www.combinatorics.ir

Transactions on Combinatorics
ISSN (print): 2251-8657, ISSN (on-line): 2251-8665
Vol. 3 No. 3 (2014), pp. 51-59.

www.ui.ac.ir

A NOTE ON THE ZERO DIVISOR GRAPH OF A LATTICE

T. TAMIZH CHELVAM* AND S. NITHYA

Communicated by Dariush Kiani

Abstract. Let L be a lattice with the least element 0. An element $x \in L$ is a zero divisor if $x \wedge y = 0$ for some $y \in L^* = L \setminus \{0\}$. The set of all zero divisors is denoted by $Z(L)$. We associate a simple graph $\Gamma(L)$ to L with vertex set $Z(L)^* = Z(L) \setminus \{0\}$, the set of non-zero zero divisors of L and distinct $x, y \in Z(L)^*$ are adjacent if and only if $x \wedge y = 0$. In this paper, we obtain certain properties and diameter and girth of the zero divisor graph $\Gamma(L)$. Also we find a dominating set and the domination number of the zero divisor graph $\Gamma(L)$.

1. Introduction

Let L be a lattice with the least element 0. We associate a simple graph $\Gamma(L)$ to L with the vertex set $Z(L)^* = Z(L) \setminus \{0\}$, the set of non-zero zero divisors of L and distinct $x, y \in Z(L)^*$ are adjacent if and only if $x \wedge y = 0$. There are many papers which interlink graph theory and lattice theory [5, 9, 10, 12, 13, 14]. These papers discuss the properties of graphs derived from partially ordered sets and lattices. In the work of Filipov [12], the adjacency between two elements is defined through the comparability relation between two elements of a poset.

S. K. Nimbhokar, M. P. Wasadikar and M. M. Pawar [14] have introduced the notion of coloring in graphs derived from lattices. In [10], E. Estaji and K. Khashyarmanesh associated to any finite lattice L, a simple graph $G(L)$ whose vertex set is $Z(L)^*$ and two vertices x and y are adjacent if and only if $x \wedge y = 0$. They studied the structure of $G(L)$, connections between the zero divisor graphs of

MSC(2010): Primary: 05C69, 06B99; Secondary: 05C15, 05C25.

Keywords: zero divisor graph, lattice, atomic lattice, ideal, dominating set.

Received: 6 September 2013, Accepted: 12 June 2014.

∗Corresponding author.

The work reported here is supported by the Major Research Project F. No. 37-267/2009(SR) awarded to authors by the University Grants Commission(UGC), Government of India.

lattices and rings and obtained some basic properties of the zero divisor graph of a lattice. The zero divisor graph of various algebraic structures has been studied by several authors [1, 2, 3, 4, 8]. We now recall some definitions. Let $(L, \vee, \wedge)$ be a lattice with the least element 0. Then $a \in L$ is called an *atom* if there is no $y \in L$ such that $0 < y < a$. The set of all atoms of L is denoted by $\mathcal{A}(L)$. The lattice L is called *atomic* if for any $x \in L$, there exists an element $a \in \mathcal{A}(L)$ such that $a \leq x$. An element $x \in L$ is *join-irreducible* if $x = a \vee b$ implies $x = a$ or $x = b$ for $a, b \in L$. The set of all join-irreducible elements in L is denoted by $\mathcal{J}(L)$. Let $(P, \leq)$ be a poset. P is said to satisfy the *ascending chain condition(ACC)*, if given any sequence $x_1 \leq x_2 \leq \cdots \leq x_n \leq \cdots$ of elements of P there exists $k \in \mathbb{N}$ such that $x_k = x_{k+1} = \cdots$. The dual of the ascending chain condition is the *descending chain condition(DCC)*. For $x \in L$, *the annihilator of* x is defined by $Ann(x) = \{y \in L : x \wedge y = 0\}$. A nonempty subset I of L is called an *ideal* if $x, y \in I$ implies $x \vee y \in I$ and $\ell \in L$, $x \in I$ and $\ell \leq x$ imply $\ell \in I$. A proper ideal I of L is said to be *prime* if $a, b \in L$ and $a \wedge b \in I$ imply $a \in I$ or $b \in I$. Suppose L is a lattice, for $x, y \in L$, we write $x \perp y$ if $x \wedge y = 0$ and $Ann(x) \cap Ann(y) = \{0\}$. The undefined terms and notations are from [7].

Let $G = (V, E)$ be a graph. For a graph G, then by $\overline{G}$ we mean the complement of G. We say that G is *connected* if there is a path between any two vertices of G, otherwise G is called disconnected. The *neighbourhood* of a vertex u is the set $N(u)$ consisting of all vertices v which are adjacent with u. For vertices x and y of G, let $d(x, y)$ be the length of a shortest path from x to y. The *diameter* of G is $diam(G) = sup\{d(x, y) : x, y \in V(G)\}$. The *eccentricity* of a vertex v is defined as $e(v) = max\{d(v, y) : y \in V(G)\}$ and the *radius* of G is given by $rad(G) = min\{e(x) : x \in V(G)\}$. The length of a smallest cycle in a graph G is called as *girth* and it is denoted by $gr(G)$. The *clique number* $\omega(G)$ is the cardinality of the maximum possible complete subgraph of G. An *induced subgraph* is a subgraph H of G with vertex set S where vertices are adjacent in H precisely when adjacent in G and it is denoted by $< S >$. A set of edges $M \subseteq E$ is said to be *perfect matching* in G if every vertex in G is adjacent to exactly one edge in M. A *split graph* is a graph whose vertices can be partitioned into two sets V_1 and V_2, where $< V_1 >$ is complete and V_2 is an independent set. A subgraph H is said to be *spanning subgraph* of G if $V(H) = V(G)$. A subset S of V is a called *dominating set* if every $v \in V - S$ is adjacent to some vertex in S. The *domination number* $\gamma(G)$ is the cardinality of the smallest possible dominating set in G. A dominating set S is called an *independent dominating set* if the induced subgraph $< S >$ is disconnected. The *independent domination number* $i(G)$ is the cardinality of the smallest possible independent dominating set. A dominating set S is called a *connected dominating set* if the induced subgraph $< S >$ is connected. The *connected domination number* $\gamma_c(G)$ is the cardinality of the smallest possible connected dominating set.

For basic definitions in graph theory we refer to [6]. Some of the results are well-known in the literature and they are listed for future reference.

Theorem 1.1. ([11, Problem 5.13(b)]). *Let L be a Boolean algebra. Then L is finite if and only if its set of atoms is finite.*

Theorem 1.2. ([11, Theorem 5.4]). *Every finite Boolean algebra is atomic.*

Theorem 1.3. ([11, Problem 5.13(a)]). *Let L be an atomic Boolean algebra. Then 1 is the lub of the set of all atoms.*

Proposition 1.4. ([10, Proposition 2.2]) *If L is a lattice, then $diam(\Gamma(L)) \leq 3$.*

Theorem 1.5. ([14, Theorem 3.2]). *Let L be a distributive lattice with 0 and $\omega(\Gamma(L)) < \infty$. Then L has only a finite number of distinct minimal prime ideals, P_i, $1 \leq i \leq n$. These ideals satisfy $\bigcap_{i=1}^{n} P_i = 0$ and $\bigcap\limits_{i \neq j} P_i \neq 0$ for all j. Further, no element of $L - \bigcup_{i=1}^{n} P_i$ is a zero divisor and every minimal prime ideal of L has the form $Ann(x)$ for some $x \in L$.*

Lemma 1.6. ([14, Lemma 2.1]). *If a distributive complemented lattice contains an infinite increasing chain, then clique $(\Gamma(L)) = \infty$.*

Lemma 1.7. ([14, Lemma 3.4]). *If for some $x, y \in L$, $Ann(x)$ and $Ann(y)$ are distinct prime ideals then $x \wedge y = 0$.*

2. Properties of the zero divisor graph on a lattice

In this section, we prove certain basic properties of the zero divisor graph $\Gamma(L)$ from a lattice L.

Proposition 2.1. *Let L be a lattice satisfying DCC.*

(i) If $x, y \in L^*$ and $x \wedge y = 0$, then there exists a non zero $z \in \mathcal{J}(L)$ such that $z \in N(x)$ and $z \notin N(y)$.

(ii) If $\Gamma(L)$ is complete, then $Z(L) \subseteq \mathcal{J}(L)$.

Proof. (i) Let $x, y \in L^*$ and $x \wedge y = 0$.

Let $S = \{z \in Z(L)^* : z \leq y \text{ and } z \wedge x = 0\}$. Clearly S is non empty as $y \in S$. Since L satisfies DCC, there exists a minimal element $z \in S$. Suppose $z = y_1 \vee y_2$ with $y_1 < z$ and $y_2 < z$. Then $y_1 \wedge x = 0 = y_2 \wedge x$ imply $y_1, y_2 \in S$, a contradiction to the minimality of z. Hence z is join-irreducible and z satisfies the required property in $\Gamma(L)$.

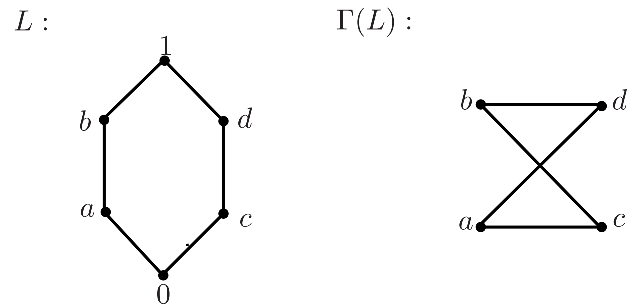
(ii) Assume that $\Gamma(L)$ is complete. For $x, y \in Z(L)^*$, the set S defined in the proof of (i) is a singleton set for every $x \in Z(L)^*$. Again by the proof of (i), $Z(L) \subseteq \mathcal{J}(L)$. □

Theorem 2.2. *Let L be a distributive lattice. Then $\Gamma(L)$ is a complete bipartite graph if and only if there exist prime ideals P_1 and P_2 in L such that $P_1 \cap P_2 = \{0\}$.*

Proof. Assume that $\Gamma(L)$ is a complete bipartite graph with bipartition (V_1, V_2). Set $P_1 = V_1 \cup \{0\}$ and $P_2 = V_2 \cup \{0\}$. Clearly $P_1 \cap P_2 = \{0\}$. Let $x_1, x_2 \in P_1$. If $x_1 = 0$ or $x_2 = 0$, then $x_1 \vee x_2 \in P_1$. Let $x_1, x_2 \neq 0$. Since $\Gamma(L)$ is complete bipartite, for any $y \in V_2$, $x_1 \wedge y = 0$ and $x_2 \wedge y = 0$ and so $(x_1 \vee x_2) \wedge y = 0$. Then $x_1 \vee x_2 \in P_1$. If $z \in L - \{0\}$, $x \in P_1$ and $z \leq x$, then for any $y_1 \in P_2$, $z \wedge y_1 = 0$ and so $z \in P_1$. Thus P_1 is an ideal. Let $x_1 \wedge x_2 \in P_1$. Then for any $y \in V_2$, $x_1 \wedge x_2 \wedge y = 0$. If $x_2 \wedge y = 0$, then $x_2 \in P_1$. If $x_2 \wedge y \neq 0$, since $x_2 \wedge y \in V_2$, we conclude that $x_1 \in P_1$. Thus P_1 is prime. Similarly P_2 is a prime ideal.

Conversely assume that there exist prime ideals P_1 and P_2 such that $P_1 \cap P_2 = \{0\}$. Set $V_1 = P_1 - \{0\}$ and $V_2 = P_2 - \{0\}$. For $x, y \in Z(L)^*$, let $x \wedge y = 0$. Since P_1 and P_2 are prime ideals, without loss of generality $x \in P_1$ and $y \in P_2$. Then $x \in V_1$ and $y \in V_2$. Clearly $x \wedge y \in P_1 \cap P_2$ and so $x \wedge y = 0$. Hence x and y are adjacent in $\Gamma(L)$. From this we get that $Z(L)^* = V_1 \cup V_2$ and $\Gamma(L)$ is a complete bipartite graph. □

Consider the lattice L given below. It contains two prime ideals $P_1 = \{0, a, b\}$, $P_2 = \{0, c, d\}$ and $P_1 \cap P_2 = \{0\}$. Note that the zero divisor graph $\Gamma(L)$ is a complete bipartite graph.



Theorem 2.3. *If L is a distributive lattice with $\omega(\Gamma(L)) < \infty$, then the radius of $\Gamma(L)$ is at most 2.*

Proof. By Theorem 1.5, $Z(L) = \bigcup_{i=1}^{n} Ann(y_i), y_i \in L^*$. Let $x \in Z(L)^*$. Then $x \in Ann(y_m)$, for some $1 \leq m \leq n$ and so $x \wedge y_m = 0$. By Lemma 1.6, for $j \neq m$, $y_m \wedge y_j = 0$. Thus the distance from any y_j to x is 1 if $j = m$ or 2 if $j \neq m$. Hence the radius of $\Gamma(L)$ is at most 2. □

From Theorem 2.3, there is a non zero $x \in L$ such that $x \wedge y = 0$ or $Ann(x) \cap Ann(y) \neq \{0\}$ for every $y \in Z(L)$.

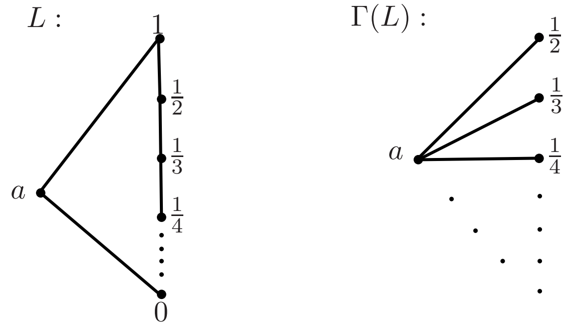
Proposition 2.4. *If L is a non atomic lattice, then $\Gamma(L)$ contains either a connected split induced subgraph or a complete bipartite induced subgraph.*

Proof. Since L is non atomic, there exists a non zero element $z \in L$ such that $a \nleq z$, for any atom a. Now there is some z_0 such that $0 < z_0 < z$. Similarly, there is some z_1 such that $0 < z_1 < z_0$. Proceeding like this, we obtain a decreasing chain $z > z_0 > z_1 > \ldots$.

If $\mathcal{A}(L) \neq \emptyset$, then there exists $x \in L^*$ such that $x \wedge a = 0$, for all $a \in \mathcal{A}(L)$. As mentioned above, there exists a decreasing chain $x > x_0 > x_1 > \cdots$. Clearly $\{x, x_0, x_1, \ldots\}$ is independent and $< \mathcal{A}(L) >$ is complete. Clearly $a \wedge x_i = 0$ for every i. Hence $< \{x, x_0, x_1, \ldots\} \cup \mathcal{A}(L) >$ is a connected split induced subgraph.

Suppose $\mathcal{A}(L) = \emptyset$. Let $x, y \in L^*$ such that $x \wedge y = 0$. As constructed above, one can get decreasing chains $C_1 : x > x_0 > x_1 > \cdots$ and $C_2 : y > y_0 > y_1 > \cdots$ corresponding to x and y respectively. Let $V_1 = \{x, x_0, x_1, \ldots, \}$ and $V_2 = \{y, y_0, y_1, \ldots, \}$. Then $< V_1 \cup V_2 >$ is a complete bipartite induced subgraph. □

Consider the lattice $L = \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \cup \{a\}$ given below. It is a non atomic lattice and its zero divisor graph $\Gamma(L)$ is a complete bipartite graph.



A distributive complemented lattice L is called a *Boolean lattice.*

Proposition 2.5. *If L is a Boolean lattice, then $\Gamma(L)$ has a perfect matching. Moreover the edge independent number $\beta_1 = \frac{|Z(L)^*|}{2}$.*

Proof. Let $M = \{(x, x') \in E(\Gamma(L)) : x, x' \in V(\Gamma(L))\}$. Clearly every vertex is an end vertex of an edge in M. Since each non zero element has a unique complement, M is an edge independent set and so M is a perfect matching. This also establishes the last part of the statement. □

Lemma 2.6. *Let L be a Boolean lattice. Then L is finite if and only if every vertex of $\Gamma(L)$ is of finite degree.*

Proof. $\Rightarrow$ part is trivial.

Conversely assume that every vertex has finite degree. Assume to the contrary, L is finite. Let $x, y \in L^*$ and $x \wedge y = 0$. Then $y \wedge z \in Ann(x)$ for every $z \in L^*$. Since $Ann(x)$ is finite, there exists a set $J = \{z \in L^* : z \wedge y = t$ for some $t \in Ann(x)\}$ is infinite. If $t = 0$, $J \subset Ann(y)$ and so y has infinite degree, a contradiction. If $t \neq 0$, $t \wedge z' = (z \wedge y) \wedge z' = 0$ for all $z \in J$. Then t has infinite degree, a contradiction. □

S. K. Nimbhokar, M. P. Wasadikar and M. M. Pawar[14] proved that, if a Boolean lattice L contains an infinite increasing chain, then $\omega(\Gamma(L)) = \infty$. Now we prove that the same result by relaxing the existence of an infinite increasing chain.

Theorem 2.7. *Let L be a Boolean lattice. Then L is finite if and only if $\omega(\Gamma(L))$ is finite.*

Proof. $\Rightarrow$ part is trivial.

Assume that $\omega(\Gamma(L))$ is finite. Suppose L contains an infinite increasing chain, by Lemma 1.6, $\Gamma(L)$ has an infinite clique, a contradiction. Suppose $C : \ell_0 > \ell_1 > \ell_2 > \cdots$ with $\ell_i \in L^*$ is an infinite decreasing chain in L. Let $y_i = \ell_i \wedge \ell'_{i+1}$. Since $\ell_i \neq 0$, $y_i \neq 0$ for every i. Suppose $y_i = y_j$ for some $i \neq j$. Then $i < i+1 \leq j$ and so $\ell_{i+1} \geq \ell_j$ and we get $y_i = \ell_i \wedge \ell'_{i+1} = \ell_j \wedge \ell'_{j+1} \wedge \ell'_{i+1} = 0$, as $\ell'_{i+1} \leq \ell'_j$, a contradiction. Thus y_i are non zero and distinct. For every $i < i+1 \leq j$, $\ell'_{i+1} \wedge \ell_j = \ell'_{i+1} \wedge \ell_{i+1} \wedge \ell_j = 0$ implies $y_i \wedge y_j = 0$. Thus $< \{y_i : i = 1, 2, 3, \ldots\} >$ is an infinite clique, a contradiction. Hence all chains in L are finite and so L is atomic which implies that $\mathcal{A}(L) \neq \emptyset$.

If there exists a vertex of infinite degree, by Lemma 2.6, L is infinite. By Theorem 1.1, $\mathcal{A}(L)$ is an infinite set and hence $< \mathcal{A}(L) >$ is an infinite clique, a contradiction. $\square$

3. **Domination in** $\Gamma(L)$

In this section, we study certain domination properties in $\Gamma(L)$. More specifically, we characterize all dominating sets in $\mathcal{A}(L)$ where L is a bounded distributive lattice. Further we obtain independent domination number for $\overline{\Gamma(L)}$, where L is any lattice. Also we obtain domination number for $\Gamma(L)$, where L is a finite Boolean lattice.

Lemma 3.1. *Let L be any lattice and $\mathcal{A}(L)$ be the set of all atoms in L. Assume that $S \subseteq \mathcal{A}(L)$ with $|S| > 1$. If $\bigvee\limits_{s \in S} s = 1$, then S is a dominating set in $\Gamma(L)$.*

Proof. Assume that $\bigvee\limits_{s \in S} s = 1$. Let $x \in V(\Gamma(L)) - S$. If $s \leq x$ for every $s \in S$, then x is the lub of elements in S, which is a contradiction to the assumption. Thus $s \nleq x$ for some $s \in S$ and so $s \wedge x = 0$. Hence S dominates all the vertices in $\Gamma(L)$. $\square$

Remark 3.2. Note that the condition $\bigvee\limits_{s \in S} s = 1$ in Lemma 3.1 is not a necessary one. For, consider the lattice L (Figure 1) and its zero-divisor graph $\Gamma(L)$ (Figure 2) given below. Note that $S = \{a\}$ is a dominating set in $\Gamma(L)$, whereas $a \neq 1$.

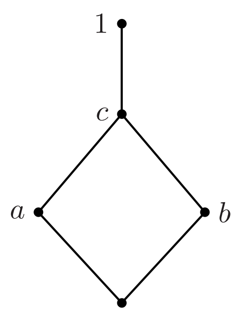


Figure 1

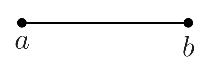


Figure 2

Theorem 3.3. *Let L be a lattice with $\mathcal{A}(L) \neq \emptyset$. Then $\mathcal{A}(L)$ is a minimal dominating set in $\overline{\Gamma(L)}$ if and only if L is atomic.*

Proof. Assume that $\mathcal{A}(L)$ is a minimal dominating set in $\overline{\Gamma(L)}$. Suppose L is not atomic. Then there exists $x \in L^*$ such that $a \nleq x$ for every $a \in \mathcal{A}(L)$ and so $x \wedge a = 0$. Thus $\mathcal{A}(L)$ is not a dominating set, a contradiction. Conversely, by the definition of an atomic lattice, $\mathcal{A}(L)$ is a dominating set in $\overline{\Gamma(L)}$. Let $a \in \mathcal{A}(L)$ and $D = \mathcal{A}(L) - \{a\}$. Then $a \wedge b = 0$ for every $b \in D$ and hence a is not dominated by D in $\overline{\Gamma(L)}$. Hence $\mathcal{A}(L)$ is minimal. $\square$

Theorem 3.4. *Let L be a lattice. Then there exist $x, y \in L^*$ such that $\{x, y\}$ is a minimum independent dominating set of $\overline{\Gamma(L)}$ if and only if $x \perp y$. In particular $\gamma(\overline{\Gamma(L)}) = i(\overline{\Gamma(L)}) = 2$.*

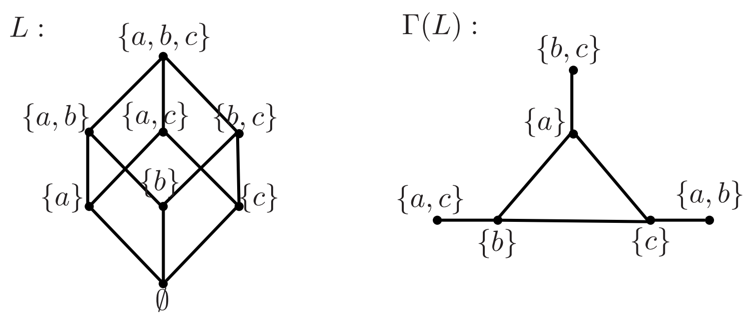
Proof. Let $x, y \in L^*$ such that $\{x, y\}$ is a minimum independent dominating set of $\overline{\Gamma(L)}$. Clearly $x \wedge y = 0$ in $\Gamma(L)$. Suppose $Ann(x) \cap Ann(y) \neq \{0\}$, there exists $0 \neq z \in Ann(x) \cap Ann(y)$ such that $z \wedge x = 0 = z \wedge y$. Thus z is not dominated by x and y in $\overline{\Gamma(L)}$, a contradiction. Conversely, let $x, y \in L^*$ such that $x \perp y$. Then $x \wedge y = 0$ and $Ann(x) \cap Ann(y) = \{0\}$. i.e., x and y are adjacent and there is no vertex which is adjacent to both x and y in $\Gamma(L)$. Note that every $v \in V - \{x, y\}$ is adjacent to either x or y in $\overline{\Gamma(L)}$. Since $\Gamma(L)$ is connected and $x \wedge y = 0$, $\{x, y\}$ is a minimum independent dominating set of $\overline{\Gamma(L)}$. □

Theorem 3.5. *If L is a finite Boolean lattice and $|\mathcal{A}(L)| > 1$, then $\gamma(\Gamma(L)) = 1$ or $|\mathcal{A}(L)|$.*

Proof. Let $\mathcal{A}(L) = \{a_1, a_2, \ldots, a_n\}$. By Theorems 1.2 and 1.3, $\bigvee_{i=1}^{n} a_i = 1$. Note that in L, every non zero element $x \notin \mathcal{A}(L)$ can be represented as the join of two or more atoms uniquely. If $|\mathcal{A}(L)| = 2$, then $L = \{0, a_1, a_2, a_1 \vee a_2 = 1\}$ and so $Z(L)^* = \{a_1, a_2\}$. Hence $\gamma(\Gamma(L)) = 1$. Suppose $|\mathcal{A}(L)| > 2$. Since $\bigvee_{i=1}^{n} a_i = 1$, by Lemma 3.1, $\mathcal{A}(L)$ is a dominating set in $\Gamma(L)$ and so $\gamma(\Gamma(L)) \leq |\mathcal{A}(L)|$.

Let S be a dominating set in $\Gamma(L)$. Let $x_j = \bigvee_{i=1, i\neq j}^{n} a_i$ for $1 \leq j \leq n$. Note that $x_j \notin \mathcal{A}(L)$ and is adjacent to only a_j for $1 \leq j \leq n$. If $S \subseteq L^* - \mathcal{A}(L)$, then S must contain all x_j for $1 \leq j \leq n$ and so $|S| \geq |\mathcal{A}(L)|$. If S intersects $\mathcal{A}(L)$, say $S \cap \mathcal{A}(L) = \{a_1, a_2, \ldots, a_k\}$, then $\{a_1, a_2, \ldots, a_k, x_{k+1}, \ldots, x_n\} \subseteq S$ and so $|S| \geq |\mathcal{A}(L)|$. Hence $\gamma(\Gamma(L)) = |\mathcal{A}(L)|$. □

Example 3.6. Consider the lattice L constructed out of the power set of $\{a, b, c\}$ with set inclusion. Note that $\mathcal{A}(L) = \{\{a\}, \{b\}, \{c\}\}$ and so $|\mathcal{A}(L)| > 1$. Note that $\gamma(\Gamma(L)) = |\mathcal{A}(L)| = 3$.



Theorem 3.7. *Let L be a distributive lattice and $diam(\Gamma(L)) = 2$. Then the following are equivalent:*

(i) *For every $x \in V(\Gamma(L))$, $N(x)$ is a connected dominating set in $\Gamma(L)$.*

(ii) *Let $x, y \in Z(L)^*$. Then there exists a non zero z such that $x \wedge z = z \wedge y = 0$.*

Proof. Assume that (i) is true. Let $x, y \in Z(L)^*$. Since $diam(\Gamma(L)) = 2$, it is enough to discuss the case that $x \wedge y = 0$. Since $diam(\Gamma(L)) = 2$, there exists $z \in V(\Gamma(L)) - \{x, y\}$ such that $z \in N(x)$ or $z \in N(y)$. Let $z \in N(x)$. Since $< N(x) >$ is connected, there exists a path in $< N(x) >$ lies between y and z which implies there exists $(y \neq) z_1 \in N(x)$ such that $z_1 \in N(y)$. Similarly if $z \in N(y)$, then there exists $(x \neq) z_2 \in N(y)$ such that $z_2 \in N(x)$. Conversely assume that (ii) is true. Since $diam(\Gamma(L)) = 2$, the neighbourhood $N(x)$ of any vertex x is a dominating set of $\Gamma(L)$. Let $x_1, x_2 \in N(x)$. If $x_1 \wedge x_2 = 0$, then x_1 and x_2 are adjacent. Suppose $x_1 \wedge x_2 \neq 0$. By our assumption, there exist $y_1, y_2 \in L^*$ such that $x \wedge y_1 = y_1 \wedge x_1 = 0$ and $x \wedge y_2 = y_2 \wedge x_2 = 0$. Then $x_1 - y_1 - x_1 \wedge x_2 - y_2 - x_2$ is a path and y_1, y_2, $x_1 \wedge x_2 \in N(x)$. Hence $< N(x) >$ is connected. $\square$

In view of Lemma 4.1 and Theorem 3.7, we have the following.

Corollary 3.8. *Let L be a distributive lattice and $diam(\Gamma(L)) = 2$. Then the following are equivalent:*

(i) *$Z(L)$ is an ideal of L.*

(ii) *Let $x, y \in Z(L)$. Then there exists a non zero z such that $x \wedge z = z \wedge y = 0$.*

(iii) *For every $x \in V(\Gamma(L))$, $N(x)$ is a connected dominating set in $\Gamma(L)$ and hence $\gamma_c(\Gamma(L)) \leq \delta(\Gamma(L))$.*

4. Diameter and Girth of $\Gamma(L)$

In this section, we study the diameter and girth of $\Gamma(L)$.

Lemma 4.1. *Let L be a distributive lattice and $diam(\Gamma(L)) = 2$. Then $Z(L)$ is an ideal of L if and only if for $x, y \in Z(L)$, there exists a non zero z such that $x \wedge z = z \wedge y = 0$.*

Proof. Assume that $Z(L)$ is an ideal and $x, y \in Z(L)$, then $x \vee y \in Z(L)$. Since $diam(\Gamma(L)) = 2$, there exists $t \in Z(L)$ such that $x - t - x \vee y$ is a path in $\Gamma(L)$. Then $t \wedge x = t \wedge y = 0$. This t satisfies the required conditions. Conversely, given any $x, y \in Z(L)$, their mutual annihilator z annihilates $x \vee y$. For $\ell \in L$, $z \in Z(L)$ and $\ell \leq z$, imply $\ell \in Z(L)$. Hence $Z(L)$ is an ideal. $\square$

Lemma 4.2. *Let $\Gamma(L)$ be the zero divisor graph of a distributive lattice L. Suppose $\Gamma(L)$ is not a complete bipartite graph. If $\Gamma(L)$ contains a complete bipartite spanning subgraph, then $Z(L)$ is an ideal of L.*

Proof. Let $\ell \in L$, $z \in Z(L)$ and $\ell \leq z$. Clearly $\ell \in Z(L)$. Assume that $x, y \in Z(L)$, $x \neq y$ and non zero. Since $\Gamma(L)$ has a complete bipartite spanning subgraph, there exist non empty sets V_1 and V_2 such that $V_1 \cup V_2 = Z(L)^*$, $V_1 \cap V_2 = \emptyset$ and $p \wedge q = 0$ for all $p \in V_1$ and $q \in V_2$. If $x, y \in V_1$, then for any $q \in V_2$, $q \wedge (x \vee y) = 0$ so $x \vee y \in Z(L)^*$. Now suppose $x \in V_1$ and $y \in V_2$. Since $\Gamma(L)$ is not complete bipartite and contains a complete bipartite spanning subgraph, there is an edge connecting two distinct vertices $p_1, p_2 \in V_1$. Now $p_1 \wedge (y \vee p_2) = (p_1 \wedge y) \vee (p_1 \wedge p_2) = 0$ and so $y \vee p_2 \in Z(L)^*$. If $y \vee p_2 \in V_1$, then for any $q \in V_2$, we have $0 = q \wedge (y \vee p_2) = (q \wedge y) \vee (q \wedge p_2) = q \wedge y$. Thus $q \wedge (x \vee y) = (q \wedge x) \vee (q \wedge y) = 0$ and so $x \vee y \in Z(L)$. If $y \vee p_2 \in V_2$, then $0 = x \wedge (y \vee p_2) = (x \wedge y) \vee (x \wedge p_2) = x \wedge p_2$ and so $p_2 \wedge (x \vee y) = (p_2 \wedge x) \vee (p_2 \wedge y) = 0$ and hence $x \vee y \in Z(L)$. Hence $Z(L)$ is an ideal of L. $\square$

In view of Lemma 4.1 and Lemma 4.2, we have the following result.

Theorem 4.3. *Let $\Gamma(L)$ be the zero divisor graph of a distributive lattice L. Suppose $\Gamma(L)$ is not a complete bipartite graph. If $\Gamma(L)$ contains a complete bipartite spanning subgraph, then for all $x, y \in Z(L)$, there exists a non zero z such that $x \wedge z = z \wedge y = 0$.*

Remark 4.4. For any L, any two atoms are adjacent in $\Gamma(L)$. If L is a lattice with at least three atoms, then $gr(\Gamma(L)) = 3$. Hence any lattice L with $gr(\Gamma(L) = 4$ contains at the maximum two atoms.

Acknowledgment

The authors express their sincere thanks to the referees for many valuable comments, which improved the exposition a lot.

References

[1] D. D. Anderson and M. Naseer, Beck's coloring of a commutative ring, *J. Algebra*, **159** (1993) 500-514.

[2] D. F. Anderson and P. Livingston, The zero divisor graph of a commutative ring, *J. Algebra*, **217** (1999) 434-447.

[3] D. F. Anderson, R. Levy and J. Shapiro, Zero divisor graphs, von Neumann regular rings and Boolean algebras, *J. Pure Appl. Algebra*, **180** (2003) 221-241.

[4] I. Beck, Coloring of Commutative rings, *J. Algebra*, **116** (1988) 208-226.

[5] B. Bollobos and I. Rival, The maximal size of the covering graph of a lattice, *Algebra Universalis*, **9** (1979) 371-373.

[6] G. Chartrand and P. Zhang, *Introduction to Graph theory*, Wadsworth and Brooks/Cole, Monterey, CA, 1986.

[7] B. A. Davey and H. A. Priestley, *Introduction to Lattices and Order*, Cambridge University Press, New York, 2002.

[8] F. R. Demeyer, T. Mckenzie and K. Schneider, The zero divisor graph of a commutative semigroup, *Semigroup Forum*, **65** (2002) 206-214.

[9] D. Duffus and I. Rival, Path length in the covering graph of a lattice, *Discrete Math.*, **19** (1977) 139-158.

[10] E. Estaji and K. Khashyarmanesh, The zero divisor graph of a lattice, *Results Math.*, **61** (2012) 1-11, DOI 10.1007/s00025-010-0067-8.

[11] E. Mendelson, *Boolean algebra and Switching Circuits*, Tata McGraw-Hill, New Delhi, 2004.

[12] N. D. Filipov, Comparability graphs of partially ordered sets of different types, *Colloq. Math. Soc. János Bolyai*, **33** (1980) 373-380.

[13] E. Gedeonová, Lattices whose covering graphs are S-graphs, *Colloq Math. Soc. János Bolyai*, **33** (1980) 407-435.

[14] S. K. Nimbhorkar, M. P. Wasadikar and M. M. Pawar, Coloring of lattices, *Math. Slovaca*, **60**(2010) 419-434.

T. Tamizh Chelvam

Department of Mathematics, Manonmaniam Sundaranar University, Tirunelveli 627012, Tamil Nadu, India

Email:tamche59gmail.com

S. Nithya

Department of Mathematics, Manonmaniam Sundaranar University, Tirunelveli 627012, Tamil Nadu, India

Email:yathnis@yahoo.com