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MINIMUM FLOWS IN THE TOTAL GRAPH OF A FINITE COMMUTATIVE RING

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Abstract. Let R be a commutative ring with zero-divisor set $Z(R)$. The total graph of R, denoted by $T(\Gamma(R))$, is the simple (undirected) graph with vertex set R where two distinct vertices are adjacent if their sum lies in $Z(R)$. This work considers minimum zero-sum k-flows for $T(\Gamma(R))$. Both for $|R|$ even and the case when $|R|$ is odd and $Z(G)$ is an ideal of R it is shown that $T(\Gamma(R))$ has a zero-sum 3-flow, but no zero-sum 2-flow. As a step towards resolving the remaining case, the total graph $T(\Gamma(\mathbb{Z}_n))$ for the ring of integers modulo n is considered. Here, minimum zero-sum k-flows are obtained for $n = p^r q^s$ (where p and q are primes, r and s are positive integers). Minimum zero-sum k-flows as well as minimum constant-sum k-flows in regular graphs are also investigated.

1. **Introduction**

For a graph G, a map $f : E(G) \longrightarrow \mathbb{Z}$ is called a flow. A c-sum flow of an unoriented graph G is a flow of G such that for every vertex $v \in V(G)$ the sum of the values of all edges incident with v is a fixed constant c. For a digraph, we instead require that for every vertex the sum of the values of the incoming edges minus the sum of the values of the outgoing edges equals c. For a natural number k, a c-sum k-flow is a c-sum flow with values from the set $\{\pm 1, \pm 2, \ldots, \pm(k-1)\}$. Akbari et al. [2] conjectured that if a graph G admits a zero-sum flow, then G admits a zero-sum 6-flow. This conjecture is known as the zero-sum conjecture (ZSC). In the same paper, the validity of ZSC is verified for bipartite graphs. Moreover, it is shown that if G is an r-regular graph with r even and $r \geq 4$, then G admits a zero-sum 3-flow. Further study of zero-sum flows in regular graphs [3] reveals that if G is an r-regular graph, with $r \geq 3$, then G admits a zero-sum 7-flow. Those results

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are improved in [4] where it is shown that every r-regular graph, $r \geq 3, r \neq 5$, admits a zero-sum 5-flow. Motivated by the above results, this paper considers the following natural problem "what is the smallest natural number k such that a graph G admits a c-sum k-flow, but G does not admit a c-sum $(k-1)$-flow?". Such a flow is called a minimum c-sum k-flow. Minimum flows for complete digraphs and for complete bipartite digraphs are investigated in [13]. For a thorough study of flows in general, we refer the reader to [15].

Let R be a commutative ring with zero-divisor set $Z(R)$. The total graph of R, denoted by $T(\Gamma(R))$, is the simple (undirected) graph with vertex set R and edge set $E = \{xy : x, y \in R, x \neq y,\ x + y \in Z(R)\}$, [5]. We restrict ourselves to finite graphs, hence we assume R is finite. Maimani et al. [11] proved the following theorem.

Theorem 1.1. *Let R be a commutative ring. Then the degree of a vertex x in $T(\Gamma(R))$ is either $|Z(R)|$ or $|Z(R)| - 1$. In particular, if $2 \in Z(R)$, then $T(\Gamma(R))$ is a $(|Z(R)| - 1)$-regular graph.*

A complete characterization of $T(\Gamma(R))$ when $Z(R)$ is an ideal of R is given in the following theorem.

Theorem 1.2. [5] *Let R be a commutative ring such that $Z(R)$ is an ideal of R. Let $|Z(R)| = \lambda$, $|R/Z(R)| = \mu$. Then*

$$T(\Gamma(R)) = \begin{cases} \bigcup\limits_{i=1}^{\mu} K_\lambda & \text{if } 2 \in Z(R); \\ K_\lambda \cup \left(\bigcup\limits_{i=1}^{\frac{\mu-1}{2}} K_{\lambda,\lambda} \right) & \text{if } 2 \in R - Z(R). \end{cases}$$

There also exist some results for the case when $Z(R)$ is not an ideal of R. In this case, as shown by Anderson and Badawi, $T(\Gamma(R))$ is connected with diameter 2. Moreover, Akbari et al. have obtained the following result:

Theorem 1.3. [1] *Let R be a finite commutative ring such that $Z(R)$ is not an ideal. Then $T(\Gamma(R))$ is Hamiltonian.*

Let us also point out some results for the case where $R = \mathbb{Z}_n$, the ring of integers modulo n. In [9] the independence number and clique number of $T(\Gamma(\mathbb{Z}_n))$ are computed. Domination in $T(\Gamma(\mathbb{Z}_n))$ is the subject of study of [7] and domination in the total graph of a commutative ring is studied in [8]. For more results on the total graph of a commutative ring the reader may refer to [1], [6], [11], [14].

Despite the rigorous studies of the total graph of a commutative ring, little is known about flows in such graphs. In this paper, we construct minimum zero-sum k-flows for the total graphs of finite commutative rings. If $|R|$ is even, then this follows in a straightforward manner from existing results. The same is true when $|R|$ is odd and $Z(G)$ is an ideal of R. The case when $|R|$ is odd and $Z(G)$ is not an ideal of R is much more intricate. Here we consider the special case $R = \mathbb{Z}_n$ and construct minimum zero-sum k-flows for $n = p^r q^s$, where p, q are primes and r, s are positive integers. As will be explained below, the structure of $T(\Gamma(\mathbb{Z}_n))$ involves spanning subgraphs that are unions of regular graphs. Hence we elaborate on minimum zero-sum k-flows of regular graphs in Section 2. Minimum c-sum k-flows of regular graphs are investigated in Section 3. Finally, Section 4 is dedicated to the study of minimum zero-sum k-flows of $T(\Gamma(\mathbb{Z}_n))$ for $n = p^r q^s$, where p, q are odd primes and r, s are positive integers.

2. Minimum Zero-sum k-flows in Regular Graphs

In this section, minimum zero-sum k-flows of regular graphs will be constructed. The following lemma characterizes connected graphs which admit zero-sum 2-flows:

Lemma 2.1. [1] *A connected graph with at least one vertex has a zero-sum 2-flow if and only if every vertex has even degree and the number of edges is even.*

Since a graph G is Eulerian if and only if the degree of each vertex of G is even, the following corollary is obvious:

Corollary 2.2. *If a graph is Eulerian with an even number of edges, then it has a zero-sum 2-flow.*

Applying Lemma 2.1 to the complete graph K_n and the complete bipartite graph $K_{n,n}$, we obtain:

Corollary 2.3. *(i) K_n has a zero-sum 2-flow if and only if $4|(n-1)$.*
(ii) $K_{n,n}$ has a zero-sum 2-flow if and only if $2|n$.

Zero-sum k-flows in regular graphs are considered in [2], [3], [4]. The following lemma improves on those results in the sense that it gives minimum zero-sum k-flows for such graphs.

But first let us recall some basic facts about factors in graphs. A factor of a graph G is a spanning subgraph of G. A k-factor of a graph is a spanning k-regular subgraph, a k-factorization partitions the edges of the graph into disjoint k-factors. A graph G is said to be k-factorable if it admits a k-factorization. In particular, a one-factor is a perfect matching. A two-factor is a collection of cycles that spans all vertices of the graph. A near-one-factor is a set of independent edges which cover all but one vertex. A set of near-one-factors which covers every edge precisely once is called a near-one-factorization. Every $2k$-regular graph has a two-factorization [12]. The complete bipartite graph $K_{n,n}$ is one-factorable, [10], the complete graph K_{2n} is one-factorable, and K_{2n+1} is near-one-factorable of even degree of regularity.

Lemma 2.4. *(i) Every $4k$-regular graph has a zero-sum 2-flow.*
(ii) Every $(4k+1)$-regular graph with a one-factor has a zero-sum 3-flow, but no zero-sum 2-flow.
(iii) Every $(4k+2)$-regular graph has a zero-sum 3-flow, but no zero-sum 2-flow.
(iv) Every $(4k+3)$-regular graph with a one-factor has a zero-sum 3-flow, but no zero-sum 2-flow.

Proof. (i) Clear.

(ii) Removing a one-factor, the remaining graph has a two-factorization with an even number of two factors. Assign value 2 to one two-factor and -1 to another. Half of the remaining two-factors shall be assigned value 1, the other half receives value -1. Finally, assign value -2 to the one-factor. Keeping in mind that every two-factor has two edges incident to each vertex, we get a zero-sum 3-flow.

(iii) We construct a zero-sum 3-flow as follows. Take any two-factorization of the graph. Assign value 2 to one factor and value -1 to two other factors. For the remaining even number of of factors, assign value 1 to one half and -1 to the other half.

(iv) We proceed similarly as in part (ii). Assign value -2 to the one-factor and value 1 to one two-factor. For the remaining even number of two-factors, assign values 1 and -1 evenly. □

Application of Corollary 2.3 and Lemma 2.4 to the graphs K_n and $K_{n,n}$ yields:

Corollary 2.5. *(i) K_n has a zero-sum 3-flow. It has a zero-sum 2-flow if and only if $4|(n-1)$.*

(ii) $K_{n,n}$ has a zero-sum 3-flow. It has a zero-sum 2-flow if and only $2|n$.

We now turn our attention to total graphs. First of all, we shall expose a trivial case. R is an integral domain if and only if $Z(R) = \{0\}$. In this case, if $T(\Gamma(R))$ contains at least one edge, then according to Theorem 1.2 the graph $T(\Gamma(R))$ is essentially a one-factor (plus maybe an additional isolated vertex). Consequently, $T(\Gamma(R))$ has no zero-sum flow. In the following, this case will be tacitly excluded.

Assume that $|R| > 2$ is even. Then $|Z(R)|$ is even and $2 \in Z(R)$. Hence, by Theorem 1.1 the graph $T(\Gamma(R))$ is regular of degree $|Z(R)| - 1$. At this point it makes sense to require $|Z(R)| > 2$. We now consider two cases:

Case 1: If $Z(R)$ is an ideal of R, then it follows from Theorem 1.2 that $T(\Gamma(R))$ is a union of complete graphs $K_{|Z(R)|}$. Corollary 2.5(i) asserts that in this case $T(\Gamma(R))$ has a zero-sum 3-flow, but no zero-sum 2-flow.

Case 2: If $Z(R)$ is not an ideal of R, then $T(\Gamma(R))$ is Hamiltonian by Theorem 1.3. Consider a Hamilton circuit of $T(\Gamma(R))$. Since it has an even number of edges we can derive a 1-factor from it. In view of Lemma 2.4 we see that $T(\Gamma(R))$ has a zero-sum 3-flow, but no zero-sum 2-flow.

Summing up, we obtain the following result:

Theorem 2.6. *If $|R|$ is even and $|Z(R)| > 2$, then $T(\Gamma(R))$ has a zero-sum 3-flow, but no zero-sum 2-flow*

3. Constant-Sum k-Flows in Regular Graphs

The following theorem presents a class of graphs which admit 2-sum 2-flows.

Theorem 3.1. *An r-regular connected graph with an odd number of vertices has a 2-sum 2-flow if and only if $r \equiv 2 \mod 4$.*

Proof. For $r = 2$, simply assign 1 to all edges (circuit). Let $r > 2$. Consider an arbitrary vertex. To get sum 2, there must be exactly two more edges with value 1 among its incident edges than there are with value -1. Moreover, due to the handshaking lemma, and the odd number of vertices, the number of 1-edges and the number of -1-edges incident to any vertex must both be even. This yields $r = (2s + 2) + 2s = 4s + 2$, with $s > 0$ for the degree of regularity. The converse claim is clear. □

Corollary 3.2. *K_{2n+1} has a 2-sum 2-flow if and only if n is odd.*

Theorem 3.3. *(i) An even-regular graph with an odd number of vertices cannot have an odd-sum 3-flow.*

(ii) An odd-regular graph cannot have an even-sum 2-flow.

Proof. (i) Suppose we have an odd-sum 3-flow for this graph. Consider an arbitrary vertex. The incident edge values are from the set $\{1, -1, 2, -2\}$ and add up to an odd number. In order to achieve an odd sum, at least one of these values must occur an odd number of times. Since the vertex degree is even, it must be either two or four of the values occurring an odd number of times. We exclude the cases $\{1, -1\}$, $\{2, -2\}$ and $\{1, -1, 2, -2\}$ as the mentioned values since any sum of odd multiples of those numbers would be even (but needs to be odd). What remains are the cases $\{1, -2\}$ and $\{-1, 2\}$. Since these value sets are disjoint, it follows that each case must occur an even number of times among all vertices, owing to the handshaking lemma. But then the number of vertices would be even, a contradiction.

(ii) Consider an arbitrary vertex. Without loss of generality, there must be an even number of 1-valued edges and an odd number of -1-valued edges incident to this vertex. But this cannot yield an even sum. □

Corollary 3.4. *K_{2n+1} has no 1-sum 3-flow.*

Lemma 3.5. *A 2-regular connected graph (i.e. a circuit) with an odd number of vertices has a c-sum k-flow if and only if c is even, $c > 0$, and $k > c/2$.*

Proof. Place an integer value a on some edge. Its adjacent edges must then have value $c - a$, their adjacent edges in turn a, and so on. Due to the odd number of edges, we hence require $a = c - a$ so that $2a = c$. □

4. Minimum zero-sum k-flows of $T(\Gamma(R))$ when $|R|$ is odd

In Theorem 2.6 we have considered the case when $|R|$ is even. If $|R|$ is odd and $Z(R)$ is an ideal, then according to Theorem 1.1 the graph $T(\Gamma(R))$ is a union of complete graphs K_λ and complete bipartite graphs $K_{\lambda,\lambda}$ for $\lambda = |Z(R)|$. The degree of K_λ is even if and only if the degree of $K_{\lambda,\lambda}$ is odd. Hence by virtue of Corollary 2.5 we obtain:

Theorem 4.1. *If $|R|$ is odd and $Z(R)$ is an ideal with $|Z(R)| > 2$, then $T(\Gamma(R))$ has a zero-sum 3-flow, but no zero-sum 2-flow.*

Let us turn to the remaining case when $|R|$ is odd and $Z(R)$ is not an ideal. Then $Z(R)$ is a union of at least two prime ideals of R, say $I_1, \ldots, I_r$ for some $r \geq 2$. Hence the graph $T(\Gamma(R))$ assumes a layered structure. To see this, let H_k be the spanning subgraph of $T(\Gamma(\mathbb{Z}_n))$, where two vertices x and y are adjacent if $x + y \in I_k$. Obviously, $T(\Gamma(R))$ is the (not necessarily edge disjoint) overlay of the graphs $H_1, \ldots, H_r$. Since $|R|$ is odd we can derive from Theorem 1.2 that each graph H_k takes the form

$$H_k = K_\lambda \cup \left(\bigcup_{i=1}^{\frac{\mu-1}{2}} K_{\lambda,\lambda} \right)$$

for suitable integers λ, μ. In the following, we shall exploit this structure.

In the sequel, we shall restrict ourselves to $R = \mathbb{Z}_n$. In view of the remaining case we require that n is odd and composite.

If p and q are distinct odd primes and $n = pq$, then $Z(\mathbb{Z}_{pq})$ has two maximal principal ideals, namely, the ideal $\langle p \rangle$ and the ideal $\langle q \rangle$. Let H_1 be the spanning subgraph of $T(\Gamma(\mathbb{Z}_n))$, where two vertices x and y are adjacent if $x + y \in \langle q \rangle$. The vertices in $\langle q \rangle$ induce the complete subgraph K_p. If $a + b \equiv 0 \pmod q$, then the coset $a + \langle q \rangle$ and the coset $b + \langle q \rangle$ together induce the complete bipartite graph $K_{p,p}$. Thus $H_1 = K_p \cup [\bigcup_{i=1}^{(q-1/2)} K_{p,p}]$. Let H_2 be the spanning subgraph of $T(\Gamma(\mathbb{Z}_n))$, where two vertices x and y are adjacent if $x + y \in \langle p \rangle$. Then, $H_2 = K_q \cup [\bigcup_{i=1}^{(p-1/2)} K_{q,q}]$. We will refer to the spanning subgraphs H_1 and H_2 as layers of the original graph $T(\Gamma(\mathbb{Z}_n))$. Note that the edges of the two layers overlap, and $E(T(\Gamma(\mathbb{Z}_n))) = E(H_1) \cup E(H_2)$. The next theorem determines minimum flow in the graph $T(\Gamma(\mathbb{Z}_{pq}))$.

Theorem 4.2. *Let $p, q > 3$ be two distinct primes. Then the graph $T(\Gamma(\mathbb{Z}_{pq}))$ has a zero-sum 3-flow, but no zero-sum 2-flow.*

Proof. The graph has no zero-sum 2-flow since it has both odd and even-degree vertices. Now, consider the two layers as described above. We are going to construct a zero-sum 3-flow for each layer in such a way that the sum of the two flows provides a zero-sum 3-flow for $T(\Gamma(\mathbb{Z}_{pq}))$. Focus on the spanning subgraph H_1 and consider the edges also present in H_2. Each $K_{q,q}$ overlaps K_p with exactly one edge, so altogether we get a near-one-factor overlapping K_p. K_q overlaps exactly one edge of each $K_{p,p}$, and so does $K_{q,q}$. Hence each $K_{p,p}$ is overlapped by a one-factor. Since $K_{p,p}$ is one-factorable, we can find a 3-flow such that one of the one-factors has values 1 only. We can arrange this to be the one-factor overlapped by K_q. In K_p we extend the overlapping near-one-factor to a two-factor. Since K_p is two factorable (p is odd), we can find a 3-flow such that one of the two factors is the constructed two-factor and has only values 1. Now focus on the layer H_2 and proceed analogously. Next, combine the flows inside each layer. Adding up the two layer flows, we finally obtain an overall 3-flow since the values in the overlapping edges always sum to 2. □

The case $p = 3$ is addressed in the next theorem. In order to prove it, we require the following lemma.

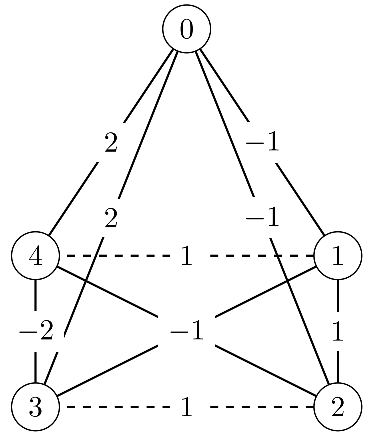


Figure 1. An assignment to the edges of K_5 so that the top (vertex 0) has edge sum two, all other vertices have sum zero.

Lemma 4.3. *There exists an assignment of values* $\{1, -1, 2, -2\}$ *to the edges of* $K_{4s+1}, s \in \mathbb{N}$*, such that for one vertex the sum of the values of its incident edges equals two, whereas for all other vertices it equals zero. Moreover, this assignment assigns only values* 1 *to some near-one-factor of the graph.*

Proof. We start with a concrete assignment for K_5 as in Figure 1.

The top vertex has edge sum two, all other vertices have sum zero. The mentioned near-one-factor is marked by dashed edges. We extend the assignment by adding $4(s-1)$ new vertices. We have to connect each vertex of K_5 with every new vertex. We partition the new edges according to the cases (a), (b) and (c) depicted in Figure 2.

In (a) assign a 2-flow to each new $K_{4,4}$. In (b) assign a 3-flow to each new K_5 such that the independent dashed edges receive value 1 each. In (c) the graph $K_{4,\ldots,4}$ is two-factorable due to its even degree. Assign a 3-flow. Altogether, this gives the required assignment for the edges of K_{4s+1}. By construction, we have a 1-valued matching of the vertices outside the upper K_5, so we can use these edges to extend the 1-valued near-one-factor. The dashed edges in (b) extend the 1-valued near-one-factor. □

Theorem 4.4. *Let* $q \neq 3$ *be a prime. Then the graph* $T(\Gamma(\mathbb{Z}_{3q}))$ *admits a zero-sum* 3*-flow, but no zero-sum* 2*-flow.*

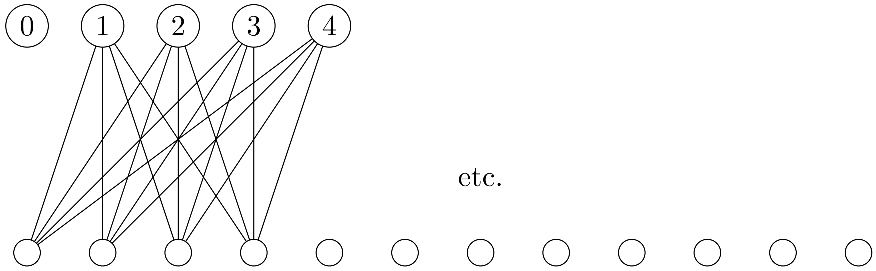
Proof. The graph consists of two layers:

$$H_1 = K_3 \cup \left(\bigcup_{i=1}^{(q-1/2)} K_{3,3} \right),$$

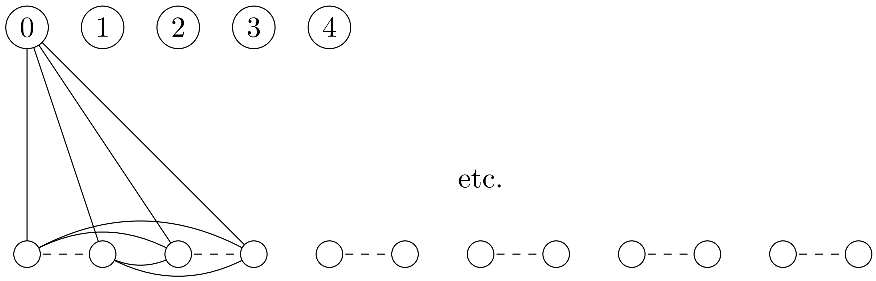
$$H_2 = K_q \cup K_{q,q}.$$

Consider the following two cases:

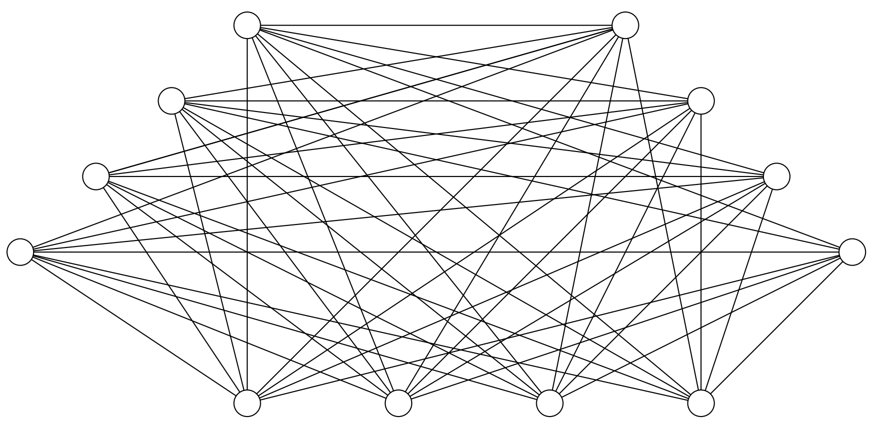
Case (i) $q = 4s+1, s \in \mathbb{N}$. Assign value 1 to the edge of K_3 that overlaps $K_{q,q}$, and assign value -1 to the other two edges. Next, assign values to K_q as in Lemma 4.3, with vertex 0 as the two-sum vertex. This assignment gives edge-sum zero for every vertex of the total graph, omitting all



(a) Building copies of $K_{4,4}$.



(b) Building copies of K_5 using the vertex 0 and each group of four new vertices.



(c) Building the complete multipartite graph $K_{4,4,\ldots,4}$.

FIGURE 2. Partitioning the edges connecting K_5 with the $4(s-1)$ new vertices.

unassigned edges. We now readily extend this assignment to a 3-flow by employing the technique in the proof Theorem 4.2 since the values of the overlapping edges are the same.

Case (ii) $q = 4s + 3, s \in \mathbb{N}$. We start with a partial assignment as shown in Figure 1. We extend it to an assignment of the same type, but for the entire K_7. The additional edges besides the considered subgraph K_5 induce the join of the null graph N_5 with K_2, as shown in Figure 3.

This is an even-degree graph, so it is two-factorable. We can easily construct a 3-flow on it such that the edge between the two bottom vertices receives value 1. Now, the rest of the proof of case (i) can be applied. The only difference is that case (i) now gives $K_{6,4}$; but this has a 3-flow. □

If $n = p^r q^s$, where p, q are distinct primes, r and s are positive integers, then the graph $T(\Gamma(\mathbb{Z}_n))$ has two layers as above. The only difference is that the overlapping edges form $p^{r-1}q^{s-1}$ one-factors

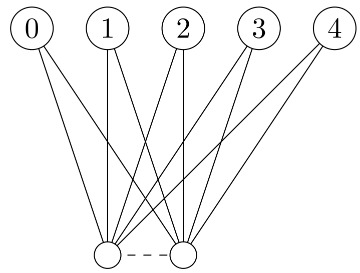


FIGURE 3. Extending the K_5 assignment to the K_7.

for each bipartite graph and $p^{r-1}q^{s-1}$ near-one-factors for the complete graphs. Thus, using the same technique as above, we get

Theorem 4.5. *Let $n = p^r q^s$, where p, q are distinct primes and r, s are positive integers. Then $T(\Gamma(\mathbb{Z}_n))$ has a zero-sum 3-flow, but no zero-sum 2-flow.*

Motivated by our findings so far, we pose the following conjectures:

Conjecture 4.6. *For every odd integer $n > 1$ the graph $T(\Gamma(\mathbb{Z}_n))$ has a zero-sum 3-flow, but no zero-sum 2-flow.*

Conjecture 4.7. *If $|R|$ is odd, $|Z(R)| > 2$ and $Z(R)$ is not an ideal, then $T(\Gamma(R))$ has a zero-sum 3-flow, but no zero-sum 2-flow.*

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