if } n \equiv 0 \pmod 2 \end{cases} \quad \textit{if and only if } G = F_{0,\lfloor\frac{n-1}{2}\rfloor,n-2\lfloor\frac{n-1}{2}\rfloor-1},$$

$\lambda_n(G) = -\sqrt{n-1}$ *if and only if* $G = F_{0,0,n-1} \cong S_n$.

2.3. **The second largest eigenvalue λ_2.** In this subsection, we determine the sharp upper and lower bounds of the second largest eigenvalue.

Theorem 2.10. *Let $G = F_{s,t,n-2s-2t-1} \in \mathscr{F}_n$ $(n \geq 6)$. Then $0 \leq \lambda_2(G) \leq 1$, and*

$$\lambda_2(G) = 0 \text{ if and only if } G = F_{0,0,n-1} \cong S_n,$$

$$\lambda_2(G) = 1 \text{ if and only if } G \in \mathscr{F}_n \backslash \{F_{0,0,n-1}, F_{0,1,n-3}, F_{1,0,n-3}\}.$$

Proof. We show that the result holds by the following three cases.

Case 1. $s + t = 0$.

Then $G = F_{0,0,n-1}$ and $\lambda_2(F_{0,0,n-1}) = 0$ by the proof Theorem 2.8.

Case 2. $s + t = 1$.

Note that

$$P(F_{0,1,n-3}, \lambda) = \lambda^{n-4}[\lambda^4 - (n-1)\lambda^2 + n - 3],$$

and

$$P(F_{1,0,n-3}, \lambda) = \lambda^{n-4}(\lambda + 1)[\lambda^3 - \lambda^2 - (n-1)\lambda + n - 3].$$

Then $\lambda_2(F_{0,1,n-3}) = \sqrt{\frac{n-1-\sqrt{n^2-6n+13}}{2}} < 1$ and $0 < \lambda_2(F_{1,0,n-3}) < 1$.

Case 3. $s + t \geq 2$.

Let $G \in \mathscr{F}_n \backslash \{F_{0,0,n-1}, F_{0,1,n-3}, F_{1,0,n-3}\}$. Then we have

$$P(G, \lambda) = \lambda^{n-2s-2t-2}(\lambda + 1)^{s+t-1}(\lambda - 1)^{s+t-1}[\lambda^4 - (n-t)\lambda^2 + n - 2s - 2t - 1].$$

Clearly, 1 is a root of $P(G, \lambda)$. By Lemma 2.5, $\lambda_2(F_{s,t,n-2s-2t-1}) = 1$. □

3. Spread among all firefly graphs

Recall that $S(G) = \lambda_1(G) - \lambda_n(G)$. By Corollaries 2.7 and 2.9, we get the following theorem, and later we propose a problem.

Theorem 3.1. *Let $G = F_{s,t,n-2s-2t-1} \in \mathscr{F}_n$ $(n \geq 8)$. Then*

$$\begin{cases} \sqrt{2n+2} \leq S(G) < \frac{1+\sqrt{4n-3}}{2} + \sqrt{n-1}, & \text{if } n \equiv 1 \pmod 2; \\ \sqrt{n+2+\sqrt{n^2+4n-12}} \leq S(G) < \lambda_1(F_{\frac{n-2}{2},0,1}) + \sqrt{n-1}, & \text{if } n \equiv 0 \pmod 2). \end{cases}$$

The equality holds if and only if $G = F_{0,\lfloor\frac{n-1}{2}\rfloor,n-2\lfloor\frac{n-1}{2}\rfloor-1}$.

Lemma 3.2. ([7]) *Let G be a connected graph with $u \in V(G)$. Let $G_{k,l}$ $(k \geq l \geq 1)$ be a graph obtained from G by attaching two hanging paths P_k and P_l at the vertex u (i.e. by identifying u first with one pendent vertex of P_k and then with one pendent vertex of P_l). Then for $l \geq 2$, $\lambda_n(G_{k,l}) \leq \lambda_n(G_{k+l-1,1})$.*

Theorem 3.3. *Let $G = F_{s,t,n-2s-2t-1} \in \mathscr{F}_n$ $(n \geq 8)$. Then*

$$S(G) \leq \max\left\{S(F_{s,0,n-2s-1}) : 0 \leq s \leq \lfloor\tfrac{n-1}{2}\rfloor\right\}.$$

Proof. By Lemmas 2.2∼3.2, for $t \geq 1$, we have

$\lambda_1(F_{s,t,n-2s-2t-1}) < \lambda_1(F_{s,0,n-2s-1})$, $\lambda_n(F_{s,0,n-2s-1}) \leq \lambda_n(F_{s,t,n-2s-2t-1})$.

Then $S(F_{s,t,n-2s-2t-1}) < S(F_{s,0,n-2s-1})$ for $t \geq 1$. □

With the help of MATLAB, by Theorem 3.3, we get the graph $F_{s,t,n-2s-2t-1} \in \mathscr{F}_n$ with maximum spread for $6 \leq n \leq 23$ (see Table 1).

Table 1. The graph $F_{s,t,n-2s-2t-1} \in \mathscr{F}_n$ with maximum spread for $6 \leq n \leq 23$.

n	$\max\{S(F_{s,t,n-2s-2t-1})\}$	n	$\max\{S(F_{s,t,n-2s-2t-1})\}$	n	$\max\{S(F_{s,t,n-2s-2t-1})\}$
6	$S(F_{2,0,1}) = 4.6125$	12	$S(F_{4,0,3}) = 6.7321$	18	$S(F_{6,0,5}) = 8.3264$
7	$S(F_{2,0,2}) = 5.0332$	13	$S(F_{4,0,4}) = 7.0237$	19	$S(F_{6,0,6}) = 8.5635$
8	$S(F_{2,0,3}) = 5.4142$	14	$S(F_{4,0,5}) = 7.3026$	20	$S(F_{6,0,7}) = 8.7938$
9	$S(F_{3,0,2}) = 5.7714$	15	$S(F_{5,0,4}) = 7.5714$	21	$S(F_{7,0,6}) = 9.0183$
10	$S(F_{3,0,3}) = 6.1100$	16	$S(F_{5,0,5}) = 7.8315$	22	$S(F_{7,0,7}) = 9.2376$
11	$S(F_{3,0,4}) = 6.4283$	17	$S(F_{5,0,6}) = 8.0283$	23	$S(F_{7,0,8}) = 9.4515$

Hence we have the following conjecture.

Conjecture 3.4. *Let G be the graph with maximum spread among $\mathscr{F}_n$ $(n \geq 6)$. Then*

$$G = \begin{cases} F_{\frac{n}{3},0,\frac{n-3}{3}}, & \text{if } n \equiv 0 \pmod 3; \\ F_{\frac{n-1}{3},0,\frac{n-1}{3}}, & \text{if } n \equiv 1 \pmod 3; \\ F_{\frac{n-2}{3},0,\frac{n+1}{3}}, & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

4. **Separator among all firefly graphs**

Recall that the separator $S_A(G) = \lambda_1(G) - \lambda_2(G)$. In this section, we characterize the graph with maximum or minimum separator in $\mathscr{F}_n$.

Theorem 4.1. *Let $G = F_{s,t,n-2s-2t-1} \in \mathscr{F}_n$ $(n \geq 6)$. Then*

$$\begin{cases} \sqrt{\frac{n+1}{2}} - 1 \leq S_A(G) \leq \sqrt{n-1}, & \text{if } n \equiv 1 \pmod 2; \\ \frac{\sqrt{n+2+\sqrt{n^2+4n-12}}}{2} - 1 \leq S_A(G) \leq \sqrt{n-1}, & \text{if } n \equiv 0 \pmod 2. \end{cases}$$

and

$S_A(G) = \sqrt{n-1}$ *if and only if* $G \cong S_n$,

$$S_A(G) = \begin{cases} \sqrt{\frac{n+1}{2}} - 1, & \text{if } n \equiv 1 \pmod 2; \\ \frac{\sqrt{n+2+\sqrt{n^2+4n-12}}}{2} - 1, & \text{if } n \equiv 0 \pmod 2 \end{cases} \quad \text{if and only if } G = F_{0,\lfloor\frac{n-1}{2}\rfloor,n-2\lfloor\frac{n-1}{2}\rfloor-1}.$$

Proof. First, we show that $S_A(G) \leq \sqrt{n-1}$.

Case 1. $G \in \{F_{0,0,n-1}, F_{0,1,n-3}, F_{1,0,n-3}\}$.

Then $S_A(F_{0,0,n-1}) = S_A(S_n) = \sqrt{n-1}$ and

$$S_A(F_{0,1,n-3}) = \sqrt{\tfrac{n-1+\sqrt{n^2-6n+13}}{2}} - \sqrt{\tfrac{n-1-\sqrt{n^2-6n+13}}{2}} < \sqrt{n-1}.$$

Recall that $P(F_{1,0,n-3},\lambda) = \lambda^{n-4}(\lambda+1)[\lambda^3-\lambda^2-(n-1)\lambda+n-3]$. Let $f(x) = x^3-x^2-(n-1)x+n-3$. Then the function $f(x)$ has exactly three roots: $x_3 < -1 < 0 < x_2 < 1 < x_1$.

Since $f(\frac{1}{2}) = \frac{4n-21}{8} > 0$ and $f(\sqrt{n-1}+\frac{1}{2}) = \frac{8n-25-2\sqrt{n-1}}{8} > 0$ for $n \geq 6$, then $\frac{1}{2} < x_2 < 1$ and $1 < x_1 < \sqrt{n-1}+\frac{1}{2}$. Thus

$$S_A(F_{1,0,n-3}) = \lambda_1(F_{1,0,n-3}) - \lambda_2(F_{1,0,n-3}) < \sqrt{n-1}+\tfrac{1}{2}-\tfrac{1}{2} = \sqrt{n-1}.$$

Case 2. $G \in \mathscr{F}_n \backslash \{F_{0,0,n-1}, F_{0,1,n-3}, F_{1,0,n-3}\}$.

Then $\lambda_2(G) = 1$ by Theorem 2.10. By Theorem 2.6, $S_A(G) \leq \lambda_1(F_{\lfloor\frac{n-1}{2}\rfloor,0,n-2\lfloor\frac{n-1}{2}\rfloor-1}) - 1$.

Subcase 2.1. $n \equiv 1 \pmod 2$.

Then $\lambda_1(F_{\frac{n-1}{2},0,0}) = \frac{1+\sqrt{4n-3}}{2} > 1$ by Corollary 2.7. Thus $S_A(F_{\frac{n-1}{2},0,0}) = \frac{1+\sqrt{4n-3}}{2} - 1 < \sqrt{n-1} = S_A(S_n)$.

Subcase 2.2. $n \equiv 0 \pmod 2$.

Then by Lemma 2.4,

$$P(F_{\frac{n-2}{2},0,1},\lambda) = (\lambda^2-1)^{n-4}[\lambda^4 - n\lambda^2 - (n-2)\lambda + 1].$$

Since $P(F_{\frac{n-2}{2},0,1},\sqrt{n-1}+1) = (n-1+2\sqrt{n-1})^{n-4}[(\sqrt{n-1}+1)(n+2\sqrt{n-1})+1] > 0$ and $\lim\limits_{\lambda\to+\infty} P(F_{\frac{n-2}{2},0,1},\lambda) = +\infty$, $\lambda_1(F_{\frac{n-2}{2},0,1}) < \sqrt{n-1}+1$.

Then $S_A(F_{\frac{n-1}{2},0,1}) = \lambda_1(F_{\frac{n-2}{2},0,1}) - 1 < \sqrt{n-1} = S_A(S_n)$.

The result follows.

Second, we show that

$$S_A(G) \geq \begin{cases} \sqrt{\frac{n+1}{2}} - 1, & \text{if } n \equiv 1 \pmod 2; \\ \frac{\sqrt{n+2+\sqrt{n^2+4n-12}}}{2} - 1, & \text{if } n \equiv 0 \pmod 2. \end{cases}$$

Case 1. $G \in \{F_{0,0,n-1}, F_{0,1,n-3}, F_{1,0,n-3}\}$.

We only consider $G \in \{F_{0,1,n-3}, F_{1,0,n-3}\}$. By Theorems 2.6 and 2.10, for $G \in \{F_{0,1,n-3}, F_{1,0,n-3}\}$, $\lambda_1(G) > \lambda_1(F_{0,\lfloor\frac{n-1}{2}\rfloor,n-2\lfloor\frac{n-1}{2}\rfloor-1})$, $0 < \lambda_2(G) < 1$ and $\lambda_2(F_{0,\lfloor\frac{n-1}{2}\rfloor,n-2\lfloor\frac{n-1}{2}\rfloor-1}) = 1$. Thus

$$S_A(G) = \lambda_1(G) - \lambda_2(G) > \lambda_1(F_{0,\lfloor\frac{n-1}{2}\rfloor,n-2\lfloor\frac{n-1}{2}\rfloor-1}) - 1.$$

Case 2. $G \in \mathscr{F}_n \backslash \{F_{0,0,n-1}, F_{0,1,n-3}, F_{1,0,n-3}\}$.

Then by Theorem 2.6, $\lambda_1(G) \geq \lambda_1(F_{0,\lfloor\frac{n-1}{2}\rfloor,n-2\lfloor\frac{n-1}{2}\rfloor-1})$. By Theorem 2.10, $\lambda_2(G) = 1$ and $S_A(G) = \lambda_1(G) - \lambda_2(G) \geq \lambda_1(F_{0,\lfloor\frac{n-1}{2}\rfloor,n-2\lfloor\frac{n-1}{2}\rfloor-1}) - 1$.

By Corollary 2.7, the result follows. $\square$

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References

[1] J. A. Bondy and U. S. R. Murty, *Graph Theory with Applications*, American Elsevier Publishing Co., Inc., New York, 1976.

[2] F. K. Bell, D. Cvetković, P. Rowlinson and S. K. Simić, Graphs for which the least eigenvalue is minimal, I, *Linear Algebra Appl.*, **429** (2008) 234–241.

[3] F. K. Bell, D. Cvetković, P. Rowlinson and S. K. Simić, Graphs for which the least eigenvalue is minimal, II, *Linear Algebra Appl.*, **429** (2008) 2168–2179.

[4] D. Cvetković and S. Simić, On graphs whose second largest eigenvalue does not exceed $\frac{\sqrt{5}-1}{2}$, *Discrete Math.*, **138** (1995) 213–227.

[5] D. Cvetković and P. Rowlinson, The largest eigenvalue of a graph: a survey, *Linear and Multilinear Algebra*, **28** (1990) 3–33.

[6] D. Cvetković, P. Rowlinson and S. Simić, An Introduction to the Theory of Graph Spectra, **75**, Cambridge University Press, Cambridge, 2010.

[7] Y. Fan, Y. Wang and Y. Gao, Minimizing the least eigenvalues of unicyclic graphs with application to spectral spread, *Linear Algebra Appl.*, **429** (2008) 577–588.

[8] Y. Hong, Bounds on the spectra of unicyclic graphs, (Chinese), *J. East China Norm. Univ. Natur. Sci. Ed.*, **1986** no. 1 31–34.

[9] J. X. Li, J. M. Guo and W. C. Shiu, On the second largest Laplacian eigenvalues of graphs, *Linear Algebra Appl.*, **438** (2013) 2438–2446.

[10] S. C. Li and M. J. Zhang, On the signless Laplacian index of cacti with a given number of pendant vertices, *Linear Algebra Appl.*, **436** (2012) 4400–4411.

[11] Q. Li and K. Feng, On the largest eigenvalues of a graphs, (Chinese), *Acta Math. Appl. Sinica*, **2** (1979) 167–175.

[12] M. Petrović, T. Aleksić and V. Simić, On the least eigenvalue of cacti, *Linear Algebra Appl.*, **435** (2011) 2357–2364.

[13] B. Wu, E. Xiao and Y. Hong, The spectral radius of trees on k pendant vertices, *Linear Algebra Appl.*, **395** (2005) 343–349.

[14] G. H. Xu, On unicyclic graphs whose second largest eigenvalue does not exceed 1, *Discrete Appl. Math.*, **136** (2004) 117–124.

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