Transactions on Combinatorics
ISSN (print): 2251-8657, ISSN (on-line): 2251-8665
Vol. 3 No. 2 (2014), pp. 31-33.

www.combinatorics.ir

www.ui.ac.ir

DECOMPOSING HYPERGRAPHS INTO k-COLORABLE HYPERGRAPHS

G. R. OMIDI* AND K. TAJBAKHSH

Communicated by Behruz Tayfeh Rezaie

Abstract. For a given hypergraph H with chromatic number $\chi(H)$ and with no edge containing only one vertex, it is shown that the minimum number l for which there exists a partition (also a covering) $\{E_1, E_2, \ldots, E_l\}$ for $E(H)$, such that the hypergraph induced by E_i for each $1 \leq i \leq l$ is k-colorable, is $\lceil \log_k \chi(H) \rceil$.

1. **Introduction**

A *hypergraph* H is a pair $H = (V, E)$, where V is a finite nonempty set (the set of *vertices*) and E is a collection of distinct nonempty subsets of V (the set of *edges*). A *proper coloring* of H is an assignment of colors to the vertices so that no edge has the same color on all its vertices. The *chromatic number* $\chi(H)$ of H is the smallest k, such that there is a proper coloring of H, using k colors. We say that a hypergraph is *k-colorable* if its chromatic number is at most k. An *independent set* in H is a set of vertices which does not contain any edge of H as a subset. For a given hypergraph H, the set $\{E_1, E_2, \ldots, E_l\}$ with $E_i \subseteq E(H)$ and $E_i \neq \emptyset$ is called a *partition* (resp. a *covering*) for $E(H)$, if $E(H) = \bigcup_{i=1}^{l} E_i$ and $E_i \bigcap E_j = \emptyset$ for any $i \neq j$ (resp. if $E(H) = \bigcup_{i=1}^{l} E_i$). Let $p_k(H)$ (resp. $c_k(H)$) denote the minimum number l for which there exists a partition (resp. a covering) $\{E_1, E_2, \ldots, E_l\}$ for $E(H)$, where the hypergraph induced by E_i (which is denoted by $\langle E_i \rangle$) for each $1 \leq i \leq l$, is k-colorable.

Acharya [1], conjectured that for any $n \geq 1$ the minimum number of colors needed to paint the edges of the graph $G = K_n$ so that in every cycle there is a nonzero even number of edges of at least one color is $\lceil \log_2 \chi(G) \rceil$. In [2], Alon and Egawa showed that this conjecture is true for each simple

MSC(2010): Primary: 05C15; Secondary: 05C65.
Keywords: Hypergraph, Chromatic number, k-Colorable.
Received: 4 February 2014, Accepted: 21 April 2014.
∗Corresponding author.

graph G. As a corollary, they concluded that $p_2(G) = \lceil \log_2 \chi(G) \rceil$ for each simple graph G. In this note, using a similar pattern, we generalize this result as follows.

Theorem 1.1. *Let $k \geq 2$ be an integer and H be a hypergraph without any edge containing only one vertex. Then $p_k(H) = c_k(H) = \lceil \log_k \chi(H) \rceil$.*

Proof. Clearly $p_k(H) \geq c_k(H)$. Let $l = c_k(H)$ and $f = \lceil \log_k \chi(H) \rceil$, so $k^{f-1} + 1 \leq \chi(H) \leq k^f$. Let $\{E_1, E_2, \ldots, E_l\}$ be a covering of the edges where each $\langle E_i \rangle = (V(H), E_i)$ is k-colorable. Assume that $V(H) = \bigcup_{j=1}^{k} V_{ij}$ is the k-partition of $V(H)$, where for any fixed i, the sets V_{ij}'s are the color classes of $\langle E_i \rangle$. Then the partition $\{V_{1r_1} \cap \cdots \cap V_{lr_l} | 1 \leq r_i \leq k\}$ for $V(H)$ yields a proper coloring for H and hence $l \geq f$. Now let $X = \{1, 2, \ldots, k\}^f$ and $\{V_x\}_{x \in X}$ be a partition for $V(H)$ into independent subsets. We assign $x \in X$ to each vertex $v \in V_x$ and we denote by x_i the i-th entry of $x \in X$. Now for any two vertices $u \in V_x$ and $v \in V_y$ of H, let $\delta(u, v) = \infty$ if $x = y$ and $\delta(u, v) = \min\{i | x_i \neq y_i\}$, otherwise. For each edge B of H, set $\delta(B) = \min\{\delta(u, v) | u \neq v, u, v \in H\}$. Clearly for each edge $B \in E$, we have $1 \leq \delta(B) \leq f$. Now for each $1 \leq i \leq f$, let $E_i = \{B \in E | \delta(B) = i\}$. Easily we can see that $\langle E_i \rangle = (V(H), E_i)$ is a k-colorable hypergraph (in fact if V_j, for each $1 \leq j \leq k$, is the set of the vertices for which the i-th entries of their corresponding vectors in X are j, then the induced subhypergraph of V_j in $\langle E_i \rangle$ has no edge). Since $\{E_i\}_{i=1}^{f}$ is a partition for the edges of H and each $\langle E_i \rangle$ is k-colorable, we have $p_k(H) \leq f$. Hence $p_k(H) = c_k(H) = f$. ⊣

A *proper edge coloring* of a hypergraph H is an assignment of colors to the edges, so that no two edges with non-empty intersection, have the same color. The *chromatic index* $\chi'(H)$ of H is the smallest k such that there is a proper edge-coloring for H using k colors. We say that a hypergraph is *k-edge colorable* if its chromatic index is at most k. In the sequel, we give the same result on the minimum number of k-edge colorable hypergraphs we need to partition (or to cover) the edges of a given hypergraph.

Theorem 1.2. *Let H be a hypergraph with chromatic index $\chi'(H)$. Then the minimum number l for which there exists a partition (or a covering) $\{E_1, E_2, \ldots, E_l\}$ for $E(H)$, where the hypergraph induced by E_i for each $1 \leq i \leq l$ is k-edge colorable, is $\lceil \chi'(H)/k \rceil$.*

Proof. Let $\{E_i\}_{i=1}^{l}$ be a partition (or a covering) for the edges where each $\langle E_i \rangle$ is k-edge colorable. Clearly $\chi'(H) \leq \sum_{i=1}^{l} \chi'(\langle E_i \rangle) \leq lk$ and so $l \geq \lceil \chi'(H)/k \rceil$. Now we color the edges of H by colors $c_1, c_2, \ldots, c_{\chi'(H)}$ and we assume that E_i is the set of the edges with colors $(i-1)k + 1 \leq c \leq ik$ for $1 \leq i \leq \lceil \chi'(H)/k \rceil - 1$ and $E_{\lceil \chi'(H)/k \rceil}$ is the set of the remaining edges. Clearly for each $1 \leq i \leq \lceil \chi'(H)/k \rceil$, $\langle E_i \rangle$ is k-edge colorable and so $l \leq \lceil \chi'(H)/k \rceil$. ⊣

Acknowledgments

We would like to acknowledge the helpful comments from Professor Noga Alon.

References

[1] B. D. Acharya, Even edge colorings of a graph, *J. Combin. Theory Ser. B*, **35** (1983) 78–79.

[2] N. Alon and Y. Egawa, Even edge colorings of a graph, *J. Combin. Theory Ser. B*, **38** (1985) 93–94.

Gholam Reza Omidi

Department of Mathematical Sciences, Isfahan University of Technology, P.O.Box 84156-83111, Isfahan, Iran

Email: romidi@cc.iut.ac.ir

Khosro Tajbakhsh

Department of Mathematics, Faculty of Mathematical Sciences, Tarbiat Modares University, P.O.Box 14115-134, Tehran, Iran

Email: khtajbakhsh@modares.ac.ir