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PERFECT STATE TRANSFER IN UNITARY CAYLEY GRAPHS OVER LOCAL RINGS

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Abstract. In this work, using eigenvalues and eigenvectors of unitary Cayley graphs over finite local rings and elementary linear algebra, we characterize which local rings allow a PST occurring in its unitary Cayley graph. Moreover, we have some developments when R is a product of local rings.

1. **Perfect State Transfer and Unitary Cayley Graphs**

Let G be an undirected graph whose vertex set $V(G) = \{v_1, \ldots, v_n\}$. The *adjacency matrix* of G, written A_G, is the $n \times n$ matrix in which entry a_{jk} is the number of edges in G with endpoint $\{v_j, v_k\}$. Define the matrix-valued function

$$H(t) = \exp(itA_G) \quad \text{for all } t \geq 0.$$

We say there is a *perfect state transfer (PST) from vertex v_j to vertex v_k* if there is a time t such that $|H(t)_{jk}| = 1$. We note that our matrix $H(t)$ determines what is known in graph theory as a *continuous quantum walk*. For background on quantum walks, we refer the reader to [9] and [10]. A perfect state transfer in continuous-time quantum walk on graphs has received considerable attention in quantum information and computations in Physics (e.g., [2, 4]). An excellent survey of perfect state transfer graphs and related questions are given by Godsil [8]. Observe that $H(t)$ has the following properties:

(i) $H(t)$ is symmetric,
(ii) $\overline{H(t)} = H(t)^{-1}$, where $^{-}$ is the complex conjugate,
(iii) $H(t)$ is unitary, i.e., $(\overline{H(t)})^T = H(t)^{-1}$.

Thus, we have the next proposition.

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Proposition 1.1. *If we have a perfect state transfer on A_G from vertex a to vertex b at time t, then we have a perfect state transfer from vertex b to vertex a at the same time.*

The following theorem is well known in Linear Algebra.

Theorem 1.2. *Let $\mathbb{R}^n = W_1 \oplus W_2 \oplus \cdots \oplus W_k$ be an orthogonal decomposition of $\mathbb{R}^n$, where each W_j is spanned by orthogonal basis $\vec{u}_{j1}, \vec{u}_{j2}, \ldots, \vec{u}_{jm_j}$ for some $m_j \in \mathbb{N}$ and for all $j \in \{1, 2, \ldots, k\}$. For each $j \in \{1, 2, \ldots, k\}$, let E_j be the projection of $\mathbb{R}^n$ for W_j. Then the lth column of the standard matrix of E_j is given by*

$$E_j(\vec{e}_l) = \langle \vec{e}_l, \vec{u}_{j1} \rangle \frac{\vec{u}_{j1}}{||\vec{u}_{j1}||^2} + \langle \vec{e}_l, \vec{u}_{j2} \rangle \frac{\vec{u}_{j2}}{||\vec{u}_{j2}||^2} + \cdots + \langle \vec{e}_l, \vec{u}_{jm_j} \rangle \frac{\vec{u}_{jm_j}}{||\vec{u}_{jm_j}||^2}$$

for all $l \in \{1, 2, \ldots, n\}$, where $\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ is the standard basis of $\mathbb{R}^n$.

Recall that A_G is orthogonally diagonalizable. Let $\theta_1, \theta_2, \ldots, \theta_m$ be distinct eigenvalues of A_G and let E_r denote the orthogonal projection on the eigenspace belonging to θ_r for all $r \in \{1, 2, \ldots, m\}$. Here, we abuse the notation by writing E_r for its standard matrix. It follows from the Spectral Theorem (Theorem 6.25 of [5]) that

(i) $E_j E_k = \delta_{jk} E_j$, for $1 \leq j, k \leq m$,
(ii) $E_1 + E_2 + \cdots + E_r = I_n$,
(iii) $\theta_1 E_1 + \theta_2 E_2 + \cdots + \theta_m E_m = A_G$.

If f is a differentiable complex-valued function defined on the eigenvalues of A_G, then

$$f(A_G) = \sum_{r=1}^{m} f(\theta_r) E_r.$$

In particular,

$$H(t) = \exp(itA_G) = \sum_{r=1}^{m} \exp(it\theta_r) E_r.$$

For $j \in \{1, 2, \ldots, n\}$, we write $|\vec{v}_j\rangle = \vec{e}_j$, the jth column of the identity matrix I_n. The following proposition is Lemma 2.1 of [8]. It will become our main tool, so we record it below.

Proposition 1.3. *A perfect state transfer occurs in G from vertex a to vertex b at time t if and only if there is a $\gamma \in \mathbb{C}$ such that $|\gamma| = 1$ and $E_r |b\rangle = \gamma \exp(-it\theta_r) E_r |a\rangle$ for all $r \in \{1, 2, \ldots, m\}$.*

Corollary 1.4. *If there is a perfect state transfer from vertex a to vertex b, then $E_r |a\rangle = \pm E_r |b\rangle$ for all $r \in \{1, 2, \ldots, m\}$.*

Proof. It follows from the fact that $E_r|a\rangle$ and $E_r|b\rangle$ are real vectors for all $r \in \{1, 2, \ldots, m\}$. □

By Proposition 1.3, another main tool for studying perfect state transfers is the spectral decomposition of A_G.

Let R be a finite commutative ring with unity $1 \neq 0$ and let $R^\times$ denote the unit group of invertible elements of R. The *unitary Cayley graph of* R, $G_R = \mathrm{Cay}(R, R^\times)$ is the Cayley graph whose vertex set is R and edge set is $\{\{a, b\} : a, b \in R \text{ and } a - b \in R^\times\}$.

For two graphs G and H, their *weak product*, $G \otimes H$, is the graph defined on $V(G) \times V(H)$ where (a,b) is adjacent to (a',b') if and only if a is adjacent to a' in G and b is adjacent to b' in H. The adjacency matrix of $G \times H$ is $A_G \otimes A_H = \begin{bmatrix} a_{11}A_H & \dots & a_{1n}A_H \\ \vdots & \ddots & \vdots \\ a_{n1}A_H & \dots & a_{nn}A_H \end{bmatrix}$, where a_{jk} is the entry in A_G for all $j,k \in \{1,2,\dots,n\}$.

Recall that a *local ring* R is a commutative ring with unity 1 which has a unique maximal ideal M. Note that, if R is a local ring with unique maximal ideal M, then $R^\times = R \smallsetminus M$. Furthermore, every finite commutative ring is a product of local rings. The structure of G_R is presented in the next proposition.

Proposition 1.5. [1] *Let R be a finite commutative ring.*

(i) G_R is a regular graph of degree $|R^\times|$.
(ii) If R is a local ring with maximal ideal M, then G_R is a complete multipartite graph whose patite sets are the cosets of M in R. In particular, G_R is a complete graph if and only if R is a field.
(iii) If $R \cong R_1 \times \dots \times R_s$ is a product of local rings, then $G_R \cong \bigotimes_{i=1}^{s} G_{R_i}$.

As is standard, if $\theta_1,\dots,\theta_k$ are eigenvalues of a graph G with multiplicities $m_1,\dots,m_k$, respectively, we use the notation $\operatorname{Spec} G = \begin{pmatrix} \theta_1 & \dots & \theta_k \\ m_1 & \dots & m_k \end{pmatrix}$ to describe the spectrum of G. We have the following fact.

Proposition 1.6. [1, 11] *Let R be a finite local ring with maximal ideal M of size m. Then*

$$\operatorname{Spec} G_R = \begin{pmatrix} |R^\times| & -m & 0 \\ 1 & \frac{|R|}{m}-1 & \frac{|R|}{m}(m-1) \end{pmatrix}.$$

In particular, if F is a finite field, then

$$\operatorname{Spec} G_F = \begin{pmatrix} |F^\times| & -1 \\ 1 & |F^\times| \end{pmatrix}.$$

When $R = \mathbb{Z}_n$, Bašić et al. [3] have investigated a perfect state transfer on G_R. They proved that if n and $n/2$ are not square-free integers, there is a PST in $G_{\mathbb{Z}_n}$. Moreover, they showed that the only unitary Cayley graphs of the ring $\mathbb{Z}_n$ that have a PST are K_2 (path of length two) and C_4 (4-cycle).

In this work, using Propositions 1.3 and 1.6, we characterize which local rings allowing PST occurring in its unitary Cayley graph in Section 2. Further developments when R is a product of local rings are studied in Section 3.

2. PST of G_R when R is Local

Throughout this section, we let R be a finite local ring with unique maximal ideal M of size m. For $k,l \in \mathbb{N}$, we write $\mathbf{0}_{k\times l}$ and $J_{k\times l}$ for the $k \times l$ matrix whose all entries are 0 and 1, respectively. We also use $\vec{0}_k = \mathbf{0}_{k\times 1}$ and $\vec{1}_k = J_{k\times 1}$. By Proposition 1.5 (ii), we have

$$A_{G_R} = \begin{bmatrix} 0_{m\times m} & J_{m\times m} & J_{m\times m} & \cdots & J_{m\times m} \\ J_{m\times m} & 0_{m\times m} & J_{m\times m} & \cdots & J_{m\times m} \\ J_{m\times m} & J_{m\times m} & 0_{m\times m} & \cdots & J_{m\times m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ J_{m\times m} & J_{m\times m} & J_{m\times m} & \cdots & 0_{m\times m} \end{bmatrix}.$$

By Proposition 1.6, G_R has eigenvalues $\theta_1 = |R^\times|$, $\theta_2 = -m$ and $\theta_3 = 0$ with multiplicities 1, $\frac{|R|}{m} - 1$ and $\frac{|R|}{m}(m-1)$, respectively, and eigenspace spanned, respectively, by the columns of the following orthogonal matrices:

$$A_1 = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}_{|R|\times 1}, \quad A_2 = \begin{bmatrix} \vec{1}_m & \frac{1}{2}\vec{1}_m & \frac{1}{3}\vec{1}_m & & \frac{1}{\frac{|R|}{m}-1}\vec{1}_m \\ -\vec{1}_m & \frac{1}{2}\vec{1}_m & \frac{1}{3}\vec{1}_m & & \frac{1}{\frac{|R|}{m}-1}\vec{1}_m \\ \vec{0}_m & -\vec{1}_m & \frac{1}{3}\vec{1}_m & & \frac{1}{\frac{|R|}{m}-1}\vec{1}_m \\ \vec{0}_m & \vec{0}_m & -\vec{1}_m & & \frac{1}{\frac{|R|}{m}-1}\vec{1}_m \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \vec{0}_m & \vec{0}_m & \vec{0}_m & & -\vec{1}_m \end{bmatrix}_{|R|\times \frac{|R|}{m}-1},$$

and

$$A_3 = \begin{bmatrix} W & & & \\ & W & & \\ & & \ddots & \\ & & & W \end{bmatrix}_{|R|\times \frac{|R|}{m}(m-1)},$$

where

$$W = \begin{bmatrix} 1 & 1 & & 1 \\ \omega & \omega^2 & & \omega^{m-1} \\ \omega^2 & \omega^4 & & \omega^{2(m-1)} \\ \vdots & \vdots & \cdots & \vdots \\ \omega^{m-1} & \omega^{2(m-1)} & & \omega^{(m-1)(m-1)} \end{bmatrix}_{m\times(m-1)} \quad \text{and } \omega = \exp(2\pi i/m).$$

Using Theorem 1.2, the standard matrices of $E_j, j = 1,2,3$, can be directly computed. We record them in the next theorem.

Theorem 2.1. *Let R be a finite local ring with unique maximal ideal M of size m. For $j \in \{1,2,3\}$, let E_j be the orthogonal projection on the eigenspace belonging to θ_j of A_{G_R}. Then*

(i) $E_1 = \frac{1}{|R|} J_{|R|\times|R|}$,

(ii) $E_2 = \begin{bmatrix} \vec{w}_1 & \vec{w}_2 & \cdots & \vec{w}_{|R|-1} & \vec{w}_{|R|} \end{bmatrix}$, *where*

$$\vec{w}_s = \sum_{l=1}^{\frac{|R|}{m}-1} \frac{\vec{u}_l}{(l+1)m}, \vec{w}_{km+s} = \sum_{l=k}^{\frac{|R|}{m}-1} \frac{\vec{u}_l}{(l+1)m} - \frac{\vec{u}_k}{(k+1)m}, \vec{w}_{|R|-m+s} = \Big(\frac{m-|R|}{|R|m}\Big)\vec{u}_{\frac{|R|}{m}-1},$$

for all $s \in \{1, 2, \dots, m\}$*,* $k \in \{1, 2, \dots, \frac{|R|}{m} - 2\}$ *and* $\vec{u}_l$ *is the* l*th column of* A_2 *for all* $l \in \{1, 2, \dots, \frac{|R|}{m} - 1\}$*, and*

(iii) $$E_3 = \frac{1}{m}\begin{bmatrix} M & & & \\ & M & & \\ & & \ddots & \\ & & & M \end{bmatrix}_{|R|\times|R|}, \text{ where } M = \begin{bmatrix} m-1 & -1 & -1 & & -1 \\ -1 & -1 & -1 & & m-1 \\ -1 & -1 & -1 & & -1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -1 & -1 & m-1 & & -1 \\ -1 & m-1 & -1 & & -1 \end{bmatrix}_{m\times m}.$$

The above computations and Corollary 1.4 give the following necessity condition.

Theorem 2.2. *Let* R *be a finite local ring with the maximal ideal* M *of size* m*. If there is a perfect state transfer from vertex* v_j *to vertex* v_k *in the graph* G_R *for some* $1 \le j < k \le |R|$*, then* m *is* 1 *or* 2*.*

Proof. By Corollary 1.4, we have $E_3|v_j\rangle = \pm E_3|v_k\rangle$. Thus, the jth column of E_3 is equal to $\pm$ the kth column of E_3, which implies that $m - 1 = -1, 0$ or 1. Since $m > 0$, $m = 1$ or 2. □

Note that if $m = 1$, then R is a finite field. We obtain a further result on the number of elements of R in the next theorem.

Theorem 2.3. *Let* R *be the finite field with* q *elements. Then there is a perfect state transfer in the graph* G_R *if and only if* $q = 2$*.*

Proof. If $q = 2$, then $A_{G_R} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $H(t) = e^{itA_{G_R}} = \begin{bmatrix} \cos t & i\sin t \\ i\sin t & \cos t \end{bmatrix}$ for all $t \ge 0$. Thus, $H(\frac{\pi}{2}) = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$ and G_R has a perfect state transfer at $t = \dfrac{\pi}{2}$. Conversely, assume that $q \ge 3$. We have

$$E_2 = \begin{bmatrix} \vec{w}_1 & \vec{w}_2 & \cdots & \vec{w}_{q-1} & \vec{w}_q \end{bmatrix},$$

where

$$\vec{w}_1 = \sum_{l=1}^{q-1} \frac{\vec{u}_l}{l+1}, \vec{w}_s = \sum_{l=s}^{q-1} \frac{\vec{u}_l}{l+1} - \frac{\vec{u}_{s-1}}{||\vec{u}_{s-1}||^2}\,(s = 2, 3, \dots, q-1), \vec{w}_q = (\frac{1-q}{q})\vec{u}_{q-1},$$

and $\vec{u}_l$ is the lth column of A_2 for all $l \in \{1, 2, \dots, q-1\}$. Let $1 \le j < k \le q$.

Case 1. $j = 1$ and $k \in \{2, 3, \dots, q-1\}$. Since

$$\vec{w}_1 - \vec{w}_k = \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} + \cdots + \frac{1}{k}\begin{bmatrix} \frac{1}{k-1} \\ \vdots \\ \frac{1}{k-1} \\ -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \left(\frac{k-1}{k}\right)\begin{bmatrix} \frac{1}{k-1} \\ \vdots \\ \frac{1}{k-1} \\ -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

The first entry of $\vec{w}_1 - \vec{w}_k$ is nonzero. Also,

$$\vec{w}_1 + \vec{w}_k = \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} + \cdots + \frac{1}{k}\begin{bmatrix} \frac{1}{k-1} \\ \vdots \\ \frac{1}{k-1} \\ -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \frac{2}{k+1}\begin{bmatrix} \frac{1}{k} \\ \vdots \\ \frac{1}{k} \\ -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$+ \cdots + \frac{2}{q}\begin{bmatrix} \frac{1}{q-1} \\ \frac{1}{q-1} \\ \frac{1}{q-1} \\ \vdots \\ \frac{1}{q-1} \\ \frac{1}{q-1} \\ \frac{1}{q-1} \\ -1 \end{bmatrix} - \left(\frac{k-1}{k}\right)\begin{bmatrix} \frac{1}{k-1} \\ \vdots \\ \frac{1}{k-1} \\ \frac{1}{k-1} \\ -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

The last entry of $\vec{w}_1 + \vec{w}_k$ is nonzero. Hence, $\vec{w}_1 \neq \pm\vec{w}_k$.

Case 2. $j = 1$ and $k = q$. Since

$$\vec{w}_1 - \vec{w}_q = \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} + \cdots + \frac{1}{q}\begin{bmatrix} \frac{1}{q-1} \\ \vdots \\ \frac{1}{q-1} \\ -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} - \left(\frac{q-1}{q}\right)\begin{bmatrix} \frac{1}{q-1} \\ \vdots \\ \frac{1}{q-1} \\ -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

and

$$\vec{w}_1 + \vec{w}_q = \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} + \cdots + \frac{1}{q}\begin{bmatrix} \frac{1}{q-1} \\ \vdots \\ \frac{1}{q-1} \\ -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \left(\frac{q-1}{q}\right)\begin{bmatrix} \frac{1}{q-1} \\ \vdots \\ \frac{1}{q-1} \\ -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

we have the qth entry of $\vec{w}_1 - \vec{w}_q$ is -1 and of $\vec{w}_1 + \vec{w}_q$ is $\frac{-2+q}{q} \geq \frac{-2+3}{q} \neq 0$ because $q > 3$. Thus, $\vec{w}_1 \neq \pm\vec{w}_k$.

Case 3. $j,k \in \{2,\dots,q-1\}$. Since

$$\vec{w}_j - \vec{w}_k = \tfrac{1}{j+1}\begin{bmatrix} \frac{1}{j} \\ \vdots \\ \frac{1}{j} \\ \frac{1}{j} \\ -1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \cdots + \tfrac{1}{k}\begin{bmatrix} \frac{1}{k-1} \\ \vdots \\ \vdots \\ \frac{1}{k-1} \\ \frac{1}{k-1} \\ -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} - \left(\tfrac{j-1}{j}\right)\begin{bmatrix} \frac{1}{j-1} \\ \vdots \\ \frac{1}{j-1} \\ -1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} + \left(\tfrac{k-1}{k}\right)\begin{bmatrix} \frac{1}{k-1} \\ \vdots \\ \vdots \\ \frac{1}{k-1} \\ \frac{1}{k-1} \\ -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

the kth entry of $\vec{w}_j - \vec{w}_k$ is -1. Moreover,

$$\begin{aligned}\vec{w}_j + \vec{w}_k = &\tfrac{1}{j+1}\begin{bmatrix} \frac{1}{j} \\ \vdots \\ \frac{1}{j} \\ \frac{1}{j} \\ -1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \cdots + \tfrac{1}{k}\begin{bmatrix} \frac{1}{k-1} \\ \vdots \\ \vdots \\ \frac{1}{k-1} \\ \frac{1}{k-1} \\ -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \tfrac{2}{k+1}\begin{bmatrix} \frac{1}{k} \\ \vdots \\ \vdots \\ \frac{1}{j} \\ -1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \cdots + \tfrac{1}{q}\begin{bmatrix} \frac{1}{q-1} \\ \frac{1}{q-1} \\ \vdots \\ \vdots \\ \frac{1}{q-1} \\ \frac{1}{q-1} \\ -1 \end{bmatrix} \\ &- \left(\tfrac{j-1}{j}\right)\begin{bmatrix} \frac{1}{j-1} \\ \vdots \\ \frac{1}{j-1} \\ -1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} - \left(\tfrac{k-1}{k}\right)\begin{bmatrix} \frac{1}{k-1} \\ \vdots \\ \vdots \\ \frac{1}{k-1} \\ \frac{1}{k-1} \\ -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix},\end{aligned}$$

the qth row of $\vec{w}_j + \vec{w}_k$ is not equal to 0.

Case 4. $j, k = 2, \ldots, q-1, k = q$. Since

$$\vec{w}_j - \vec{w}_q = \tfrac{1}{j+1}\begin{bmatrix} \frac{1}{j} \\ \vdots \\ \frac{1}{j} \\ \frac{1}{j} \\ -1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \cdots + \tfrac{1}{q}\begin{bmatrix} \frac{1}{q-1} \\ \frac{1}{q-1} \\ \vdots \\ \frac{1}{q-1} \\ \frac{1}{q-1} \\ \vdots \\ \frac{1}{q-1} \\ \frac{1}{q-1} \\ -1 \end{bmatrix} - \left(\tfrac{j-1}{j}\right)\begin{bmatrix} \frac{1}{j-1} \\ \vdots \\ \frac{1}{j-1} \\ -1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} - \left(\tfrac{1-q}{q}\right)\begin{bmatrix} \frac{1}{q-1} \\ \frac{1}{q-1} \\ \vdots \\ \frac{1}{q-1} \\ \frac{1}{q-1} \\ \vdots \\ \frac{1}{q-1} \\ \frac{1}{q-1} \\ -1 \end{bmatrix},$$

and

$$\vec{w}_j + \vec{w}_q = \tfrac{1}{j+1}\begin{bmatrix} \frac{1}{j} \\ \vdots \\ \frac{1}{j} \\ \frac{1}{j} \\ -1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \cdots + \tfrac{1}{q}\begin{bmatrix} \frac{1}{q-1} \\ \frac{1}{q-1} \\ \vdots \\ \frac{1}{q-1} \\ \frac{1}{q-1} \\ \vdots \\ \frac{1}{q-1} \\ \frac{1}{q-1} \\ -1 \end{bmatrix} - \left(\tfrac{j-1}{j}\right)\begin{bmatrix} \frac{1}{j-1} \\ \vdots \\ \frac{1}{j-1} \\ -1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} + \left(\tfrac{1-q}{q}\right)\begin{bmatrix} \frac{1}{q-1} \\ \frac{1}{q-1} \\ \vdots \\ \frac{1}{q-1} \\ \frac{1}{q-1} \\ \vdots \\ \frac{1}{q-1} \\ \frac{1}{q-1} \\ -1 \end{bmatrix},$$

the qth entry of $\vec{w}_j - \vec{w}_q$ is not equal to 0, and of $\vec{w}_j + \vec{w}_q = \frac{-2+q}{q} \geq \frac{1}{3} > 0$. Hence, $\vec{w}_j \neq \pm\vec{w}_k$ for all $1 \leq j < k \leq q$. That is, $E_2\,|v_j\rangle \neq \pm E_2\,|v_k\rangle$, for all $1 \leq j < k \leq q$.

By Proposition 1.4, there is no perfect state transfers in G_R. □

For $m = 2$, we have $|R| = 2^k$ for some $k \geq 2$. We get the following result.

Theorem 2.4. *Let R be a finite local ring with maximal ideal M of size two. Then the graph G_R has a perfect state transfer at time $t = \dfrac{\pi}{2}$.*

Proof. Recall that G_R has three distinct eigenvalues, $\theta_1 = 2^k - 2, \theta_2 = -2$ and $\theta_3 = 0$. Choose $\gamma = -1$ and $t = \frac{\pi}{2}$. Then $\exp(-it\theta_1) = \exp(-it\theta_2) = -1$ and $\exp(-it\theta_3) = 1$. Following Proposition 1.3 and Theorem 2.1, we show

$$E_1|v_1\rangle = E_1|v_2\rangle = \gamma\exp(-it\theta_1)E_1|v_2\rangle,$$

$$E_2|v_1\rangle = E_2|v_2\rangle = \gamma\exp(-it\theta_2)E_2|v_2\rangle,$$

$$\text{and } E_3|v_1\rangle = -E_3|v_2\rangle = \gamma\exp(-it\theta_3)E_3|v_2\rangle.$$

Hence, G_R has a perfect state transfer from vertex v_1 to vertex v_2 at time $t = \dfrac{\pi}{2}$. □

We conclude all discussions in this section in the next theorem.

Theorem 2.5. *Let R be a finite local ring with maximal ideal M of size m. Then G_R has a perfect state transfer if and only if $R = \mathbb{F}_2$ or $m = 2$. In particular, a perfect state occurs at time $t = \frac{\pi}{2}$.*

Moreover, if R is a local ring with $m = 2$, it follows from [6] that $|R|$ must be 4. Thus, R is $\mathbb{Z}_4$ or $\mathbb{Z}_2[x]/(x^2)$ as shown in [12]. Hence, we conclude that:

Corollary 2.6. *Let R be a finite local ring. Then G_R has a perfect state transfer if and only if $R = \mathbb{F}_2$ or $\mathbb{Z}_4$ or $\mathbb{Z}_2[x]/(x^2)$.*

3. Further Developments

In this section, we present some results when R is a product of finite local rings. We begin with the following lemma.

Lemma 3.1. *Let G and H be undirected graphs. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be eigenvalues of G corresponding to eigenvectors $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n$, respectively, and let $\mu_1, \mu_2, \ldots, \mu_m$ be eigenvalues of H corresponding to eigenvectors $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_m$, respectively.*

(i) For $k \in \{1, 2, \ldots, n\}$ and $l \in \{1, 2, \ldots, m\}$, we have $\vec{u}_k \otimes \vec{w}_l$ is an eigenvectors of $G \otimes H$ with eigenvalue $\lambda_k \mu_l$.

(ii) Let $g_1, g_2 \in V(G)$ and $h_1, h_2 \in V(H)$. Then

$$\langle (g_2, h_2)| \exp(itA_{G\times H})|(g_1, h_1)\rangle = \sum_{k=1}^{n} \langle g_2| \Big(\sum_{l=1}^{m} \langle h_2|\vec{w}_l \vec{w}_l^T|h_1\rangle \exp(it\lambda_k\mu_l) \Big) \vec{u}_k \vec{u}_k^T|g_1\rangle.$$

Proof. (i) Let $k \in \{1, 2, \ldots, n\}$ and $l \in \{1, 2, \ldots, m\}$. Then

$$A_{G\otimes H}(\vec{u}_k \otimes \vec{w}_l) = A_G\vec{u}_k \otimes A_H\vec{w}_l = \lambda_k\vec{u}_k \otimes \mu_l\vec{w}_l = \lambda_k\mu_l(\vec{u}_k \otimes \vec{w}_l).$$

(ii) Let $g_1, g_2 \in V(G)$ and $h_1, h_2 \in V(H)$. From (i), let

$$P = \Big[\vec{u}_1 \otimes \vec{w}_1 \quad \vec{u}_1 \otimes \vec{w}_2 \quad \cdots \quad \vec{u}_1 \otimes \vec{w}_m \quad \vec{u}_2 \otimes \vec{w}_1 \quad \vec{u}_2 \otimes \vec{w}_2 \quad \cdots \quad \vec{u}_n \otimes \vec{w}_m\Big]$$

be the $nm \times nm$ matrix such that

$$\exp(itA_{G\otimes H}) = P \begin{bmatrix} e^{it\lambda_1\mu_1} & & & & & & & \\ & e^{it\lambda_1\mu_2} & & & & & & \\ & & \ddots & & & & & \\ & & & e^{it\lambda_1\mu_m} & & & & \\ & & & & e^{it\lambda_2\mu_1} & & & \\ & & & & & e^{it\lambda_2\mu_2} & & \\ & & & & & & \ddots & \\ & & & & & & & e^{it\lambda_n\mu_m} \end{bmatrix} P^T.$$

Then

$$\begin{aligned}\langle (g_2,h_2)|\exp(itA_{G\times H})|(g_1,h_1)\rangle &= \sum_{k=1}^{n}\sum_{l=1}^{m}\langle g_2|\vec{u}_k\vec{u}_k^T|g_1\rangle\langle h_2|\vec{w}_l\vec{w}_l^T|h_1\rangle\exp(it\lambda_k\mu_l)\\ &= \sum_{k=1}^{n}\langle g_2|\Big(\sum_{l=1}^{m}\langle h_2|\vec{w}_l\vec{w}_l^T|h_1\rangle\exp(it\lambda_k\mu_l)\Big)\vec{u}_k\vec{u}_k^T|g_1\rangle.\end{aligned}$$

Hence, we have the lemma. □

If R is a finite local ring and G_R has no even eigenvalues, then by Proposition 1.6, R is the finite field of 2^r elements for some $r\in\mathbb{N}$. Thus, G_R is complete and its adjacency matrix is given by

$$A_{G_R} = \begin{bmatrix} 0 & 1 & 1 & \cdots & 1\\ 1 & 0 & 1 & \cdots & 1\\ \vdots & \ddots & & & \vdots\\ 1 & 1 & 1 & \cdots & 0\end{bmatrix}.$$

It can be shown that for $j\in\{0,1,\ldots,|R|-1\}$, the vector

$$\vec{w}_j = \frac{1}{|R|}\begin{bmatrix}1 & \omega_1 & \omega_j^2 & \cdots & \omega_j^{|R|-1}\end{bmatrix}^T, \quad \text{where } \omega_j = \exp\Big(\frac{2\pi i j}{|R|}\Big),$$

is an eigenvector of A_{G_R} with eigenvalue $\mu_j = \sum_{k=1}^{|R|-1}\omega_j^k$. Note that $\mu_0 = |R|-1$ and $\mu_j = -1$ if $j\geq 1$ (which are the same results found in Proposition 1.6). Moreover, we observe that

$$\langle 0|\vec{w}_j\vec{w}_j^T|0\rangle = \frac{1}{|R|} \tag{3.1}$$

for all $j\in\{0,1,\ldots,|R|-1\}$. The following proposition was proved for circulant graphs in [7]. However, the observation above yields the same result.

Proposition 3.2. *Let G be a graph on n vertices with perfect state transfer at time t_G so that*

$$t_G\,\mathrm{Spec}\,G \subseteq \mathbb{Z}\pi := \{a\pi : a\in\mathbb{Z}\}.$$

Then $G\otimes G_R$ has a perfect state transfer at time t_G if R is a finite local ring and G_R has no even eigenvalues (that is, R is the finite field of 2^r elements for some $r\in\mathbb{N}$).

Proof. Let $\lambda_1,\lambda_2,\ldots,\lambda_n$ be eigenvalues of G corresponding to eigenvectors $\vec{u}_1,\vec{u}_2,\ldots,\vec{u}_n$, respectively. Assume that G has a perfect state transfer at time t_G from vertex g_1 to vertex g_2. By Lemma 3.1 (ii) and Eq. (3.1), we have

$$\begin{aligned}
\langle (g_1,0)|e^{it_G A_{G\times H}}|(g_2,0)\rangle &= \sum_{k=1}^{n}\langle g_1|\sum_{j=0}^{|R|-1}\langle 0|\vec{w}_j\vec{w}_j^T|0\rangle \exp(it_G\lambda_k\mu_j)\vec{u}_k\vec{u}_k^T|g_2\rangle\\
&= \frac{1}{|R|}\sum_{k=1}^{n}\langle g_1\Big(\exp(it_G\lambda_k\mu_0)+\sum_{j=1}^{|R|-1}\exp(it_G\lambda_k\mu_j)\Big)\vec{u}_k\vec{u}_k^T|g_2\rangle\\
&= \frac{1}{|R|}\sum_{k=1}^{n}\langle g_1\Big(\exp(it_G\lambda_k(|R|-1))+\sum_{j=1}^{|R|-1}\exp(it_G\lambda_k(-1))\Big)\vec{u}_k\vec{u}_k^T|g_2\rangle\\
&= \sum_{k=1}^{n}\langle g_1\exp(-it_G\lambda_k)\vec{u}_k\vec{u}_k^T|g_2\rangle \qquad (\text{because } t_G\lambda_k\in\mathbb{Z}\pi)\\
&= \langle g_1|\exp(-it_GA_G)|g_2\rangle.
\end{aligned}$$

Since a perfect state transfer occurs from vertex g_1 to vertex g_2,

$$|\langle g_1|\exp(-it_GA_G)|g_2\rangle| = |\langle g_1|\exp(it_GA_G)|g_2\rangle| = 1,$$

so we have a perfect state transfer from vertex $(g_1,0)$ to vertex $(g_2,0)$. □

Theorem 2.5 and Proposition 3.2 give the following theorem.

Theorem 3.3. *Let $\mathbb{F}_{2^r}$ be the finite field with 2^r elements and R a finite local ring with $m=2$. Then $G_R\otimes G_{\mathbb{F}_{2^r}}$ has a perfect state transfer. Moreover, let $m\in\mathbb{N}$ and $\mathbb{F}_{2^{r_1}},\mathbb{F}_{2^{r_2}},\dots,\mathbb{F}_{2^{r_m}}$ be the finite fields with $2^{r_1},2^{r_2},\dots,2^{r_m}$ elements, respectively. Then $G_R\otimes G_{\mathbb{F}_{2^{r_1}}}\otimes\cdots\otimes G_{\mathbb{F}_{2^{r_m}}}$ has a perfect state transfer.*

Proof. Since R is a local ring with $m=2$, by Theorem 2.5, we have a perfect state transfer at time $t=\frac{\pi}{2}$. It follows from Proposition 1.6 that $t\,\mathrm{Spec}\,G_R\subseteq\mathbb{Z}\pi$. Hence, Proposition 3.2 inductively gives the desired results. □

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