Transactions on Combinatorics
ISSN (print): 2251-8657, ISSN (on-line): 2251-8665
Vol. 3 No. 3 (2014), pp. 35-41.

www.combinatorics.ir
www.ui.ac.ir

ON ASSOCIATION SCHEMES WITH COMMUTATIVE THIN THIN RESIDUE

JAVAD BAGHERIAN

Communicated by Alireza Abdollahi

Abstract. The main result of this paper gives a characterization of association schemes having commutative thin thin residue. This gives a generalization of Ito's Theorem on finite groups for association schemes.

1. Introduction

The theory of association schemes is a generalization of the theory of finite groups. So, it is natural to ask which group theoretic results can be generalized to association schemes. As an important results in the character theory of finite groups is Ito's Theorem. This theorem is as the following:

Theorem 1.1. *(Ito) Let G be a finite group and A be an abelian normal subgroup of G. Then $\chi(1)$ divides $|G : A|$ for all $\chi \in \mathrm{Irr}(G)$.*

The above theorem gives a characterization of finite groups which contain an abelian normal subgroup (See [7, Theorem 6.15]). It is known that every finite group G can be identify to the thin association scheme $(G, \widetilde{G})$ where $\widetilde{G} = \{\widetilde{g} | g \in G\}$ and $\widetilde{g} = \{(h, k) \in G \times G | hg = k\}$. Moreover, for every $\chi \in \mathrm{Irr}(G)$, $\chi(1) = m_\chi$, where m_χ is the multiplicity of χ in the decomposition of standard character of association scheme $(G, \widetilde{G})$. Since normal subgroups of G are strongly normal closed subsets of association scheme $(G, \widetilde{G})$, the following theorem generalize Ito's Theorem in the theory of finite groups (see Section 2 for definitions):

Theorem 1.2. *Let (X, S) be an association scheme and T a strongly normal closed subset of S. If T is an abelian group, then for every $\chi \in \mathrm{Irr}(S)$, m_χ divides $|X|/|T|$.*

MSC(2010): Primary: 05E30; Secondary: 20C99.
Keywords: association scheme, strongly normal, thin residue.
Received: 24 March 2014, Accepted: 24 May 2014.

In this paper as a main result we give a proof for Theorem 1.2 in Section 3.

2. **Preliminaries**

Let us first state some necessary definitions and notation. For details, we refer the reader to [8] for the background of association schemes. Throughout this paper, $\mathbb{C}$ denotes the complex numbers.

Definition 2.1. *Let X be a finite set and S be a partition of $X \times X$. Then (X, S) is called an association scheme (or shortly scheme) if the following properties hold:*

(i) $1_X \in S$, *where* $1_X := \{(x, x)|x \in X\}$.

(ii) *For every $s \in S$, s^* is also in S, where $s^* := \{(x, y)|(y, x) \in s\}$.*

(iii) *For every $g, h, k \in S$, there exists a nonnegative integer λ_{ghk} such that for every $(x, y) \in k$, there exist exactly λ_{ghk} elements $z \in X$ with $(x, z) \in g$ and $(z, y) \in h$.*

For each $s \in S$, we call $n_s = \lambda_{ss^*1_X}$ the *valency* of s. For any nonempty subset H of S, put $n_H = \sum_{h\in H} n_h$. We call n_S the *order* of (X, S). Clearly, $n_S = |X|$.

Let H and K be nonempty subsets of S. We define HK to be the set of all elements $t \in S$ such that there exist element $h \in H$ and $k \in K$ with $\lambda_{hkt} \neq 0$. The set HK is called the *complex product* of H and K. If one of factors in a complex product consists of a single element s, then one usually writes s for $\{s\}$. A scheme (X, S) is called *commutative* if for all $g, h, k \in S$, $\lambda_{ghk} = \lambda_{hgk}$.

A nonempty subset H of S is called a *closed subset* if $HH \subseteq H$. For a closed subset H of S we define $\mathrm{O}_\vartheta(H) = \{h \in H|n_h = 1\}$, called the *thin radical* of H. Note that $\mathrm{O}_\vartheta(H)$ is a closed subset of S. In fact $\mathrm{O}_\vartheta(H)$ is a group with respect to the relational product. The closed subset H is called *thin* if $\mathrm{O}_\vartheta(H) = H$. A closed subset H of S is called *strongly normal*, denoted by $H \lhd^\sharp S$, if $sHs^* = H$ for any $s \in S$. We put $\mathrm{O}^\vartheta(S) = \cap_{H\lhd^\sharp S} H$ and call it the *thin residue* of H. One can see that $\mathrm{O}^\vartheta(S) = \langle \cup_{s\in S} ss^* \rangle$.

Let H be a closed subset of S. For every $h \in H$ we define $xh = \{y \in X|(x, y) \in h\}$. Put $X/\!\!/H = \{xH|x \in X\}$ and $S/\!\!/H = \{s^H|s \in S\}$, where $xH = \bigcup_{h\in H} xh$ and $s^H = \{(xH, yH)|y \in xHsH\}$. Then $(X/\!\!/H, S/\!\!/H)$ is a scheme, called the *quotient scheme* of (X, S) over H. Note that a closed subset H is strongly normal iff the quotient scheme $(X/\!\!/H, S/\!\!/H)$ is a group with respect to the relational product iff $ss^* \subseteq H$, for every $s \in S$.

Let (X, S) be a scheme. For every $s \in S$, let σ_s be the adjacency matrix of s. For any nonempty subset H of S, we put $\sigma_H := \{\sigma_h|h \in H\}$. For convenience σ_{1_X} is denoted by 1. It is known that $\mathbb{C}S = \bigoplus_{s\in S} \mathbb{C}\sigma_s$, the *adjacency algebra* of (X, S), is a semisimple algebra. The set of irreducible characters of S is denoted by $\mathrm{Irr}(S)$. We denote by e_χ, the central primitive idempotent of $\mathbb{C}S$ corresponding to χ. An irreducible character $\chi \in \mathrm{Irr}(S)$ is called *faithful* if $\mathrm{Ker}(\chi) = \{1_X\}$, where $\mathrm{Ker}(\chi) = \{s \in S|\chi(\sigma_s) = n_s\chi(1)\}$. One can see that $1_S \in \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}S, \mathbb{C})$ such that $1_S(\sigma_s) = n_s$ is an irreducible character of $\mathbb{C}S$, which is called the *principal* character. In [4], Hanaki has shown that the irreducible characters of $S/\!\!/\mathrm{O}^\vartheta(S)$ can be consider as irreducible characters of S.

Let Γ_S be a representation of $\mathbb{C}S$ which sends σ_s to itself for every $s \in S$. Let γ_S be the character afforded by Γ_S. Then one can see that $\gamma_S(1) = |X|$ and $\gamma_S(\sigma_s) = 0$ for every $1_X \neq s \in S$. Consider the following irreducible decomposition of γ_S,

$$\gamma_S = \sum_{\chi \in \mathrm{Irr}(S)} m_\chi \chi.$$

Then we call m_χ the *multiplicity* of χ and $\{m_\chi | \chi \in \mathrm{Irr}(S)\}$, the set of *multiplicities* of (X, S). One can see that $m_{1_S} = 1$ and $|X| = \sum_{\chi \in \mathrm{Irr}(S)} m_\chi \chi(1)$ (see [8, section 4]). Moreover, for every $\chi \in \mathrm{Irr}(S)$, $\chi \in \mathrm{Irr}(S/\!\!/\mathrm{O}^\vartheta(S))$ if and only if $m_\chi = \chi(1)$ (see [6]).

Let (X, S) be a scheme and T a closed subset of S. Suppose that L is a $\mathbb{C}T$-module which affords the character φ, and V is a $\mathbb{C}S$-module which affords the character χ. Then V is a $\mathbb{C}T$-module which affords the restriction χ_T of χ to $\mathbb{C}T$, and $L^S = L \otimes_{\mathbb{C}T} \mathbb{C}S$ is a $\mathbb{C}S$-module which affords the induction φ^S of φ. For all characters χ, ψ of $\mathbb{C}T$ we define

$$(\chi, \psi)_T = \sum_{\varphi \in \mathrm{Irr}(T)} a_\varphi b_\varphi,$$

where $\chi = \sum_{\varphi \in \mathrm{Irr}(T)} a_\varphi \varphi$ and $\psi = \sum_{\varphi \in \mathrm{Irr}(T)} b_\varphi \varphi$.

Theorem 2.2 (See [2].). *Let (X, S) be a scheme and T a closed subset of S. Suppose that $\varphi \in \mathrm{Irr}(T)$. Then*

$$\frac{n_S}{n_T} m_\varphi = \sum_{\chi \in \mathrm{Irr}(S)} (\varphi^S, \chi)_S m_\chi.$$

Let (X, S) be a scheme and T be a strongly normal closed subset of S. Put $G = S/\!\!/T$. Let φ be an irreducible character of T and L be an irreducible $\mathbb{C}T$ module affording φ. Consider the induction of L to S. Then one can see that

$$L^S = L \otimes_{\mathbb{C}T} \mathbb{C}S = \bigoplus_{s^T \in S/\!\!/T} L \otimes \mathbb{C}(TsT).$$

The *stabilizer* $G\{L\}$ of L in G is defined by

$$G\{L\} = \{s^T \in S/\!\!/T | L \otimes \mathbb{C}(TsT) \cong L\}.$$

One can see that $G\{L\}$ is a subgroup of G.

Theorem 2.3. *(See [2].) Let (X, S) be a scheme and T be a strongly normal closed subset of S. Fix an irreducible character φ of $\mathbb{C}T$. Suppose that $U/\!\!/T$ is the stabilizer of φ in $S/\!\!/T$. Put*

$$A = \{\psi \in \mathrm{Irr}(U) | (\psi_T, \varphi) \neq 0\}, \ B = \{\chi \in \mathrm{Irr}(S) | (\chi_T, \varphi) \neq 0\}.$$

Then there exists a bijection $\tau : A \to B$ such that $\tau(\psi) = \psi^S$. Moreover, $\psi = \tau^{-1}(\chi)$ is the unique element of A where $(\chi_U, \psi) \neq 0$.

3. Proof of the main theorem

Let (X,S) be a scheme and T be a strongly normal closed subset of S such that $T \subseteq \mathrm{O}_\vartheta(S)$ and commutative. Let $\chi \in \mathrm{Irr}(S)$ and $\varphi \in \mathrm{Irr}(T)$ such that $(\chi_T, \varphi) \neq 0$. Consider the induction of $\mathbb{C}e_\varphi$ to S. Then we have

$$\mathbb{C}e_\varphi \otimes_{\mathbb{C}T} \mathbb{C}S = \bigoplus_{s^T \in S/\!\!/T} \mathbb{C}e_\varphi \otimes \mathbb{C}(TsT).$$

Let $H/\!\!/T$ be the stabilizer of $\mathbb{C}e_\varphi$ in $S/\!\!/T$.

Lemma 3.1. *Suppose that $H/\!\!/T = S/\!\!/T$. Then m_χ divides n_S/n_T.*

Proof. Since $H/\!\!/T = S/\!\!/T$ it follows that for every $s \in S$, $\mathbb{C}e_\varphi \otimes \mathbb{C}(TsT) \cong \mathbb{C}e_\varphi$. Let $s \in S - T$. Put $T_s = \{t \in T | st = s\}$. We show that $T_s \leq \mathrm{Ker}(\varphi)$. To do so, let $t \in T_s$. Then for every $t' \in T$ we have $(st')t = (st)t' = st'$. So $< t >$ acts trivially on $TsT = sT$. This implies that $t \in \mathrm{Ker}(\mathbb{C}e_\varphi \otimes \mathbb{C}(TsT))$. Since $\mathbb{C}e_\varphi \otimes \mathbb{C}(TsT) \cong \mathbb{C}e_\varphi$ it follows that $t \in \mathrm{Ker}(\varphi)$.

Now let $s \in S - T$. Since T is strongly normal we have $s^*s \subseteq T$. Let $t \in s^*s$. Since $\lambda_{s^*st}n_t = \lambda_{st^*s}n_s$ we get $st^* = s$ and then $s = st$. So $t \in T_s \leq \mathrm{Ker}(\varphi)$. This shows that $s^*s \subseteq \mathrm{Ker}(\varphi)$ and then $\mathrm{O}^\vartheta(S) \subseteq \mathrm{Ker}(\varphi)$. Since

$$S/\!\!/\mathrm{Ker}(\varphi) \simeq (S/\!\!/\mathrm{O}^\vartheta(S))/\!\!/(\mathrm{Ker}(\varphi)/\!\!/\mathrm{O}^\vartheta(S)),$$

it follows that $S/\!\!/\mathrm{Ker}(\varphi)$ is a finite group and hence from [8, Theorem 2.2.3] we have $\mathrm{Ker}(\varphi)$ is a strongly normal closed subset of S. Put $K = \mathrm{Ker}(\varphi)$. Since

$$(\varphi^S)_T = \frac{n_S}{n_T}\varphi$$

and $(\varphi^S, \chi) \neq 0$ we conclude that $\chi_T = e\varphi_T$, for some positive integer e. Thus $\chi_K = f\varphi_K$, for some positive integer f. It follows that $K < \mathrm{Ker}(\chi)$ and hence $\chi \in \mathrm{Irr}(S/\!\!/K)$. Since $T/\!\!/K$ is an abelian normal closed subset of $S/\!\!/K$, from Theorem1.1 we have

$$m_\chi = \chi(1) | \frac{\frac{n_S}{n_K}}{\frac{n_T}{n_K}} = \frac{n_S}{n_T}.$$

This completes the proof. □

Lemma 3.2. *If $\{T/\!\!/T\} \leq H/\!\!/T < S/\!\!/T$, then m_χ divides n_S/n_T.*

Proof. From Theorem 2.3 it follows that there exists $\psi \in \mathrm{Irr}(H)$ such that $(\psi_H, \varphi) \neq 0$ and $\chi = \psi^S$. Since the stabilizer of $\mathbb{C}e_\varphi$ in $H/\!\!/T$ is $H/\!\!/T$, from Lemma 3.1 we have

$$m_\psi | \frac{n_H}{n_T}$$

and hence

$$\frac{n_H}{n_T} = qm_\psi, \tag{3.1}$$

for some positive integer q. On the other hand, since $\chi = \psi^S$ we have

$$m_\chi = \frac{n_S}{n_H} m_\psi$$

and then from equality (3.1) we obtain

$$qm_\chi = \frac{n_S}{n_H}\frac{n_H}{n_T} = \frac{n_S}{n_T}.$$

So m_χ divides n_S/n_T. This completes the proof. □

Proof of Theorem 1.2. Let $\chi \in \mathrm{Irr}(S)$ and $\varphi \in \mathrm{Irr}(T)$ such that $(\chi_T, \varphi) \neq 0$. Suppose that $H/\!\!/T$ is the stabilizer of $\mathbb{C}e_\varphi$ in $S/\!\!/T$. Then either $H/\!\!/T = S/\!\!/T$ or $\{T/\!\!/T\} \leq H/\!\!/T < S/\!\!/T$. So the result follows from Lemmas 3.1 and 3.2, respectively. □

Corollary 3.3. *Let (X, S) be a commutative scheme and T a thin strongly normal closed subset of S. Then for every $\chi \in \mathrm{Irr}(S)$, m_χ divides n_S/n_T.*

Remark 3.4. *The abelian condition for T in Theorem 1.2 is a necessary condition. In the example below we give a scheme with a nonabelian strongly normal closed subset which the conclusion of Theorem 1.2 does not hold for (X, S).*

Example 3.5. (This example is [5, as12, No. 42].)
Let (X, S) be a scheme of order 12 with the following basic matrix

$$\sum_{i=0}^{7} i\sigma_{g_i} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 6 & 6 & 7 & 7 & 7 \\ 1 & 0 & 4 & 5 & 2 & 3 & 7 & 7 & 7 & 6 & 6 & 6 \\ 2 & 5 & 0 & 4 & 3 & 1 & 7 & 7 & 7 & 6 & 6 & 6 \\ 3 & 4 & 5 & 0 & 1 & 2 & 7 & 7 & 7 & 6 & 6 & 6 \\ 5 & 2 & 3 & 1 & 0 & 4 & 6 & 6 & 6 & 7 & 7 & 7 \\ 4 & 3 & 1 & 2 & 5 & 0 & 6 & 6 & 6 & 7 & 7 & 7 \\ 6 & 7 & 7 & 7 & 6 & 6 & 0 & 4 & 5 & 1 & 2 & 3 \\ 6 & 7 & 7 & 7 & 6 & 6 & 5 & 0 & 4 & 2 & 3 & 1 \\ 6 & 7 & 7 & 7 & 6 & 6 & 4 & 5 & 0 & 3 & 1 & 2 \\ 7 & 6 & 6 & 6 & 7 & 7 & 1 & 2 & 3 & 0 & 4 & 5 \\ 7 & 6 & 6 & 6 & 7 & 7 & 2 & 3 & 1 & 5 & 0 & 4 \\ 7 & 6 & 6 & 6 & 7 & 7 & 3 & 1 & 2 & 4 & 5 & 0 \end{pmatrix}$$

where $S = \{s_0, s_1, \ldots, s_7\}$. Then from [5] the character table of the complex adjacency algebra of S is as follows:

	σ_{s_0}	σ_{s_1}	σ_{s_2}	σ_{s_3}	σ_{s_4}	σ_{s_5}	σ_{s_6}	σ_{s_7}	m_χ
χ_1	1	1	1	1	1	1	3	3	1
χ_2	1	1	1	1	1	1	-3	-3	1
χ_3	1	-1	-1	-1	1	1	3	-3	1
χ_4	1	-1	-1	-1	1	1	-3	3	1
χ_5	2	0	0	0	-1	-1	-0	0	4

One can see that $T = \{s_0, \ldots, s_5\}$ is a strongly normal closed subset of S which is isomorphic to the symmetric group S_3. Consider irreducible character χ_5 of S. Then $m_{\chi_5} \nmid n_S/n_T$ and so conclusion of Theorem 1.2 does not hold for (X, S).

Let (X, S) be a scheme. Let χ be a character of S. Put $Z(\chi) = \{s \in S || \chi(\sigma_s)| = n_s\chi(1)\}$. Then $Z(\chi)$ is a closed subset of S containing $\mathrm{Ker}(\chi)$; see [3]

Corollary 3.6. *Let (X, S) be a scheme and χ an irreducible character of S. If $Z(\chi)$ is a strongly normal closed subset of S, then m_χ divides $n_S/n_{Z(\chi)}$.*

Proof. First we assume that χ is a faithful character. Then from [1, Theorem 3.1], $Z(\chi)$ is cyclic as a finite group. So $Z(\chi)$ is an abelian strongly normal closed subset of S and Theorem 1.2 shows that m_χ divides $n_S/n_{Z(\chi)}$.

Now we assume that χ is not a faithful character of S. Put $Z = Z(\chi)$ and $K = \mathrm{Ker}(\chi)$. Then from [1, Theorem 2.1], χ can be considered as a faithful irreducible character of $S/\!\!/K$. So there exists a faithful irreducible character χ' of $S/\!\!/K$ such that

$$\chi'(\sigma_{s^K}) = (n_{s^K}/n_s)\chi(\sigma_s),$$

for every $s \in S$. Since

$$S/\!\!/Z \cong (S/\!\!/K)/\!\!/(Z/\!\!/K)$$

it follows that $Z/\!\!/K$ is a strongly normal closed subset of $S/\!\!/K$. Moreover, since

$$\begin{aligned} Z(\chi') &= \{s^K \in S/\!\!/K || \chi'(\sigma_{s^K})| = n_{s^K}\chi'(\sigma_{1^K})\} \\ &= \{s^K \in S/\!\!/K | (n_{s^K}/n_s)|\chi(\sigma_s)| = n_{s^K}\chi'(\sigma_{1^K})\} \\ &= \{s^K \in S/\!\!/K || \chi(\sigma_s)| = n_s\chi(1)\} \\ &= \{s^K \in S/\!\!/K | s \in Z\} = Z/\!\!/K, \end{aligned}$$

from above first case we conclude that $m_{\chi'}$ divides n_S/n_Z. But from [4, Thoerem 4.1], $m_\chi = m_{\chi'}$ and so m_χ divides n_S/n_Z, as desired. $\square$

Acknowledgments

The author would like to thank anonymous referees for their valuable comments. This research was partially supported by the Center of Excellence for Mathematics, University of Isfahan.

References

[1] A. Hanaki, Faithful representation of association schemes, *Proc. Amer. Math. Soc.*, **139** (2011) 3191-3193.
[2] A. Hanaki, Clifford theory for association schemes, *J. Algebra*, **321** (2009) 1686-1695.
[3] A. Hanaki, Nilpotent schemes and group-like schemes, *J. Combin. Theory Ser. A*, **115** (2008) 226-236.
[4] A. Hanaki, Representations of association schemes and their factor schemes, *Graphs Combin.*, **19** (2003) 195-201.

[5] A. Hanaki, Data for Association Schemes, http://kissme.shinshu-u.ac.jp/as/data.

[6] M. Hirasaka and M. Muzychuk, Association schemes generated by a non-symmetric relation of valency 2, *Discrete Math.*, **244** (2002) 109-135.

[7] M. Isaacs, *Character Theory of Finite Groups*, Dover Publications, Inc., New York, 1994.

[8] P.-H. Zieschang, *Theory of Association Schemes*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2005.

Javad Bagherian

Department of Mathematics, University of Isfahan, P.O.Box 81746-73441, Isfahan, Iran

`bagherian@sci.ui.ac.ir`