www.combinatorics.ir

Transactions on Combinatorics
ISSN (print): 2251-8657, ISSN (on-line): 2251-8665
Vol. 3 No. 4 (2014), pp. 19-30.

www.ui.ac.ir

THE GEODETIC DOMINATION NUMBER FOR THE PRODUCT OF GRAPHS

S. ROBINSON CHELLATHURAI AND S. PADMA VIJAYA*

Communicated by Manouchehr Zaker

Abstract. A subset S of vertices in a graph G is called a geodetic set if every vertex not in S lies on a shortest path between two vertices from S. A subset D of vertices in G is called dominating set if every vertex not in D has at least one neighbor in D. A geodetic dominating set S is both a geodetic and a dominating set. The geodetic (domination, geodetic domination) number $g(G)(\gamma(G), \gamma_g(G))$ of G is the minimum cardinality among all geodetic (dominating, geodetic dominating) sets in G. In this paper, we show that if a triangle free graph G has minimum degree at least 2 and $g(G) = 2$, then $\gamma_g(G) = \gamma(G)$. It is shown, for every nontrivial connected graph G with $\gamma(G) = 2$ and $diam(G) > 3$, that $\gamma_g(G) > g(G)$. The lower bound for the geodetic domination number of Cartesian product graphs is proved. Geodetic domination number of product of cycles (paths) are determined. In this work, we also determine some bounds and exact values of the geodetic domination number of strong product of graphs.

1. Introduction

We consider only finite, simple, connected graphs with at least two vertices. For any graph G, the set of vertices is denoted by $V(G)$ and the edge set by $E(G)$. The *order* and *size* of G are denoted by p and q respectively. For a vertex $v \in V(G)$, the *open neighborhood* $N(v)$ is the set of all vertices adjacent to v, and $N[v] = N(v) \cup \{v\}$ is the *closed neighborhood* of v. The *degree* $d(v)$ of a vertex v is defined by $d(v) = |N(v)|$. The *minimum* and *maximum degree* of a graph G are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. For $X \subseteq V(G)$, let $G[X]$ be the subgraph of G induced by X, $N(X) = \bigcup_{x \in X} N(x)$ and $N[X] = \bigcup_{x \in X} N[x]$. If G is a connected graph, then the *distance* $d(x, y)$ is the length of a shortest $x - y$ path in G. The *diameter* of a connected graph G is defined

MSC(2010): Primary: 05C12; Secondary: 05C69.

Keywords: Cartesian product, strong product, geodetic number, domination number, geodetic domination number.

Received: 6 March 2014, Accepted: 27 June 2014.

*Corresponding author.

by $diam(G) = max_{x,y \in V(G)} d(x,y)$. A graph is said to be *triangle free* if it does not contain cycles of length 3. The *complement* of a graph G is the graph $\overline{G}$ with the same vertex set such that two vertices of $\overline{G}$ are adjacent if and only if they are not adjacent in G. An $x-y$ path of length $d(x,y)$ is called an $x-y$ *geodesic*. A vertex v is said to *lie* on an $x-y$ geodesic P if v is an internal vertex of P. The closed interval $I[x,y]$ consists of x, y and all vertices lying on some $x-y$ geodesic of G, and for a nonempty set $S \subseteq V(G)$, $I[S] = \bigcup_{x,y \in S} I[x,y]$.

If G is a connected graph, then a set S of vertices is a *geodetic set* if $I[S] = V(G)$. The *geodetic number* $g(G)$ of G is the minimum cardinality of a geodetic set of G. The geodetic number was introduced in [2, 7] and further studied in [5]. A vertex of G is an *extreme* if the subgraph induced by its neighborhood is complete. The set of all extreme vertices of G is denoted by $Ext(G)$. It is easily seen that every extreme vertex belongs to every geodetic set [2, 7, 5]. A vertex in a graph G dominates itself and its neighbors. A set of vertices S in a graph G is a *dominating set* if $N[S] = V(G)$. The *domination number* $\gamma(G)$ of G is the minimum cardinality of a dominating set of G. The domination number was introduced in [8]. A set of vertices S in a graph G is a *geodetic dominating set* if S is both a geodetic and a dominating set of G. The *geodetic domination number* $\gamma_g(G)$ of G is the minimum cardinality of a geodetic dominating set of G. The geodetic domination number was introduced in [4, 6]. It is easily seen that every extreme vertex belongs to every geodetic dominating set.

The following theorems are used in the sequel.

Theorem 1.1. [4, 6] *If G is a connected graph of order $p \geq 2$, then $2 \leq max\{\gamma(G), g(G)\} \leq \gamma_g(G) \leq p$.*

Theorem 1.2. [4] *Let G be a connected graph of order $p \geq 2$. Then $\gamma_g(G) = 2$ if and only if there exists a geodetic set $S = \{u, v\}$ of G such that $d(u,v) \leq 3$.*

2. Geodetic Domination Number of Certain classes of Graphs

The geodetic domination numbers of some standard graphs can be easily found, and are given as follows:

2.1 The complete graph K_p of p vertices has $\gamma_g(K_p) = p$.
2.2 The star graph $K_{1,p-1}$ of p vertices has $\gamma_g(K_{1,p-1}) = p-1$.
2.3 The complete bipartite graph $K_{p,q}$ on $p+q$ vertices with $p, q \geq 2$ have $\gamma_g(K_{p,q}) = min\{p, q, 4\}$.
2.4 The wheel graph W_p of p vertices has $\gamma_g(W_p) = \left\lceil \frac{p-1}{2} \right\rceil$, for $p \geq 5$.
2.5 The cycle C_p of p vertices has $\gamma_g(C_p) = \left\lceil \frac{p}{3} \right\rceil, p \geq 6$.
2.6 The path P_p of p vertices has $\gamma_g(P_p) = \left\lceil \frac{p+2}{3} \right\rceil$.
2.7 The Petersen graph G has $\gamma_g(G) = 4$.
2.8 The complement of a path graph $\overline{P}_p$ of p vertices has $\gamma_g(\overline{P}_p) = 3$, $p \geq 5$.
2.9 The complement of a cycle graph $\overline{C}_p$ of p vertices has $\gamma_g(\overline{C}_p) = 3$, $p \geq 5$.

3. Basic Results

In this section we give some basic results of geodetic domination number. Further, we look at some relationships between the geodetic domination number and other parameters.

Theorem 3.1. *Let G be a triangle free connected graph with minimum degree $\delta \geq 2$. If $g(G) = 2$, then $\gamma_g(G) = \gamma(G)$.*

Proof. Let S be a minimum dominating set of G, so that $\gamma(G) = |S|$. Since $g(G) = 2$, there exists a pair of vertices $x, y \in V(G)$ such that every vertex of $G - \{x, y\}$ lies on the $x - y$ geodesic in G. Therefore $\{x, y\}$ is a minimum geodetic set of G. We show that S is a geodetic set of G. We consider two cases:

Case(i): Let $\{x, y\} \subseteq S$. Then it is clear that every vertex of $G - S$ lies on the $x - y$ geodesic in G where $x, y \in S$. Hence S is a geodetic dominating set of G.

Case(ii): Let $\{x, y\} \nsubseteq S$. Then at least one of x and y is not in S. Suppose $x \notin S$. Since $\delta \geq 2$, x has at least two neighbors, say u and v. Since G is a triangle free graph, $uv \notin E(G)$. Therefore x lies on the $u - v$ geodesic in G. Since S is a dominating set of G, at least one of u and v belongs to S. If $u, v \in S$, then S is a geodetic dominating set of G. If $u \in S$ and $v \notin S$, then there exists a vertex $v' \in S$ such that $v \in N(v')$. Therefore x lies on a $u - v'$ geodesic in G. Hence S is a geodetic dominating set of G.

In both the cases, $\gamma_g(G) \leq |S| = \gamma(G)$. Hence, by Theorem 1.1 we conclude that $\gamma_g(G) = \gamma(G)$. □

Remark 3.2. *The converse of the Theorem 3.1 is false. For the cycle C_7, $\gamma_g(C_7) = \gamma(C_7)$ but $g(C_7) \neq 2$.*

Remark 3.3. *Theorem 3.1 is not true if G is not a triangle free graph. For the wheel graph W_5, $g(W_5) = 2$, $\gamma_g(W_5) = 2$ and $\gamma(W_5) = 1$, so that $\gamma_g(G) \neq \gamma(G)$.*

Theorem 3.4. [4] *If G is a connected graph with $\gamma(G) = 1$, then $\gamma_g(G) = g(G)$.*

Theorem 3.5. *Let G be a connected graph with $diam(G) \leq 3$. If $\gamma(G) = 2$, then $\gamma_g(G) = g(G)$.*

Proof. If $G = K_{p,q}$, $(p, q \geq 2)$, then $\gamma(G) = 2$ and $\gamma_g(G) = g(G) = min\{p, q, 4\}$. So we only consider $G \neq K_{p,q}$. Since $\gamma(G) = 2$ and $diam(G) \leq 3$, there exists at least two pair of non-adjacent vertices of G and $diam(G) \neq 1$. Let S be a minimum geodetic set of G, so that $g(G) = |S|$. We show that S is a dominating set of G. Let $x \notin S$. Since S is a geodetic set of G, there exists a pair of vertices $u, v \in S$ such that $x \in I(u, v) \subseteq I[S]$. Suppose $x \notin N(S)$. Then $d(u, v) > 3$, which is a contradiction. Hence S is a geodetic dominating set of G and $\gamma_g(G) \leq |S| = g(G)$. Hence, by Theorem 1.1 we conclude that $\gamma_g(G) = g(G)$. □

Remark 3.6. *The converse of the Theorem 3.5 is false. For the wheel graph W_5, $\gamma_g(W_5) = g(W_5)$ but $\gamma(W_5) = 1 \neq 2$.*

Theorem 3.7. *If G is a connected graph with $\gamma(G) = 2$ and $diam(G) > 3$, then $\gamma_g(G) > g(G)$.*

Proof. Let S be a minimum geodetic set of G, so that $g(G) = |S|$. Let $X = V(G) - N[S]$. Suppose $X = \phi$, then $V(G) = N[S]$ and S is a geodetic dominating set of G. Suppose S is a minimum dominating set of G, then $\gamma_g(G) = 2$. By Theorem 1.2, $diam(G) \leq 3$ which is a contradiction. Therefore $\gamma_g(G) > 2$ and S is not a minimum dominating set of G. Let $d = diam(G)$ and P be the diametral path joining a pair of vertices $u, v \in S$. Let P be $u = u_0, u_1, u_2, \ldots, u_d = v$. Since S is a geodetic dominating set of G with $\gamma_g(G) > 2$ and $d > 3$, there exists at least one vertex $u_i \in S$, which contradicts to S is a minimum geodetic set of G. Hence $X \neq \phi$. Let $x \in X$. Then $x \notin N[S]$. That is, x is not dominated by any element of S. Therefore S is not a geodetic dominating set of G. Hence $\gamma_g(G) > g(G)$. □

Theorem 3.8. [6] Let G be a connected graph of order $p \geq 3$. Then $\gamma_g(G) = p - 1$ if and only if $G = K_1 + \cup m_j K_j$ where $\sum m_j \geq 2$.

Theorem 3.9. *Let G be a connected graph of order $p \geq 3$. If $\gamma_g(G) = p - 1$, then G contains exactly one cut vertex of degree $p - 1$.*

Proof. This follows from Theorem 3.8. □

Remark 3.10. *The converse of the Theorem 3.9 is false. For the graph G with 6 vertices given in the Figure 3.1, G contains a cut vertex v of degree $p - 1 = 5$ but $\gamma_g(G) = 4 \neq p - 1$.*

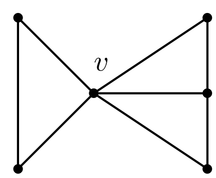

Figure 3.1: G

Remark 3.11. *If G has exactly one vertex of degree $p - 1$, then $\gamma_g(G) \neq p - 1$. For the wheel graph W_p, $\gamma_g(W_p) \neq p - 1$.*

4. The Geodetic Domination Number of Cartesian Product of Graphs

Definition 4.1. *Cartesian product of two graphs G and H, denoted by $G \times H$, is the graph with vertex set $V(G) \times V(H)$ specified by putting (u_1, v_1) adjacent to (u_2, v_2) if and only if $u_1 = u_2$ and $v_1v_2 \in E(H)$ or $v_1 = v_2$ and $u_1u_2 \in E(G)$.*

Definition 4.2. *For $u_i \in V(G)$, set $H_i = \{u_i\} \times H$ is a layer of H and for $v_j \in V(H)$, set $G_j = G \times \{v_j\}$ is a layer of G. The mappings $\Pi_G : (u, v) \mapsto u$ and $\Pi_H : (u, v) \mapsto v$ from $V(G \times H)$ onto G and H respectively are called projections. For a set $S \subseteq V(G \times H)$, we define the G-projection on G as $\Pi_G(S) = \{u \in V(G) : (u, v) \in S$ for some $v \in V(H)\}$ and the H-projection $\Pi_H(S) = \{v \in V(H) : (u, v) \in S$ for some $u \in V(G)\}$. Often, we write S_1 and S_2 as shorthand for $\Pi_G(S)$ and $\Pi_H(S)$ respectively.*

Lemma 4.3. [3, 13] *Let $G = (V, E)$ be the Cartesian product $G \times H$ of connected graphs $G = (V_1, E_1)$ and $H = (V_2, E_2)$. If $S \subseteq V$, then $I[S] \subseteq I[S_1] \times I[S_2]$.*

Lemma 4.4. *Let $G = (V, E)$ be the Cartesian product $G \times H$ of connected graphs $G = (V_1, E_1)$ and $H = (V_2, E_2)$. Then*

(i). $N[a] \subseteq N[a_1] \times N[a_2]$ for all $a \in V$

(ii). If $S \subseteq V$, then $N[S] \subseteq N[S_1] \times N[S_2]$

Theorem 4.5. *Let G and H be connected graphs. Then $\gamma_g(G \times H) \geq max\{\gamma_g(G), \gamma_g(H)\}$. Equality holds if G and H are complete graphs.*

Proof. Let S be a minimum geodetic dominating set of $G \times H$. Then by Lemma 4.3 and Lemma 4.4, $V(G \times H) = I[S] \subseteq I[S_1] \times I[S_2]$ and $V(G \times H) = N[S] \subseteq N[S_1] \times N[S_2]$. Therefore S_1 and S_2 are geodetic dominating sets of G and H respectively, so that $\gamma_g(G) \leq |S_1|$ and $\gamma_g(H) \leq |S_2|$. So, $\gamma_g(G \times H) = |S| \geq max\{|S_1|, |S_2|\} \geq max\{\gamma_g(G), \gamma_g(H)\}$. Now consider complete graphs G and H with vertex sets $V(G) = \{u_1, u_2, \ldots, u_p\}$ and $V(H) = \{v_1, v_2, \ldots, v_q\}$, so that $\gamma_g(G) = p$ and $\gamma_g(H) = q$. Without loss of generality, we assume that $p \geq q$. Let $S' = \{(u_1, v_1), (u_2, v_2), \ldots, (u_q, v_q), (u_{q+1}, v_q), (u_{q+2}, v_q), \ldots, (u_p, v_q)\}$. It is straightforward to verify that S' is a geodetic dominating set of $G \times H$ and $|S'| = p$. Hence $\gamma_g(G \times H) \leq |S'| = p = max\{\gamma_g(G), \gamma_g(H)\} \leq \gamma_g(G \times H)$, so equality holds. □

Corollary 4.6. *For every nontrivial connected graph G, $\gamma_g(G) \leq \gamma_g(G \times K_n)$.*

From the Theorem 4.5, $\gamma_g(K_n \times K_2) = \gamma_g(K_n)$. In the construction of $G \times K_2$, we have two copies $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ of G, where if v_1v_2 is an edge of $G \times K_2$ and $v_i \in V_i$, $i = 1, 2$, then v_1 and v_2 are said to correspond to each other. In the proof of the following theorem, it is useful to observe that every $u - v$ geodesic in $G \times K_2$, where both $u, v \in V_i$ for either $i = 1$ or $i = 2$, lies completely in G_i.

Theorem 4.7. *Let G be a connected graph of order at least 3 and diameter at most 2. Then G contains a minimum geodetic dominating set S with a vertex x such that every vertex of G lies on some $x - w$ geodesic in G for some $w \in S$ if and only if $\gamma_g(G) = \gamma_g(G \times K_2)$.*

Proof. Let $G \times K_2$ be formed from two copies G_1 and G_2 of G and S be a minimum geodetic dominating set of G_1 such that S contains a vertex x with the property that every vertex of G_1 lies on some $x - w$ geodesic in G_1 for some $w \in S$. Let D consist of x together with those vertices of G_2 corresponding to those vertices in $S - \{x\}$. Thus $|D| = |S|$. We show that D is a geodetic dominating set of $G \times K_2$. Let $v \notin D$ be a vertex of $G \times K_2$. First, assume that $v \in V_1$. Since $I[S] = V_1$ and $diam(G_1) \leq 2$, it follows that $v \in I(x, w) \subseteq I[S]$ and either $w \neq v$ or $w = v$. Since w' is the corresponding vertex of $w \in S$, $w' \in D$ and $v \in I(x, w') \subseteq I[D]$. Since $N[S] = V_1$ and $diam(G_1) \leq 2$, $v \in N(D)$ whenever $w \neq v$ or $w = v$. Therefore D is a geodetic dominating set of $G \times K_2$. Next, assume that $v \in V_2$. From the property, it is clear that $v \in I[x', w']$, where $w' \in D$ and x' is the vertex in V_2 corresponding

to x. Since $diam(G_2) \leq 2$, $v \in I[x, w'] \subseteq I[D]$ and $v \in N(w') \subseteq N[D]$. Therefore D is a geodetic dominating set of $G \times K_2$. Now, $\gamma_g(G \times K_2) \leq |D| = |S| = \gamma_g(G)$. Hence, by Corollary 4.6 we conclude that $\gamma_g(G) = \gamma_g(G \times K_2)$.
Conversely, assume that $\gamma_g(G) = \gamma_g(G \times K_2)$ where $G \times K_2$ is formed from two copies of G_1 and G_2 of G. Let D be a minimum geodetic dominating set of $G \times K_2$. Clearly $D \cap V_i \neq \phi$, $i = 1, 2$. Let $x \in D \cap V_1$ and let S consist of vertices of $D \cap V_1$ together with those vertices of G_1 corresponding to those vertices in $D \cap V_2$. It is clear that S is a geodetic dominating set of G_1 and $|S| = |D|$. Since D is a minimum geodetic dominating set of $G \times K_2$ and $\gamma_g(G) = \gamma_g(G \times K_2)$, S is a minimum geodetic dominating set of G_1. We show that every vertex of G_1 lies on some $x - w$ geodesic for some $w \in S$. Suppose that there exists a vertex $v \in V_1$ such that $v \notin I(x, w)$ for all $w \in S$. Then $v \notin N(x)$ and $d(x, v) = d(x, w) + d(w, v) > 2$. This contradicts to $diam(G_1) \leq 2$. □

Remark 4.8. *Theorem 4.7 is false if G is a connected graph with $p \geq 3$ and $diam(G) > 2$ satisfies the property that G contains a minimum geodetic dominating set S with a vertex x such that every vertex of G lies on some $x - w$ geodesic in G for some $w \in S$. For the cycle graph C_6, geodetic dominating set of C_6 satisfies the property, but $\gamma_g(C_6) \neq \gamma_g(C_6 \times K_2)$.*

Remark 4.9. *Theorem 4.7 is false if G is a connected graph with $p \geq 3$ and $diam(G) \leq 2$ does not satisfy the property that G contains a minimum geodetic dominating set S with a vertex x such that every vertex of G lies on some $x - w$ geodesic in G for some $w \in S$. For the wheel graph W_7, geodetic dominating set of W_7 does not satisfy the property and $\gamma_g(W_7) \neq \gamma_g(W_7 \times K_2)$.*

We emphasize that throughout this section the vertices of a cycle C_n or a path P_n are always denoted by $0, 1, 2, \ldots, n-1$. This notation is used to formulate the proof of the following theorems. From the construction of minimum dominating set D given in the lemma in [11], the following lemma follows.

Lemma 4.10. *For any integers m and n,*

(i). *Let $m \geq 3$. Then there exists a minimum geodetic dominating set S of $C_m \times C_n$ such that for every $i \in V(C_n)$, $|V((C_m)_i) \cap S| \leq m - 1$, where $(C_m)_i$ is the i^{th} layer of C_m in $C_m \times C_n$.*

(ii). *Let $m \geq 2$. Then there exists a minimum geodetic dominating set S of $P_m \times P_n$ such that for every $i \in V(P_n)$, $|V((P_m)_i) \cap S| \leq m - 1$, where $(P_m)_i$ is the i^{th} layer of P_m in $P_m \times P_n$.*

Theorem 4.11. [10, 11] *For any integer n,*

(i). $g(C_3 \times C_n) = \begin{cases} 3, & \text{if } n \text{ is even} \\ 4, & \text{if } n \text{ is odd.} \end{cases}$

(ii). $\gamma(C_3 \times C_n) = n - \left\lfloor \frac{n}{4} \right\rfloor, n \geq 4.$

Theorem 4.12. *For any integer $n \geq 4$, $\gamma_g(C_3 \times C_n) = n - \left\lfloor \frac{n}{4} \right\rfloor$.*

Proof. Let $n \geq 4$ and S consist of vertices $(1, i)$, $i \equiv 0(mod4)$ and vertices $(0, i), (2, i)$, $i \equiv 2(mod4)$, $0 \leq i \leq n-1$. If $n \equiv 2(mod4)$, then add the vertex $(0, n-1)$ to the set S. It is straightforward to verify that S is a geodetic dominating set of $C_3 \times C_n$ with $|V((C_3)_i) \cap S| \leq 2$ for all $i \in V(C_n)$. Since $|V((C_3)_i) \cap S| \leq 2$, there exists at least one C_3-layer which contains no vertex of S. Let s be the number of C_3-layers which contain no vertex of S (empty layers). Since no two empty layers are adjacent and there are $\lceil \frac{s}{2} \rceil$ C_3-layers with precisely two vertices from S, $|S| = 2\lceil \frac{s}{2} \rceil + (n - s - \lceil \frac{s}{2} \rceil) = n - s + \lceil \frac{s}{2} \rceil$. We consider two cases to prove $|S| = n - \lfloor \frac{n}{4} \rfloor$.

Case(i): Suppose $n \not\equiv 2(mod4)$. Then $s = \lfloor \frac{n}{2} \rfloor$. Therefore,

$$\begin{aligned} |S| &= n - \left\lfloor \frac{n}{2} \right\rfloor + \left\lceil \frac{\lfloor \frac{n}{2} \rfloor}{2} \right\rceil \\ &= n - \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor - \left\lfloor \frac{\lfloor \frac{n}{2} \rfloor}{2} \right\rfloor \\ &= n - \left\lfloor \frac{n}{4} \right\rfloor \end{aligned}$$

Case(ii): Suppose $n \equiv 2(mod4)$. Then $s = \frac{n}{2} - 1$, an integer. Therefore,

$$\begin{aligned} |S| &= n - \frac{n}{2} + 1 + \left\lceil \frac{\frac{n}{2} - 1}{2} \right\rceil \\ &= n - \frac{n}{2} + 1 + \frac{n}{2} - 1 + \left\lfloor \frac{n-2}{4} \right\rfloor \\ &= n - \left\lfloor \frac{(n-2)}{4} \right\rfloor \\ &= n - \left\lfloor \frac{n}{4} \right\rfloor \text{ (since } n \equiv 2(mod4) \text{)} \end{aligned}$$

Hence, in both the cases, $\gamma_g(C_3 \times C_n) \leq |S| = n - \lfloor \frac{n}{4} \rfloor$. By Theorem 1.1 and Theorem 4.11, we conclude that $\gamma_g(C_3 \times C_n) = n - \lfloor \frac{n}{4} \rfloor$. □

Theorem 4.13. [10, 11] *For any integer n,*

(*i*). $g(C_4 \times C_n) = \begin{cases} 2, & \text{if } n \text{ is even} \\ 3, & \text{if } n \text{ is odd.} \end{cases}$

(*ii*). $\gamma(C_4 \times C_n) = n$, $n \geq 4$.

Theorem 4.14. *For any integer* $n \geq 4$, $\gamma_g(C_4 \times C_n) = n$.

Proof. Let $n \geq 4$ and S consist of vertices $(0, i)$, $i \equiv 0(mod2)$ and vertices $(2, i)$, $i \equiv 1(mod2)$, $0 \leq i \leq n-1$. Let

$$S = \begin{cases} (0,0),(0,2),(0,4),\ldots,(0,n-2),(2,1),(2,3),\ldots,(2,n-1), & \text{if n is even} \\ (0,0),(0,2),(0,4),\ldots,(0,n-1),(2,1),(2,3),\ldots,(2,n-2), & \text{if n is odd} \end{cases}$$

It is clear that S is a geodetic dominating set of $C_4 \times C_n$ and $|V((C_4)_i) \cap S| = 1$ for all $i \in V(C_n)$. Since each C_4-layer contain exactly one vertex of S, $|S| = n$. So $\gamma_g(C_4 \times C_n) \leq |S| = n$. Hence, by Theorem 1.1 and Theorem 4.13 we conclude that $\gamma_g(C_4 \times C_n) = n$. □

Theorem 4.15. [10, 11] *For any integer n,*

(i). $g(P_2 \times P_n) = 2$.
(ii). $\gamma(P_2 \times P_n) = \lceil \frac{n+1}{2} \rceil$.

Theorem 4.16. *For any positive integer $n \geq 2$, $\gamma_g(P_2 \times P_n) = \lceil \frac{n+1}{2} \rceil$.*

Proof. Let S consist of vertices $(0, i)$, $i \equiv 0(mod4)$ and vertices $(1, i)$, $i \equiv 2(mod4)$, $0 \leq i \leq n-1$. If $n \equiv 2(mod4)$, then add the vertex $(1, n-1)$ to the set S. If $n \equiv 0(mod4)$, then add the vertex $(0, n-1)$ to the set S. It is straightforward to verify that S is a geodetic dominating set of $P_2 \times P_n$ Such that $|V((P_2)_i) \cap S| \leq 1$ for all $i \in V(P_n)$. Since $|V((P_2)_i) \cap S| \leq 1$, there exists at least one P_2-layer which contains no vertex of S and $|S| = n - s$, where s is the number of empty layers. If n is even, then $s = \lfloor \frac{n}{2} \rfloor - 1$. Therefore, $|S| = n - \lfloor \frac{n}{2} \rfloor + 1 = \lceil \frac{n}{2} \rceil + 1$. If n is odd, then $s = \lfloor \frac{n}{2} \rfloor$. Therefore, $|S| = n - \lfloor \frac{n}{2} \rfloor = \lceil \frac{n}{2} \rceil$. Therefore,

$$\begin{aligned}\gamma_g(P_2 \times P_n) &\leq |S| \\ &= \begin{cases} \lceil \frac{n}{2} \rceil + 1, & \text{if n is even} \\ \lceil \frac{n}{2} \rceil, & \text{if n is odd} \end{cases} \\ &= \left\lceil \frac{n+1}{2} \right\rceil\end{aligned}$$

By Theorem 1.1 and Theorem 4.15, we conclude that $\gamma_g(P_2 \times P_n) = \lceil \frac{n+1}{2} \rceil$. □

Observation 4.17. *The geodetic domination numbers of $P_i \times P_j$, $1 \leq i \leq 6$, can be easily found and are given as follows*

(i). $\gamma_g(P_1 \times P_j) = \left\lceil \frac{j+2}{3} \right\rceil$, $j \geq 2$
(ii). $\gamma_g(P_2 \times P_j) = \left\lceil \frac{j+1}{2} \right\rceil$, $j \geq 2$
(iii). $\gamma_g(P_3 \times P_j) = \left\lfloor \frac{3j+4}{4} \right\rfloor$, $j \geq 2$
(iv). $\gamma_g(P_4 \times P_j) = \begin{cases} j+1, & if j = 1,2,3,5,6,9 \\ j, & otherwise for j \geq 1 \end{cases}$
(v). $\gamma_g(P_5 \times P_j) = \begin{cases} \left\lfloor \frac{6j+6}{5} \right\rfloor, & if j = 2,3,7 \\ \left\lfloor \frac{6j+8}{5} \right\rfloor, & otherwise for j \geq 2 \end{cases}$
(vi). $\gamma_g(P_6 \times P_j) = \begin{cases} \left\lfloor \frac{10j+10}{7} \right\rfloor, & if j \geq 6 \text{ and } j \equiv 1 (\mod 7) \\ \left\lfloor \frac{10j+12}{7} \right\rfloor, & otherwise if j \geq 4 \end{cases}$

5. Geodetic Domination Number of Strong Product graphs

Definition 5.1. *The strong product of graphs G and H, denoted by $G \otimes H$, has vertex set $V(G) \times V(H)$, where two distinct vertices (u_1, v_1) and (u_2, v_2) are adjacent with respect to the strong product if*

(i). $u_1 = u_2$ *and* $v_1 v_2 \in E(H)$ *or*

(ii). $v_1 = v_2$ *and* $u_1u_2 \in E(G)$ *or*
(iii). $u_1u_2 \in E(G)$ *and* $v_1v_2 \in E(H)$

Definition 5.2. *The mappings* $\Pi_G : (u, v) \mapsto u$ *and* $\Pi_H : (u, v) \mapsto v$ *from* $V(G \otimes H)$ *onto* G *and* H *respectively are called projections. For a set* $S \subseteq V(G \otimes H)$, *we define the* G*-projection in* G *as* $\Pi_G(S) = \{u \in V(G) : (u, v) \in S \text{ for some } v \in V(H)\}$ *and the* H*-projection* $\Pi_H(S) = \{v \in V(H) : (u, v) \in S \text{ for some } u \in V(G)\}$

Theorem 5.3. [3, 12] *Let* G *and* H *be connected graphs. Then* $Ext(G \otimes H) = Ext(G) \times Ext(H)$.

Theorem 5.4. [3, 12, 14] *Let* $S_1 \subseteq V(G)$ *and* $S_2 \subseteq V(H)$ *for graphs* G *and* H, *then*
(i). $I_G[S_1] \times I_H[S_2] \subseteq I_{G\otimes H}[S_1 \times S_2]$
(ii). $N_G[S_1] \times N_H[S_2] = N_{G\otimes H}[S_1 \times S_2]$

Theorem 5.5. [3, 12] *Let* G *and* H *be nontrivial connected graphs. Then* $g(G \otimes H) \geq 4$.

Theorem 5.6. [12] *Let* G *and* H *be connected graphs and* S *be a geodetic set of* $G \otimes H$. *Then* $\Pi_G(S)$ *is a geodetic set of* G *or* $\Pi_H(S)$ *is a geodetic set of* H.

Theorem 5.7. [12] *Let* G *and* H *be connected graphs and* S *be a geodetic set of* $G \otimes H$. *If* $Ext(G) \neq \phi$, *then* $\Pi_H(S)$ *is a geodetic set of* H.

Theorem 5.8. *Let* $S_1 \subseteq V(G)$ *and* $S_2 \subseteq V(H)$ *for graphs* G *and* H. *If* S_1 *and* S_2 *are geodetic dominating sets of* G *and* H *respectively, then* $S_1 \times S_2$ *is a geodetic dominating set of* $G \otimes H$.

Proof. Let S_1 and S_2 be a geodetic dominating set of G and H respectively. Then we have that $I_G[S_1] = V(G)$, $N_G[S_1] = V(G)$, $I_H[S_2] = V(H)$ and $N_H[S_2] = V(H)$. Now $V(G \otimes H) = V(G) \times V(H) = I_G[S_1] \times I_H[S_2] \subseteq I_{G\otimes H}[S_1 \times S_2]$ and $V(G \otimes H) = V(G) \times V(H) = N_G[S_1] \times N_H[S_2] = N_{G\otimes H}[S_1 \times S_2]$. Hence $S_1 \times S_2$ is a geodetic dominating set of $G \otimes H$. □

Theorem 5.9. *Let* G *and* H *be connected graphs and* S *be a geodetic dominating set of* $G \otimes H$. *Then* $\Pi_G(S)$ *is a geodetic dominating set of* G *or* $\Pi_H(S)$ *is a geodetic dominating set of* H.

Proof. Assume that both $S_1 = \Pi_G(S)$ and $S_2 = \Pi_H(S)$ are not a dominating set of G and H respectively, and consider $x \in V(G) - N_G[S_1]$ and $y \in V(H) - N_H[S_2]$. Since S is a geodetic dominating set of $G \otimes H$, there exists $(g, h) \in S$ such that $(x, y) \in N_{G\otimes H}[(g, h)] \subseteq N_{G\otimes H}[S]$. By Theorem 5.4(ii), $x \in N_G[g] \subseteq N_G[S_1]$ and $y \in N_H[h] \subseteq N_H[S_2]$, which is a contradiction. Hence, by Theorem 5.6, $\Pi_G(S)$ is a geodetic dominating set of G or $\Pi_H(S)$ is a geodetic dominating set of H. □

Theorem 5.10. *Let* G *and* H *be connected graphs and* S *be a geodetic dominating set of* $G \otimes H$. *If* $Ext(G) \neq \phi$, *then* $\Pi_H(S)$ *is a geodetic dominating set of* H.

Proof. Let S be a geodetic dominating set of $G \otimes H$ and $(x, y) \notin S \subseteq V(G \otimes H)$. Let $Ext(G) \neq \phi$. Then by Theorem 5.7, $\Pi_H(S)$ is a geodetic set of H. We show that $\Pi_H(S)$ is a dominating set of H. Since S is a geodetic dominating set of $G \otimes H$, there exists $(g,h) \in S$ such that $(x,y) \in N_{G\otimes H}[(g,h)] \subseteq N_{G\otimes H}[S]$. By Theorem 5.4(ii), $y \in N_H[h] \subseteq N_H[\Pi_H(S)]$ for any $x \in V(G)$. Hence the result follows. □

Theorem 5.11. *Let G and H be connected graphs such that $Ext(G) \neq \phi$. Then $\gamma_g(H) \leq \gamma_g(G \otimes H)$.*

Proof. Let $Ext(G) \neq \phi$ and S be a geodetic dominating set of $G \otimes H$. Then by Theorem 5.10, $\Pi_H(S)$ is a geodetic dominating set of H. So that $\gamma_g(H) = min\{|\Pi_H(S)| : \Pi_H(S)$ is a geodetic dominating set of $H\} \leq min\{|S| : S$ is a geodetic dominating set of $G \otimes H\} = \gamma_g(G \otimes H)$. □

Corollary 5.12. *Let G and H be connected graphs such that $Ext(G) \neq \phi$ and $Ext(H) \neq \phi$. Then $max\{\gamma_g(G), \gamma_g(H)\} \leq \gamma_g(G \otimes H)$.*

Proof. This follows from Theorem 5.11. □

Corollary 5.13. *Let G be a connected graph and an integer $n \geq 2$. Then*

(i). $\gamma_g(G) \leq \gamma_g(G \otimes K_n)$
(ii). $\gamma_g(G) \leq \gamma_g(G \otimes K_{1,n})$

Proof. This follows from Theorem 5.11. □

Theorem 5.14. *Let G and H be nontrivial graphs. Then $\gamma_g(G \otimes H) \geq 4$.*

Proof. This follows from Theorem 1.1 and Theorem 5.5. □

Theorem 5.15. *For any two graphs G and H, $min\{\gamma_g(G), \gamma_g(H)\} \leq \gamma_g(G \otimes H) \leq \gamma_g(G) \cdot \gamma_g(H)$ and both bounds are sharp.*

Proof. Let S_1 and S_2 be a minimum geodetic dominating set of G and H respectively, so that $\gamma_g(G) = |S_1|$ and $\gamma_g(H) = |S_2|$. By Theorem 5.8, $S_1 \times S_2$ is a geodetic dominating set of $G \otimes H$. Therefore $\gamma_g(G \otimes H) \leq |S1 \times S2| = |S1| \cdot |S2| = \gamma_g(G) \cdot \gamma_g(H)$. To show the sharpness, take $G = K_m$ and $H = K_n$. Then, $\gamma_g(K_m \otimes K_n) = \gamma_g(K_{mn}) = mn = \gamma_g(K_m) \cdot \gamma_g(K_n)$. Now let S be a minimum geodetic dominating set of $G \otimes H$. According to Theorem 5.9, without loss of generality, we assume that $\Pi_G(S)$ is a geodetic dominating set of G. Hence $min\{\gamma_g(G), \gamma_g(H)\} \leq \gamma_g(G) \leq |\Pi_G(S)| \leq |S| = \gamma_g(G \otimes H)$. To show the sharpness, take $G = K_{m,n}$ and $H = K_p$ with $m, n, p \geq 4$. Then $\gamma_g(K_{m,n} \otimes K_p) = 4 = min\{\gamma_g(K_{m,n}), \gamma_g(K_p)\}$. □

Theorem 5.16. [3, 9, 14] *For any positive integers m and n,*

(i). $\gamma(P_m \otimes P_n) = \gamma(P_m).\gamma(P_n) = \lceil \frac{m}{3} \rceil . \lceil \frac{n}{3} \rceil$.
(ii). $g(P_m \otimes P_n) = g(P_m).g(P_n) = 2.2 = 4$.

Corollary 5.17. *For any integers $m, n \geq 4$, $\lceil \frac{m}{3} \rceil . \lceil \frac{n}{3} \rceil \leq \gamma_g(P_m \otimes P_n) \leq \lceil \frac{m+2}{3} \rceil . \lceil \frac{n+2}{3} \rceil$.*

Proof. This follows from Theorem 5.15 and Theorem 5.16. □

Theorem 5.18. *Let G be a connected graph with a geodetic dominating set S satisfies the following condition (A): for all $x \in S$, there exists $y, z \in S - \{x\}$ such that $x \in I_G[y, z]$. Then, for every vertex $k \in V(K_n)$, $S \times \{k\}$ is a geodetic dominating set of $G \otimes K_n$.*

Proof. Let S be a geodetic dominating set of G satisfying the condition (A) and take an arbitrary vertex $(u, v) \in V(G \otimes K_n)$. If $u \in S$, then there exists a pair of vertices $x, y \in S - \{u\}$ such that $u \in I_G[x, y]$. Hence, $d_{G\otimes K_n}((x, k), (y, k)) = d_G(x, y) = d_G(x, u) + d_G(u, y) = d_{G\otimes K_n}((x, k), (u, v)) + d_{G\otimes K_n}((u, v), (y, k))$. Hence $(u, v) \in I_{G\otimes K_n}[(x, k), (y, k)] \subseteq I_{G\otimes K_n}[S \times \{k\}]$. Since $v \in V(K_n)$, $(u, v) \in N_{G\otimes K_n}[(u, k)] \subseteq N_{G\otimes K_n}[S \times \{k\}]$. If $u \notin S$, then, since S is a geodetic dominating set G, there exists $x, y \in S$ such that $u \in I_G[x, y]$ and there exists $z \in S$ such that $u \in N_G[z]$. By the similar argument, we conclude that $S \times \{k\}$ is a geodetic dominating set of $G \otimes K_n$ for every $k \in V(K_n)$. □

Corollary 5.19. *Let G be a connected graph with a geodetic dominating set S satisfying the condition (A). Then for every $n \geq 2$, $\gamma_g(G \otimes K_n) \leq \gamma_g(G)$.*

Proof. This follows from Theorem 5.18. □

Theorem 5.20. *Let G be a connected graph with a minimum geodetic dominating set S satisfies the condition (A). Then for every $n \geq 2$, $\gamma_g(G \otimes K_n) = \gamma_g(G)$.*

Proof. This follows from Corollary 5.13 and Corollary 5.19. □

Example 5.21. (i). *Consider the complete bipartite graph $K_{m,n}$ with $m, n \geq 4$ and the complete graph K_p with $p \geq 2$. Notice that if $V(K_{m,n}) = \{u_1, u_2, \ldots, u_m\} \cup \{v_1, v_2, \ldots, v_n\}$, then the set $\{u_1, u_2, v_1, v_2\}$ is a minimum geodetic dominating set satisfying condition (A). Hence $\gamma_g(K_{m,n} \otimes K_p) = \gamma_g(K_{m,n}) = 4$.*

(ii). *Consider the cycle C_n with $n \geq 12$ and the complete graph K_m with $m \geq 2$. Notice that if $V(C_n) = \{c_0, c_1, c_2, \ldots, c_{n-1}\}$, then the set $\{c_0, c_3, c_6, \ldots, c_{3t}\}$, where $3t$ is the largest value of multiple of 3 which is less than n, is a minimum geodetic dominating set satisfying condition (A). Hence $\gamma_g(K_m \otimes C_n) = \gamma_g(C_n) = \lceil \frac{n}{3} \rceil$.*

Theorem 5.22. *Let G and H be connected graphs. Then $Ext(G)$ and $Ext(H)$ are geodetic dominating sets of G and H respectively if and only if $Ext(G \otimes H)$ is a geodetic dominating set of $G \otimes H$.*

Proof. Let $Ext(G)$ and $Ext(H)$ be geodetic dominating sets of G and H respectively. By Theorem 5.3 and Theorem 5.8, $Ext(G\otimes H) = Ext(G) \times Ext(H)$ is a geodetic dominating set of $G\otimes H$. Conversely, let $Ext(G\otimes H)$ be a geodetic dominating set of $G\otimes H$. Since $Ext(G) \neq \phi$ and $Ext(H) \neq \phi$, by Theorem 5.3 and Theorem 5.10, $Ext(G)$ and $Ext(H)$ are geodetic dominating sets of G and H respectively. □

Corollary 5.23. *Let G and H be connected graphs. Then $Ext(G)$ and $Ext(H)$ are geodetic dominating sets of G and H respectively if and only if $\gamma_g(G \otimes H) = \gamma_g(G).\gamma_g(H)$.*

Proof. This follows from Theorem 5.22. □

Remark 5.24. *For any integers m and n,*

(i). $\gamma_g(K_m \otimes K_n) = \gamma_g(K_m).\gamma_g(K_n) = mn$.

(ii). $\gamma_g(K_m \otimes K_{1,n}) = \gamma_g(K_m).\gamma_g(K_{1,n}) = mn$.

(iii). $\gamma_g(K_{1,m} \otimes K_{1,n}) = \gamma_g(K_{1,m}).\gamma_g(K_{1,n}) = mn$.

Acknowledgments

The authors are thankful to the anonymous referees, whose comments and remarks helped to improve this paper significantly.

References

[1] B. Bresar, S. Klavzar and A. T. Horvat, On the geodetic number and related metric sets in Cartesian product graphs, *Discrete Math.*, **308** (2008) 5555-5561.

[2] F. Buckley and F. Harary, *Distance in Graphs*, Addison-Wesley Publishing Company, Redwood City, CA, 1990.

[3] J. Caceres, C. Hernando, M. Mora and I. M. Pelayo, On the geodetic and hull numbers in strong product graphs, *Comput. Math. Appl.*, **60** (2010) 3020–3031.

[4] H. Escuadro, R. Gera, A. Hansberg, N. Jafari Rad and L. Volkmann, Geodetic Domination in Graphs, *J. Combin. Math. Combin. Comput.*, **77** (2011) 89-101.

[5] G. Chartrand, F. Harary and P. Zhang, On the Geodetic Number of a Graph, *Networks*, **39** (2002) 1-6.

[6] A. Hansberg and L. Volkmann, On the geodetic and geodetic domination numbers of a graph, *Discrete Math.*, **310** (2010) 2140-2146.

[7] F. Harary, E. Loukakis and C. Tsouros, The geodetic number of a graph, *Math. Comput. Modelling*, **17** (1993) no. 11 89-95.

[8] T. W. Haynes, S. T. Hedetniemi and P. J. Slater, *Fundamentals of domination in graphs*, **208**, Marcel Dekker, Inc, New York, 1998.

[9] M. S. Jacobson and L. F. Kinch, On the domination number of products of graphs I, *Ars Combin.*, **18** (1984) 33-44.

[10] J. Cao, B. Wu and M. Shi, The geodetic Number of $C_m \times C_n$, *IEEE*, (2009).

[11] S. Klavzar and N. Seifter, Dominating Cartesian products of cycles, *Discrete Appl. Math.*, **59** (1995) 129-136.

[12] A. P. Santhakumaran and S. V. Ullas Chandran, The geodetic number of strong product graphs, *Discuss. Math. Graph Theory* , **30** no. 4 (2010) 687-700.

[13] T. Jiang, I. Pelayo and D. Pritikin, Geodesic convexity and Cartesian products in graphs, 2004.

[14] I. G. Yero and J. A. Rodriguez-velazquez, Domination and Roman domination in graphs product, Manuscript.

[15] I. G. Yero and J. A. Rodriguez-velazquez, Roman domination in cartesian product graphs and strong product graphs, *Applicable Analysis and Discrete Mathematics*, **7** (2013) 262-274.

S. Robinson Chellathurai

Department of Mathematics, Scott Christian College, P.O.Box 629 001, Nagercoil, India

Email: `robinchel@rediffmail.com`

S. Padma Vijaya

Department of Mathematics, University College of Engineering Nagercoil, Anna University, Tirunelveli Region, P.O.Box 629 004, Nagercoil, India

Email: `padmaberry@yahoo.com`