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DECOMPOSING HYPERGRAPHS INTO k-COLORABLE HYPERGRAPHS

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ABSTRACT. For a given hypergraph H with chromatic number $\chi(H)$ and with no edge containing only one vertex, it is shown that the minimum number l for which there exists a partition (also a covering) $\{E_1, E_2, \ldots, E_l\}$ for E(H), such that the hypergraph induced by E_i for each $1 \leq i \leq l$ is k-colorable, is $\lceil \log_k \chi(H) \rceil$.

1. Introduction

A hypergraph H is a pair H = (V, E), where V is a finite nonempty set (the set of vertices) and E is a collection of distinct nonempty subsets of V (the set of edges). A proper coloring of H is an assignment of colors to the vertices so that no edge has the same color on all its vertices. The chromatic number $\chi(H)$ of H is the smallest k, such that there is a proper coloring of H, using k colors. We say that a hypergraph is k-colorable if its chromatic number is at most k. An independent set in H is a set of vertices which does not contain any edge of H as a subset. For a given hypergraph H, the set $\{E_1, E_2, \ldots, E_l\}$ with $E_i \subseteq E(H)$ and $E_i \neq \emptyset$ is called a partition (resp. a covering) for E(H), if $E(H) = \bigcup_{i=1}^{l} E_i$ and $E_i \cap E_j = \emptyset$ for any $i \neq j$ (resp. if $E(H) = \bigcup_{i=1}^{l} E_i$). Let $p_k(H)$ (resp. $c_k(H)$) denote the minimum number l for which there exists a partition (resp. a covering) $\{E_1, E_2, \ldots, E_l\}$ for E(H), where the hypergraph induced by E_i (which is denoted by $\langle E_i \rangle$) for each $1 \leq i \leq l$, is k-colorable.

Acharya [1], conjectured that for any $n \ge 1$ the minimum number of colors needed to paint the edges of the graph $G = K_n$ so that in every cycle there is a nonzero even number of edges of at least one color is $\lceil \log_2 \chi(G) \rceil$. In [2], Alon and Egawa showed that this conjecture is true for each simple

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graph G. As a corollary, they concluded that $p_2(G) = \lceil \log_2 \chi(G) \rceil$ for each simple graph G. In this note, using a similar pattern, we generalize this result as follows.

Theorem 1.1. Let $k \ge 2$ be an integer and H be a hypergraph without any edge containing only one vertex. Then $p_k(H) = c_k(H) = \lceil \log_k \chi(H) \rceil$.

Proof. Clearly $p_k(H) \ge c_k(H)$. Let $l = c_k(H)$ and $f = \lceil \log_k \chi(H) \rceil$, so $k^{f-1} + 1 \le \chi(H) \le k^f$. Let $\{E_1, E_2, \ldots, E_l\}$ be a covering of the edges where each $\langle E_i \rangle = (V(H), E_i)$ is k-colorable. Assume that $V(H) = \bigcup_{j=1}^k V_{ij}$ is the k-partition of V(H), where for any fixed i, the sets V_{ij} 's are the color classes of $\langle E_i \rangle$. Then the partition $\{V_{1r_1} \cap \cdots \cap V_{lr_l} | 1 \le r_i \le k\}$ for V(H) yields a proper coloring for H and hence $l \ge f$. Now let $X = \{1, 2, \ldots, k\}^f$ and $\{V_x\}_{x \in X}$ be a partition for V(H) into independent subsets. We assign $x \in X$ to each vertex $v \in V_x$ and we denote by x_i the i-th entry of $x \in X$. Now for any two vertices $u \in V_x$ and $v \in V_y$ of H, let $\delta(u, v) = \infty$ if x = y and $\delta(u, v) = \min\{i | x_i \neq y_i\}$, otherwise. For each edge B of H, set $\delta(B) = \min\{\delta(u, v) | u \neq v, u, v \in H\}$. Clearly for each edge $B \in E$, we have $1 \le \delta(B) \le f$. Now for each $1 \le i \le f$, let $E_i = \{B \in E | \delta(B) = i\}$. Easily we can see that $\langle E_i \rangle = (V(H), E_i)$ is a k-colorable hypergraph (in fact if V_j , for each $1 \le j \le k$, is the set of the vertices for which the i-th entries of their corresponding vectors in X are j, then the induced subhypergraph of V_j in $\langle E_i \rangle$ has no edge). Since $\{E_i\}_{i=1}^f$ is a partition for the edges of H and each $\langle E_i \rangle$ is k-colorable, we have $p_k(H) \le f$. Hence $p_k(H) = c_k(H) = f$.

A proper edge coloring of a hypergraph H is an assignment of colors to the edges, so that no two edges with non-empty intersection, have the same color. The chromatic index $\chi'(H)$ of H is the smallest k such that there is a proper edge-coloring for H using k colors. We say that a hypergraph is k-edge colorable if its chromatic index is at most k. In the sequel, we give the same result on the minimum number of k-edge colorable hypergraphs we need to partition (or to cover) the edges of a given hypergraph.

Theorem 1.2. Let H be a hypergraph with chromatic index $\chi'(H)$. Then the minimum number l for which there exists a partition (or a covering) $\{E_1, E_2, \ldots, E_l\}$ for E(H), where the hypergraph induced by E_i for each $1 \le i \le l$ is k-edge colorable, is $[\chi'(H)/k]$.

Proof. Let $\{E_i\}_{i=1}^l$ be a partition (or a covering) for the edges where each $\langle E_i \rangle$ is k-edge colorable. Clearly $\chi'(H) \leq \sum_{i=1}^l \chi'(\langle E_i \rangle) \leq lk$ and so $l \geq \lceil \chi'(H)/k \rceil$. Now we color the edges of H by colors $c_1, c_2, \ldots, c_{\chi'(H)}$ and we assume that E_i is the set of the edges with colors $(i-1)k+1 \leq c \leq ik$ for $1 \leq i \leq \lceil \chi'(H)/k \rceil - 1$ and $E_{\lceil \chi'(H)/k \rceil}$ is the set of the remaining edges. Clearly for each $1 \leq i \leq \lceil \chi'(H)/k \rceil$, $\langle E_i \rangle$ is k-edge colorable and so $l \leq \lceil \chi'(H)/k \rceil$.

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