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ON LICT SIGRAPHS

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Abstract. A signed graph (marked graph) is an ordered pair $S = (G, \sigma)$ $(S = (G, \mu))$, where $G = (V, E)$ is a graph called the underlying graph of S and $\sigma : E \to \{+, -\}$ $(\mu : V \to \{+, -\})$ is a function. For a graph G, $V(G), E(G)$ and $C(G)$ denote its vertex set, edge set and cut-vertex set, respectively. The lict graph $L_c(G)$ of a graph $G = (V, E)$ is defined as the graph having vertex set $E(G) \cup C(G)$ in which two vertices are adjacent if and only if they correspond to adjacent edges of G or one corresponds to an edge e_i of G and the other corresponds to a cut-vertex c_j of G such that e_i is incident with c_j. In this paper, we introduce lict sigraphs, as a natural extension of the notion of lict graph to the realm of signed graphs. We show that every lict sigraph is balanced. We characterize signed graphs S and S' for which $S \sim L_c(S)$, $\eta(S) \sim L_c(S)$, $L(S) \sim L_c(S')$, $J(S) \sim L_c(S')$ and $T_1(S) \sim L_c(S')$, where $\eta(S)$, $L(S)$, $J(S)$ and $T_1(S)$ are negation, line graph, jump graph and semitotal line sigraph of S, respectively, and $\sim$ means switching equivalence.

1. Introduction

By a graph $G = (V, E)$, we mean a finite, undirected graph without loops or multiple edges. For graph theoretic terminology, we refer to [8]. For a graph $G, V(G), E(G)$ and $C(G)$ denote its vertex set, edge set and cut-vertex set, respectively.

A signed graph or a sigraph is an ordered pair $S = (G, \sigma)$, where $S^u = G = (V, E)$ is a graph called the underlying graph of S and $\sigma : E \to \{+, -\}$ is a function. A cycle in a signed graph is said to be positive if the product of its edges is positive. A cycle which is not positive is said to be negative. A

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signed graph is said to be balanced if every cycle in it is positive [9]. Otherwise, it is called unbalanced.

A marking of vertices of S is a function $\mu : V \to \{+, -\}$. A signed graph S together with a marking μ is denoted by S_μ [4]. Given a signed graph S one can easily define a marking μ of vertices of S as follows:

For any vertex $v \in V(S)$, $\mu(v) = \prod_{uv \in E(S)} \sigma(uv)$, the marking μ of vertices of S is called canonical marking of S. The signed graphs have interesting connections with many classical mathematical systems [15].

The following characterization of balanced signed graphs is well known.

Theorem 1.1. [12] *A signed graph $S = (G, \sigma)$ is balanced if and only if there exists a marking μ of its vertices such that each edge uv in S satisfies $\sigma(uv) = \mu(u)\mu(v)$.*

The line sigraph (or $\times$-line sigraph) of a signed graph S is a sigraph $L(S)$ (or $L_\times(S)$) defined on the line graph $L(S^u)$ by assigning to each edge ef of $L(S^u)$, the product of signs of the adjacent edges e and f of S [2].

Proposition 1.2. [2] *The line sigraph of a signed graph is balanced.*

The jump graph $J(G)$ of a graph G is the graph whose vertices are edges of G and where two vertices of $J(G)$ are adjacent if and only if they are nonadjacent in G. Equivalently, $J(G)$ is the complement of line graph $L(G)$ [6].

The jump sigraph of a signed graph $S = (G, \sigma)$ is a signed graph $J(S) = (J(G), \sigma')$, where for any edge ee' in $J(G)$, $\sigma'(ee') = \sigma(e)\sigma(e')$ [3].

Proposition 1.3. [3] *The jump signed graph of a signed graph is balanced.*

The semitotal line graph $T_1(G)$ of a graph $G = (V, E)$ is the graph whose vertex set is $V \cup E$ and two vertices are adjacent in $T_1(G)$ if and only if they are adjacent edges of G or one is a vertex of G and the other is an edge incident with it [13].

The semitotal line sigraph[7] of a sigraph $S = (G, \sigma)$ is a signed graph $T_1(S) = (T_1(G), \sigma')$ where for any edge uv of $T_1(G)$,

$$\sigma'(uv) = \begin{cases} \sigma(u)\sigma(v), & \text{if} \quad u, v \in E(G); \\ \sigma(u), & \text{if} \quad u \in E(G), v \in V(G). \end{cases}$$

The concept of switching a signed graph was introduced in [1]. Its deeper mathematical aspects are found in [16]. Switching S with respect to a marking μ is the operation of changing the sign of every edge of S to its opposite whenever its end vertices are of opposite signs. The signed graph obtained in this way is denoted by $S_\mu(S)$ and is called $\mu -$ *switched* signed graph or just switched signed graph. Two signed graphs $S_1 = (G, \sigma)$ and $S_2 = (G^{'}, \sigma^{'})$ are said to be isomorphic, written as $S_1 \cong S_2$, if there exists a graph isomorphism $f : G \to G^{'}$ such that for any edge $e \in G$, $\sigma(e) = \sigma^{'}(f(e))$. Further, a signed graph $S_1 = (G, \sigma)$ switches to a signed graph $S_2 = (G^{'}, \sigma^{'})$ (or that S_1 and S_2 are switching equivalent), written $S_1 \sim S_2$, whenever there exists a marking μ of vertices of S_1 such that $S_\mu(S_1) \cong S_2$. Note that $S_1 \sim S_2$ implies that $G \cong G^{'}$, since the definition of switching does not involve change of adjacencies in the underlying graphs of the respective signed graphs.

Two signed graphs $S_1 = (G, \sigma)$ and $S_2 = (G^{'}, \sigma^{'})$ are said to be weakly isomorphic or cycle isomorphic [14] if there exists an isomorphism $\phi : G \to G^{'}$ such that the sign of every cycle Z in S_1 equals to the sign of $\phi(Z)$ in S_2.

The following result is well known.

Theorem 1.4. [14] *Two signed graphs S_1 and S_2 with the same underlying graph are switching equivalent if and only if they are cycle isomorphic.*

One of the important operations on signed graphs involves changing signs of their edges. The negation $\eta(S)$ of S is a signed graph obtained from S by negating the sign of every edge of S, that is, by changing the sign of each edge to its opposite [10].

The lict graph $L_c(G)$ of a graph $G = (V, E)$ is defined as the graph having the vertex set $E(G) \bigcup C(G)$ in which two vertices are adjacent if and only if they correspond to adjacent edges of G or one corresponds to an edge e_i of G and the other corresponds to a cut-vertex c_j of G and e_i is incident with c_j. This concept was introduced in [11].

Theorem 1.5. [11] *For any graph G, we have $G \cong L_c(G)$ if and only if G is a cycle.*

We can extend the notion of the lict graph to the realm of signed graphs to obtain the lict sigraph as follows,

The lict sigraph $L_c(S)$ of a signed graph $S = (G, \sigma)$ has the lict graph $L_c(G)$ as underlying graph and for any edge $uv \in L_c(G)$

$$\sigma_{L_c}(uv) = \begin{cases} \sigma(u)\sigma(v), & \text{if} \quad u, v \in E(G); \\ \sigma(v), & \text{if} \quad u \in C(G), v \in E(G). \end{cases}$$

The sigraph S and its Lict sigraph $L_c(S)$ are shown in Figure 1.

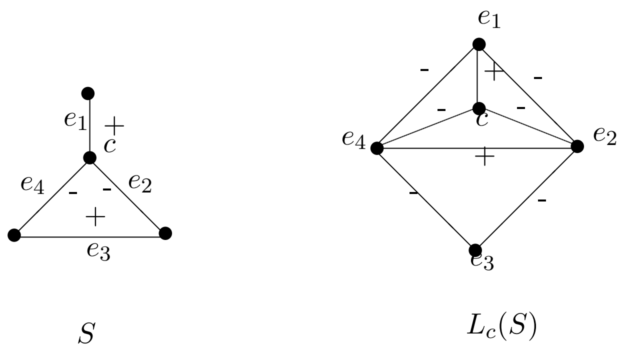

Figure 1

Theorem 1.6. [5] *The graph pair (G,H) is a solution of the equation $L(G) \cong L_c(H)$ if and only if the following are satisfied:*

(1) *Every component of H is a block, or*
(2) *$G \cong H^*$, where H^* is a graph obtained from H by adding one new vertex v_i for each cut-vertex c_i of H and inserting an edge between v_i and c_i.*

A graph in which any two distinct vertices are adjacent is called a complete graph. A complete graph on n vertices is denoted by K_n.

A bipartite graph G is a graph whose vertex set V can be partitioned into two subsets V_1 and V_2 such that every edge of G joins V_1 with V_2. If G contains every edge joining V_1 and V_2, then G is a complete bipartite graph. If V_1 and V_2 have n and m vertices in complete bipartite graph, we write $G = K_{n,m}$.

A path P_n, is an alternating sequence of distinct vertices and edges $v_0, e_1, v_1, e_2, \cdots, v_{n-1}, e_n, v_n$ beginning and ending with vertices. Further, if $v_0 = v_n$, then it is called a cycle, denoted by C_n.

The product of two graphs $G(V_1, E_1)$ and $H(V_2, E_2)$ is a graph having $V = V_1 \times V_2$ as its vertex set such that $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are adjacent if $u_1 = v_1$ and u_2 is adjacent to v_2 in H or u_1 is adjacent to v_1 in G and $u_2 = v_2$.

If G and H are graphs with the property that the identification of any vertex of G with an arbitrary vertex of H results in a unique graph up to isomorphism, then we write $G \bullet H$ for this graph.

Theorem 1.7. [5] *The following pairs (G,H) of graphs are all satisfying the graph equation $J(G) \cong L_c(H)$:*
$(K_{1,n}, nK_2), n \geq 1$; $(K_3, 3K_2)$; $(3K_2, K_3)$; $(3P_3, K_4)$; $(C_6, K_{2,3})$; $(K_{3,3}, K_{3,3})$; $(2P_3, C_4)$; (C_5, C_5); $(K_{2,3}, C_6)$; $(K_2 \cup 2P_3, K_4 - x)$, *where x is any edge of K_4;* $(G', K_2 \times P_3)$, *where G' is the graph $K_{2,3}$ together with an end edge incident with a vertex of degree 2;* $((n+1)K_2, K_{1,n}), n \geq 2$; $(P_4 \cup 2K_2, K_3 \bullet K_2)$; *and* $(P_5 \cup K_2, P_4)$, *where P_n is a path of order n.*

Theorem 1.8. [5] *The pair (G, H) is a solution of the graph equation $T_1(G) \cong L_c(H)$ if and only if (G, H) is (nK_1, nK_2), for some $n \geq 1$.*

2. Main Results

Proposition 2.1. *The lict sigraph of a signed graph is balanced.*

Proof. Let (S, σ) be a signed graph. Suppose that $\sigma^{'}$ denotes the signing of $L_c(S)$. Let the signing σ of S be the marking of the vertices of $L_c(S)$ which correspond to the edges of S and let the vertices of $L_c(S)$ that correspond to the cut-vertices of S be marked by $+$. Then by the definition of $L_c(S)$, we see that

$$\sigma^{'}(uv) = \begin{cases} \sigma(u)\sigma(v), & \text{if} \quad u, v \in E(G); \\ \sigma(v), & \text{if} \quad u \in C(G), v \in E(G). \end{cases}$$

Hence by Theorem 1.1, the result follows. □

Proposition 2.2. *Let $S = (K_p, \sigma)$ be a signed graph with $p \geq 3$. Then S is a lict sigraph if and only if it is balanced.*

Proof. If $p = 3$, then K_3 is the underlying graph of S, which is the lict graph of K_3 or $K_{1,2}$. If $p \geq 4$, then then K_p is the underlying graph, which is the lict graph of $K_{1,p-1}$. If S is balanced, then let $e_i, 1 \leq i \leq p$ be vertices of S such that e_1 is incident with even number of negative edges. Let e_1 correspond to a cut-vertex of $S^{'} = (K_{1,p-1}, \sigma^{'})$ having edges $e_i, 2 \leq i \leq p$ such that $\sigma^{'}(e_i) = \sigma(e_1 e_i), 2 \leq i \leq p$. Then it can be verified that $L_c(S') = S$. Hence, S is a lict sigraph.
The converse is true by Proposition 2.1. □

Proposition 2.3. *Let $S = (K_{m,n}, \sigma)$ be a signed graph. Then S is a lict sigraph if and only if it is balanced and $m = n = 2$.*

Proof. A cut-vertex v of any graph H together with $k \geq 2$ edges incident with it form a complete subgraph K_{k+1} in $L_c(H)$. Since $K_{m,n}$ does not contain a complete subgraph $K_l, l \geq 3$, it follows that $K_{m,n}$ is a lict graph of a block. But $L_c(H) \cong L(H)$ if and only if H is a block. Hence $K_{m,n}$ is a line graph too. Therefore, $m \leq 2$ and $n \leq 2$, since otherwise, $K_{1,3}$ would be an induced subgraph of $K_{m,n}$, which is a forbidden induced subgraph of a line graph. Also, $K_{m,n} \neq K_2$ and $K_{1,2}$, since they are not lict graphs of any graphs. Hence, G is isomorphic to $K_{2,2}$. Since S is a lict sigraph, it is balanced from Proposition 2.1.

Conversely, suppose that $S = (K_{2,2}, \sigma)$ is balanced. Since $L_c(K_{2,2}) \cong K_{2,2}$, we construct $S^{'} = (K_{2,2}, \sigma^{'})$ according to the following cases.
Case 1. S is all positive. Then let $S^{'}$ be either all positive or all negative.
Case 2. S is all negative. Then let $S^{'}$ be such that $\sigma^{'}(e_i) \neq \sigma^{'}(e_j)$, for every pair of nonadjacent edges e_i and e_j.
Case 3. $\sigma(f_i) = \sigma(f_j)$, for every pair of nonadjacent edges f_i and f_j. Then let $\sigma^{'}(e_i) \neq \sigma^{'}(e_j)$, for every pair of nonadjacent edges e_i and e_j.

Case 4. $\sigma(f_i) \neq \sigma(f_j)$, for every pair of nonadjacent edges f_i and f_j. Then let $\sigma^{'}(e) = +$ for exactly one edge or $\sigma^{'}(e) = -$ for exactly one edge.

In all the above cases, we have $L_c(S^{'}) \cong S$. Hence S is a lict sigraph. $\square$

Proposition 2.4. *Let $S = (G, \sigma)$, be a signed graph. Then $S \sim L_c(S)$ if and only if S is balanced and G is a cycle.*

Proof. Suppose that $S \sim L_c(S)$. This implies that $G \cong L_c(G)$. From Theorem 1.5, it follows that G is a cycle. Also by Proposition 2.1, $L_c(S)$ is balanced. Since $S \sim L_c(S)$, it follows by Theorem 1.4, that S is balanced.

Conversely, suppose that G is a cycle. Then by Theorem 1.5, $G \cong L_c(G)$. Now, since S is any balanced signed graph with the underlying graph G, and by Proposition 2.1, $L_c(S)$ is balanced signed graph with the underlying graph $L_c(G)$, the result follows from Theorem 1.4. $\square$

Proposition 2.5. *Let $S = (G, \sigma)$ be a signed graph. Then $\eta(S) \sim L_c(S)$ if and only if either S is unbalanced and G is an odd cycle or S is balanced and G is an even cycle.*

Proof. Suppose that $\eta(S) \sim L_c(S)$. Then $G \cong L_c(G)$ and hence by Theorem 1.5, G is a cycle. By Proposition 2.1, $L_c(S)$ is balanced. Now, if S is a balanced signed graph with underlying graph $G = C_n$, where n is odd, then $\eta(S)$ is unbalanced, by definitions. Next, if S is unbalanced signed graph with underlying graph $G = C_n$, where n is even, then also $\eta(S)$ is unbalanced. Hence in both of the cases, $\eta(S)$ being unbalanced cannot be switching equivalent to $L_c(S)$, which is balanced. Hence either S is unbalanced and G is an odd cycle or S is balanced and G is an even cycle.

Conversely, suppose that for a signed graph $S = (G, \sigma)$, either S is unbalanced and G is an odd cycle or S is balanced and G is an even cycle. Then clearly $\eta(S)$ is balanced. From Proposition 2.1, $L_c(S)$ is balanced. Also by Theorem 1.5, $G \cong L_c(G)$. Hence the result follows from Theorem 1.4. $\square$

Theorem 2.6. *Let $S = (G, \sigma)$ and $S^{'} = (H, \sigma^{'})$ be two signed graphs. Then $L(S) \sim L_c(S^{'})$ if and only if the conditions of Theorem 1.6 are satisfied.*

Proof. Suppose that $L(S) \sim L_c(S^{'})$ for signed graphs $S = (G, \sigma)$ and $S^{'} = (H, \sigma^{'})$. Then $L(G) \cong L_c(H)$. Thus, by Theorem 1.6, conditions of Theorem 1.6 are satisfied.

Conversely, suppose that conditions (1) and (2) of Theorem 1.6 hold for graphs G and H. Then by Theorem 1.6, $L(G) \cong L_c(H)$. Thus, by Proposition 1.2 and Proposition 2.1, $L(S)$ and $L_c(S^{'})$ are balanced, with underlying graphs $L(G)$ and $L_c(H)$, respectively. So, $L(S)$ and $L_c(S^{'})$ are cycle isomorphic. Hence the result follows from Theorem 1.4. $\square$

Theorem 2.7. *Let $S = (G, \sigma)$ and $S^{'} = (H, \sigma^{'})$ be two signed graphs. Then $J(S) \sim L_c(S^{'})$ if and only if (G, H) is any of the pairs mentioned in Theorem 1.7.*

Proof. Suppose that $J(S) \sim L_c(S^{'})$. Then $J(G) \cong L_c(H)$ and result follows from Theorem 1.7.

Conversely, suppose that (G, H) is any of the pairs in the statement of the Theorem 1.7. Then by Theorem 1.7, $J(G) \cong L_c(H)$. Consider any signed graphs S and $S^{'}$ with underlying graphs G and

H, respectively. By Proposition 1.3 and Proposition 2.1, $J(S)$ and $L_c(S^{'})$ are balanced and hence cycle isomorphic with underlying graphs $J(G)$ and $L_c(H)$, respectively. Hence the result follows from Theorem 1.4. □

Theorem 2.8. *Let $S = (G, \sigma)$ and $S^{'} = (H, \sigma^{'})$ be two signed graphs. Then $T_1(S) \sim L_c(S^{'})$ if and only if (G, H) is (nK_1, nK_2), for some $n \geq 1$.*

Proof. Suppose that $T_1(S) \sim L_c(S^{'})$. Then $T_1(G) \cong L_c(H)$ and the result follows from Theorem 1.8.

Conversely, suppose that (G, H) is (nK_1, nK_2), for some $n \geq 1$. Then by Theorem 1.8, $T_1(G) \cong L_c(H)$. Since both $T_1(G)$ and $L_c(H)$ are totally disconnected graphs, it follows that for any signed graphs S and $S^{'}$ with underlying graphs G and H respectively, $T_1(S) \sim L_c(S^{'})$. □

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