



## ON LIC T SIGRAPHS

V. MATHAD\* AND K. P. NARAYANKAR

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ABSTRACT. A signed graph (marked graph) is an ordered pair  $S = (G, \sigma)$  ( $S = (G, \mu)$ ), where  $G = (V, E)$  is a graph called the underlying graph of  $S$  and  $\sigma : E \rightarrow \{+, -\}$  ( $\mu : V \rightarrow \{+, -\}$ ) is a function. For a graph  $G$ ,  $V(G)$ ,  $E(G)$  and  $C(G)$  denote its vertex set, edge set and cut-vertex set, respectively. The lict graph  $L_c(G)$  of a graph  $G = (V, E)$  is defined as the graph having vertex set  $E(G) \cup C(G)$  in which two vertices are adjacent if and only if they correspond to adjacent edges of  $G$  or one corresponds to an edge  $e_i$  of  $G$  and the other corresponds to a cut-vertex  $c_j$  of  $G$  such that  $e_i$  is incident with  $c_j$ . In this paper, we introduce lict sigraphs, as a natural extension of the notion of lict graph to the realm of signed graphs. We show that every lict sigraph is balanced. We characterize signed graphs  $S$  and  $S'$  for which  $S \sim L_c(S)$ ,  $\eta(S) \sim L_c(S)$ ,  $L(S) \sim L_c(S')$ ,  $J(S) \sim L_c(S')$  and  $T_1(S) \sim L_c(S')$ , where  $\eta(S)$ ,  $L(S)$ ,  $J(S)$  and  $T_1(S)$  are negation, line graph, jump graph and semitotal line sigraph of  $S$ , respectively, and  $\sim$  means switching equivalence.

### 1. Introduction

By a graph  $G = (V, E)$ , we mean a finite, undirected graph without loops or multiple edges. For graph theoretic terminology, we refer to [8]. For a graph  $G$ ,  $V(G)$ ,  $E(G)$  and  $C(G)$  denote its vertex set, edge set and cut-vertex set, respectively.

A signed graph or a sigraph is an ordered pair  $S = (G, \sigma)$ , where  $S^u = G = (V, E)$  is a graph called the underlying graph of  $S$  and  $\sigma : E \rightarrow \{+, -\}$  is a function. A cycle in a signed graph is said to be positive if the product of its edges is positive. A cycle which is not positive is said to be negative. A

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\*Corresponding author.

signed graph is said to be balanced if every cycle in it is positive [9]. Otherwise, it is called unbalanced.

A marking of vertices of  $S$  is a function  $\mu : V \rightarrow \{+, -\}$ . A signed graph  $S$  together with a marking  $\mu$  is denoted by  $S_\mu$  [4]. Given a signed graph  $S$  one can easily define a marking  $\mu$  of vertices of  $S$  as follows:

For any vertex  $v \in V(S)$ ,  $\mu(v) = \prod_{uv \in E(S)} \sigma(uv)$ , the marking  $\mu$  of vertices of  $S$  is called canonical marking of  $S$ . The signed graphs have interesting connections with many classical mathematical systems [15].

The following characterization of balanced signed graphs is well known.

**Theorem 1.1.** [12] *A signed graph  $S = (G, \sigma)$  is balanced if and only if there exists a marking  $\mu$  of its vertices such that each edge  $uv$  in  $S$  satisfies  $\sigma(uv) = \mu(u)\mu(v)$ .*

The line sigraph (or  $\times$ -line sigraph) of a signed graph  $S$  is a sigraph  $L(S)$  (or  $L_\times(S)$ ) defined on the line graph  $L(S^u)$  by assigning to each edge  $ef$  of  $L(S^u)$ , the product of signs of the adjacent edges  $e$  and  $f$  of  $S$  [2].

**Proposition 1.2.** [2] *The line sigraph of a signed graph is balanced.*

The jump graph  $J(G)$  of a graph  $G$  is the graph whose vertices are edges of  $G$  and where two vertices of  $J(G)$  are adjacent if and only if they are nonadjacent in  $G$ . Equivalently,  $J(G)$  is the complement of line graph  $L(G)$  [6].

The jump sigraph of a signed graph  $S = (G, \sigma)$  is a signed graph  $J(S) = (J(G), \sigma')$ , where for any edge  $ee'$  in  $J(G)$ ,  $\sigma'(ee') = \sigma(e)\sigma(e')$  [3].

**Proposition 1.3.** [3] *The jump signed graph of a signed graph is balanced.*

*The semitotal line graph  $T_1(G)$  of a graph  $G = (V, E)$  is the graph whose vertex set is  $V \cup E$  and two vertices are adjacent in  $T_1(G)$  if and only if they are adjacent edges of  $G$  or one is a vertex of  $G$  and the other is an edge incident with it [13].*

The semitotal line sigraph[7] of a sigraph  $S = (G, \sigma)$  is a signed graph  $T_1(S) = (T_1(G), \sigma')$  where for any edge  $uv$  of  $T_1(G)$ ,

$$\sigma'(uv) = \begin{cases} \sigma(u)\sigma(v), & \text{if } u, v \in E(G); \\ \sigma(u), & \text{if } u \in E(G), v \in V(G). \end{cases}$$

The concept of switching a signed graph was introduced in [1]. Its deeper mathematical aspects are found in [16]. Switching  $S$  with respect to a marking  $\mu$  is the operation of changing the sign of every edge of  $S$  to its opposite whenever its end vertices are of opposite signs. The signed graph obtained in this way is denoted by  $S_\mu(S)$  and is called  $\mu$ -switched signed graph or just switched

signed graph. Two signed graphs  $S_1 = (G, \sigma)$  and  $S_2 = (G', \sigma')$  are said to be isomorphic, written as  $S_1 \cong S_2$ , if there exists a graph isomorphism  $f : G \rightarrow G'$  such that for any edge  $e \in G$ ,  $\sigma(e) = \sigma'(f(e))$ . Further, a signed graph  $S_1 = (G, \sigma)$  switches to a signed graph  $S_2 = (G', \sigma')$  (or that  $S_1$  and  $S_2$  are switching equivalent), written  $S_1 \sim S_2$ , whenever there exists a marking  $\mu$  of vertices of  $S_1$  such that  $S_\mu(S_1) \cong S_2$ . Note that  $S_1 \sim S_2$  implies that  $G \cong G'$ , since the definition of switching does not involve change of adjacencies in the underlying graphs of the respective signed graphs.

Two signed graphs  $S_1 = (G, \sigma)$  and  $S_2 = (G', \sigma')$  are said to be weakly isomorphic or cycle isomorphic [14] if there exists an isomorphism  $\phi : G \rightarrow G'$  such that the sign of every cycle  $Z$  in  $S_1$  equals to the sign of  $\phi(Z)$  in  $S_2$ .

The following result is well known.

**Theorem 1.4.** [14] *Two signed graphs  $S_1$  and  $S_2$  with the same underlying graph are switching equivalent if and only if they are cycle isomorphic.*

One of the important operations on signed graphs involves changing signs of their edges. The negation  $\eta(S)$  of  $S$  is a signed graph obtained from  $S$  by negating the sign of every edge of  $S$ , that is, by changing the sign of each edge to its opposite [10].

The lict graph  $L_c(G)$  of a graph  $G = (V, E)$  is defined as the graph having the vertex set  $E(G) \cup C(G)$  in which two vertices are adjacent if and only if they correspond to adjacent edges of  $G$  or one corresponds to an edge  $e_i$  of  $G$  and the other corresponds to a cut-vertex  $c_j$  of  $G$  and  $e_i$  is incident with  $c_j$ . This concept was introduced in [11].

**Theorem 1.5.** [11] *For any graph  $G$ , we have  $G \cong L_c(G)$  if and only if  $G$  is a cycle.*

We can extend the notion of the lict graph to the realm of signed graphs to obtain the lict sigraph as follows,

The lict sigraph  $L_c(S)$  of a signed graph  $S = (G, \sigma)$  has the lict graph  $L_c(G)$  as underlying graph and for any edge  $uv \in L_c(G)$

$$\sigma_{L_c}(uv) = \begin{cases} \sigma(u)\sigma(v), & \text{if } u, v \in E(G); \\ \sigma(v), & \text{if } u \in C(G), v \in E(G). \end{cases}$$

The sigraph  $S$  and its Lict sigraph  $L_c(S)$  are shown in Figure 1.

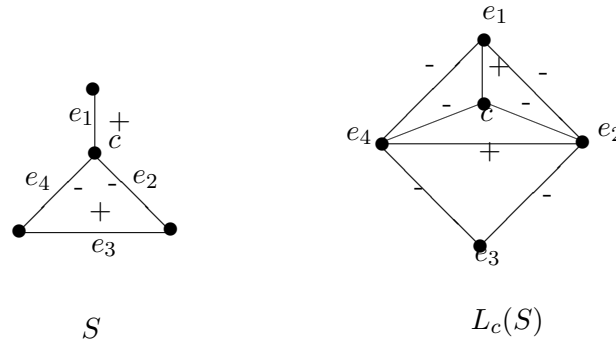


Figure 1

**Theorem 1.6.** [5] *The graph pair  $(G, H)$  is a solution of the equation  $L(G) \cong L_c(H)$  if and only if the following are satisfied:*

- (1) *Every component of  $H$  is a block, or*
- (2)  *$G \cong H^*$ , where  $H^*$  is a graph obtained from  $H$  by adding one new vertex  $v_i$  for each cut-vertex  $c_i$  of  $H$  and inserting an edge between  $v_i$  and  $c_i$ .*

A graph in which any two distinct vertices are adjacent is called a complete graph. A complete graph on  $n$  vertices is denoted by  $K_n$ .

A bipartite graph  $G$  is a graph whose vertex set  $V$  can be partitioned into two subsets  $V_1$  and  $V_2$  such that every edge of  $G$  joins  $V_1$  with  $V_2$ . If  $G$  contains every edge joining  $V_1$  and  $V_2$ , then  $G$  is a complete bipartite graph. If  $V_1$  and  $V_2$  have  $n$  and  $m$  vertices in complete bipartite graph, we write  $G = K_{n,m}$ .

A path  $P_n$ , is an alternating sequence of distinct vertices and edges  $v_0, e_1, v_1, e_2, \dots, v_{n-1}, e_n, v_n$  beginning and ending with vertices. Further, if  $v_0 = v_n$ , then it is called a cycle, denoted by  $C_n$ .

The product of two graphs  $G(V_1, E_1)$  and  $H(V_2, E_2)$  is a graph having  $V = V_1 \times V_2$  as its vertex set such that  $u = (u_1, u_2)$  and  $v = (v_1, v_2)$  are adjacent if  $u_1 = v_1$  and  $u_2$  is adjacent to  $v_2$  in  $H$  or  $u_1$  is adjacent to  $v_1$  in  $G$  and  $u_2 = v_2$ .

If  $G$  and  $H$  are graphs with the property that the identification of any vertex of  $G$  with an arbitrary vertex of  $H$  results in a unique graph up to isomorphism, then we write  $G \bullet H$  for this graph.

**Theorem 1.7.** [5] *The following pairs  $(G, H)$  of graphs are all satisfying the graph equation  $J(G) \cong L_c(H)$ :*

- $(K_{1,n}, nK_2), n \geq 1;$      $(K_3, 3K_2);$      $(3K_2, K_3);$      $(3P_3, K_4);$      $(C_6, K_{2,3});$      $(K_{3,3}, K_{3,3});$      $(2P_3, C_4);$   
 $(C_5, C_5);$      $(K_{2,3}, C_6);$      $(K_2 \cup 2P_3, K_4 - x)$ , where  $x$  is any edge of  $K_4$ ;     $(G', K_2 \times P_3)$ , where  $G'$  is the graph  $K_{2,3}$  together with an end edge incident with a vertex of degree 2;     $((n+1)K_2, K_{1,n}), n \geq 2;$   
 $(P_4 \cup 2K_2, K_3 \bullet K_2);$     and     $(P_5 \cup K_2, P_4)$ , where  $P_n$  is a path of order  $n$ .

**Theorem 1.8.** [5] *The pair  $(G, H)$  is a solution of the graph equation  $T_1(G) \cong L_c(H)$  if and only if  $(G, H)$  is  $(nK_1, nK_2)$ , for some  $n \geq 1$ .*

## 2. Main Results

**Proposition 2.1.** *The lict sigraph of a signed graph is balanced.*

*Proof.* Let  $(S, \sigma)$  be a signed graph. Suppose that  $\sigma'$  denotes the signing of  $L_c(S)$ . Let the signing  $\sigma$  of  $S$  be the marking of the vertices of  $L_c(S)$  which correspond to the edges of  $S$  and let the vertices of  $L_c(S)$  that correspond to the cut-vertices of  $S$  be marked by  $+$ . Then by the definition of  $L_c(S)$ , we see that

$$\sigma'(uv) = \begin{cases} \sigma(u)\sigma(v), & \text{if } u, v \in E(G); \\ \sigma(v), & \text{if } u \in C(G), v \in E(G). \end{cases}$$

Hence by Theorem 1.1, the result follows. □

**Proposition 2.2.** *Let  $S = (K_p, \sigma)$  be a signed graph with  $p \geq 3$ . Then  $S$  is a lict sigraph if and only if it is balanced.*

*Proof.* If  $p = 3$ , then  $K_3$  is the underlying graph of  $S$ , which is the lict graph of  $K_3$  or  $K_{1,2}$ . If  $p \geq 4$ , then  $K_p$  is the underlying graph, which is the lict graph of  $K_{1,p-1}$ . If  $S$  is balanced, then let  $e_i, 1 \leq i \leq p$  be vertices of  $S$  such that  $e_1$  is incident with even number of negative edges. Let  $e_1$  correspond to a cut-vertex of  $S' = (K_{1,p-1}, \sigma')$  having edges  $e_i, 2 \leq i \leq p$  such that  $\sigma'(e_i) = \sigma(e_1e_i), 2 \leq i \leq p$ . Then it can be verified that  $L_c(S') = S$ . Hence,  $S$  is a lict sigraph.

The converse is true by Proposition 2.1. □

**Proposition 2.3.** *Let  $S = (K_{m,n}, \sigma)$  be a signed graph. Then  $S$  is a lict sigraph if and only if it is balanced and  $m = n = 2$ .*

*Proof.* A cut-vertex  $v$  of any graph  $H$  together with  $k \geq 2$  edges incident with it form a complete subgraph  $K_{k+1}$  in  $L_c(H)$ . Since  $K_{m,n}$  does not contain a complete subgraph  $K_l, l \geq 3$ , it follows that  $K_{m,n}$  is a lict graph of a block. But  $L_c(H) \cong L(H)$  if and only if  $H$  is a block. Hence  $K_{m,n}$  is a line graph too. Therefore,  $m \leq 2$  and  $n \leq 2$ , since otherwise,  $K_{1,3}$  would be an induced subgraph of  $K_{m,n}$ , which is a forbidden induced subgraph of a line graph. Also,  $K_{m,n} \neq K_2$  and  $K_{1,2}$ , since they are not lict graphs of any graphs. Hence,  $G$  is isomorphic to  $K_{2,2}$ . Since  $S$  is a lict sigraph, it is balanced from Proposition 2.1.

Conversely, suppose that  $S = (K_{2,2}, \sigma)$  is balanced. Since  $L_c(K_{2,2}) \cong K_{2,2}$ , we construct  $S' = (K_{2,2}, \sigma')$  according to the following cases.

**Case 1.**  $S$  is all positive. Then let  $S'$  be either all positive or all negative.

**Case 2.**  $S$  is all negative. Then let  $S'$  be such that  $\sigma'(e_i) \neq \sigma'(e_j)$ , for every pair of nonadjacent edges  $e_i$  and  $e_j$ .

**Case 3.**  $\sigma(f_i) = \sigma(f_j)$ , for every pair of nonadjacent edges  $f_i$  and  $f_j$ . Then let  $\sigma'(e_i) \neq \sigma'(e_j)$ , for every pair of nonadjacent edges  $e_i$  and  $e_j$ .

**Case 4.**  $\sigma(f_i) \neq \sigma(f_j)$ , for every pair of nonadjacent edges  $f_i$  and  $f_j$ . Then let  $\sigma'(e) = +$  for exactly one edge or  $\sigma'(e) = -$  for exactly one edge.

In all the above cases, we have  $L_c(S') \cong S$ . Hence  $S$  is a licit sigraph.  $\square$

**Proposition 2.4.** *Let  $S = (G, \sigma)$ , be a signed graph. Then  $S \sim L_c(S)$  if and only if  $S$  is balanced and  $G$  is a cycle.*

*Proof.* Suppose that  $S \sim L_c(S)$ . This implies that  $G \cong L_c(G)$ . From Theorem 1.5, it follows that  $G$  is a cycle. Also by Proposition 2.1,  $L_c(S)$  is balanced. Since  $S \sim L_c(S)$ , it follows by Theorem 1.4, that  $S$  is balanced.

Conversely, suppose that  $G$  is a cycle. Then by Theorem 1.5,  $G \cong L_c(G)$ . Now, since  $S$  is any balanced signed graph with the underlying graph  $G$ , and by Proposition 2.1,  $L_c(S)$  is balanced signed graph with the underlying graph  $L_c(G)$ , the result follows from Theorem 1.4.  $\square$

**Proposition 2.5.** *Let  $S = (G, \sigma)$  be a signed graph. Then  $\eta(S) \sim L_c(S)$  if and only if either  $S$  is unbalanced and  $G$  is an odd cycle or  $S$  is balanced and  $G$  is an even cycle.*

*Proof.* Suppose that  $\eta(S) \sim L_c(S)$ . Then  $G \cong L_c(G)$  and hence by Theorem 1.5,  $G$  is a cycle. By Proposition 2.1,  $L_c(S)$  is balanced. Now, if  $S$  is a balanced signed graph with underlying graph  $G = C_n$ , where  $n$  is odd, then  $\eta(S)$  is unbalanced, by definitions. Next, if  $S$  is unbalanced signed graph with underlying graph  $G = C_n$ , where  $n$  is even, then also  $\eta(S)$  is unbalanced. Hence in both of the cases,  $\eta(S)$  being unbalanced cannot be switching equivalent to  $L_c(S)$ , which is balanced. Hence either  $S$  is unbalanced and  $G$  is an odd cycle or  $S$  is balanced and  $G$  is an even cycle.

Conversely, suppose that for a signed graph  $S = (G, \sigma)$ , either  $S$  is unbalanced and  $G$  is an odd cycle or  $S$  is balanced and  $G$  is an even cycle. Then clearly  $\eta(S)$  is balanced. From Proposition 2.1,  $L_c(S)$  is balanced. Also by Theorem 1.5,  $G \cong L_c(G)$ . Hence the result follows from Theorem 1.4.  $\square$

**Theorem 2.6.** *Let  $S = (G, \sigma)$  and  $S' = (H, \sigma')$  be two signed graphs. Then  $L(S) \sim L_c(S')$  if and only if the conditions of Theorem 1.6 are satisfied.*

*Proof.* Suppose that  $L(S) \sim L_c(S')$  for signed graphs  $S = (G, \sigma)$  and  $S' = (H, \sigma')$ . Then  $L(G) \cong L_c(H)$ . Thus, by Theorem 1.6, conditions of Theorem 1.6 are satisfied.

Conversely, suppose that conditions (1) and (2) of Theorem 1.6 hold for graphs  $G$  and  $H$ . Then by Theorem 1.6,  $L(G) \cong L_c(H)$ . Thus, by Proposition 1.2 and Proposition 2.1,  $L(S)$  and  $L_c(S')$  are balanced, with underlying graphs  $L(G)$  and  $L_c(H)$ , respectively. So,  $L(S)$  and  $L_c(S')$  are cycle isomorphic. Hence the result follows from Theorem 1.4.  $\square$

**Theorem 2.7.** *Let  $S = (G, \sigma)$  and  $S' = (H, \sigma')$  be two signed graphs. Then  $J(S) \sim L_c(S')$  if and only if  $(G, H)$  is any of the pairs mentioned in Theorem 1.7.*

*Proof.* Suppose that  $J(S) \sim L_c(S')$ . Then  $J(G) \cong L_c(H)$  and result follows from Theorem 1.7.

Conversely, suppose that  $(G, H)$  is any of the pairs in the statement of the Theorem 1.7. Then by Theorem 1.7,  $J(G) \cong L_c(H)$ . Consider any signed graphs  $S$  and  $S'$  with underlying graphs  $G$  and

$H$ , respectively. By Proposition 1.3 and Proposition 2.1,  $J(S)$  and  $L_c(S')$  are balanced and hence cycle isomorphic with underlying graphs  $J(G)$  and  $L_c(H)$ , respectively. Hence the result follows from Theorem 1.4.  $\square$

**Theorem 2.8.** *Let  $S = (G, \sigma)$  and  $S' = (H, \sigma')$  be two signed graphs. Then  $T_1(S) \sim L_c(S')$  if and only if  $(G, H)$  is  $(nK_1, nK_2)$ , for some  $n \geq 1$ .*

*Proof.* Suppose that  $T_1(S) \sim L_c(S')$ . Then  $T_1(G) \cong L_c(H)$  and the result follows from Theorem 1.8.

Conversely, suppose that  $(G, H)$  is  $(nK_1, nK_2)$ , for some  $n \geq 1$ . Then by Theorem 1.8,  $T_1(G) \cong L_c(H)$ . Since both  $T_1(G)$  and  $L_c(H)$  are totally disconnected graphs, it follows that for any signed graphs  $S$  and  $S'$  with underlying graphs  $G$  and  $H$  respectively,  $T_1(S) \sim L_c(S')$ .  $\square$

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**Veena Mathad**

Department of Studies in Mathematics, University of Mysore, Mysore, India

Email: [veena\\_mathad@rediffmail.com](mailto:veena_mathad@rediffmail.com)

**Kishori P. Narayankar**

Department of Mathematics, Mangalore University, Mangalore, India

Email: [kishori\\_pn@yahoo.co.in](mailto:kishori_pn@yahoo.co.in)