Transactions on Combinatorics
ISSN (print): 2251-8657, ISSN (on-line): 2251-8665
Vol. 3 No. 4 (2014), pp. 55-58.

www.combinatorics.ir

www.ui.ac.ir

COMPLETE SOLUTION TO A CONJECTURE OF ZHANG-LIU-ZHOU

M. TAVAKOLI*, F. RAHBARNIA, M. MIRZAVAZIRI AND A. R. ASHRAFI

Communicated by Alireza Abdollahi

Abstract. Let $d_{n,m} = [\frac{2n+1-\sqrt{17+8(m-n)}}{2}]$ and $E_{n,m}$ be the graph obtained from a path $P_{d_{n,m}+1} = v_0v_1\cdots v_{d_{n,m}}$ by joining each vertex of $K_{n-d_{n,m}-1}$ to $v_{d_{n,m}}$ and $v_{d_{n,m}-1}$, and by joining $m-n+1-\binom{n-d_{n,m}}{2}$ vertices of $K_{n-d_{n,m}-1}$ to $v_{d_{n,m}-2}$. Zhang, Liu and Zhou [On the maximal eccentric connectivity indices of graphs, Appl. Math. J. Chinese Univ., in press] conjectured that if $d_{n,m} \geqslant 3$, then $E_{n,m}$ is the graph with maximal eccentric connectivity index among all connected graph with n vertices and m edges. In this note, we prove this conjecture. Moreover, we present the graph with maximal eccentric connectivity index among the connected graphs with n vertices. Finally, the minimum of this graph invariant in the classes of tricyclic and tetracyclic graphs are computed.

1. Introduction

Unless otherwise stated, throughout this paper our notation and terminology follows West [1]. Throughout this paper, graphs means simple connected graphs without loops and multiple edges. The complete graph, path and the star of order n are denoted by K_n, P_n and S_n, respectively. The **splice** of two disjoint graphs G and H by vertices y and z, $(G \cdot H)(y, z)$, is defined by identifying the vertices y and z in the union of G and H, see Figure 1(a).

Suppose G is a graph with vertex set $V(G)$. The distance between the vertices u and v of $V(G)$ is denoted by $d(u, v)$. It is defined as the number of edges in a minimal path connecting the vertices u and v.

A **topological index** is a graph invariant applicable in chemistry. The **eccentricity** $\varepsilon_G(u)$ is defined as the largest distance between u and other vertices of G. We will omit the subscript G when

MSC(2010): Primary: 05C15; Secondary: 05C12.

Keywords: Eccentric connectivity index, Tricyclic graph, Tetracyclic graph, Graph operation.

Received: 20 January 2014, Accepted: 4 August 2014.

*Corresponding author.

the graph is clear from the context. The **eccentric connectivity** index of a graph G is defined as $\xi^c(G) = \sum_{u \in V(G)} deg(u)\varepsilon(u)$ [2]. We encourage the reader to consult papers [3, 4] for some applications and papers [5, 6, 7, 8] for the mathematical properties of this topological index.

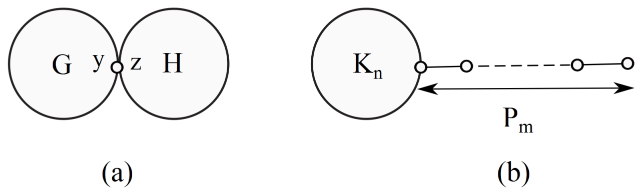

FIGURE 1. (a). The splice of two disjoint graphs G and H by vertices y and z; (b). A lollipop on $n + m - 1$ vertices.

For a connected graph G, the diameter $D(G)$ is the maximum eccentricity among vertices of G. Let $T_{n,m}$ be isomorphic to $(K_n \cdot P_m)(u, v)$ that u is a vertex of K_n and v is a pendant of P_m. This graph is called a lollipop on $n + m - 1$ vertices, Figure 1(b), [9].

2. Main Results

Let $d_{n,m} = \left[\frac{2n+1-\sqrt{17+8(m-n)}}{2}\right]$ and $E_{n,m}$ be the graph obtained from a path $P_{d_{n,m}+1} = v_0 v_1 \cdots v_{d_{n,m}}$ by joining each vertex of $K_{n-d_{n,m}-1}$ to $v_{d_{n,m}}$ and $v_{d_{n,m}-1}$, and by joining $m - n + 1 - \binom{n-d_{n,m}}{2}$ vertices of $K_{n-d_{n,m}-1}$ to $v_{d_{n,m}-2}$. In [10], Zhang, Liu and Zhou conjectured that if $d_{n,m} \geqslant 3$, then $E_{n,m}$ is the graph with maximal eccentric connectivity index among all connected graph with n vertices and m edges. The aim of this section is to prove a this conjecture. We also present the graph with maximal eccentric connectivity index among the connected graphs with n vertices. Finally, the minimum of this graph invariant in the classes of tricyclic and tetracyclic graphs are presented.

Proposition 2.1. *Let G be a graph with $n + m - 1$ vertices and $\binom{n}{2} + m - 1$ edges. Then $D(G) \leqslant D(T_{n,m})$.*

Proof. Suppose $D(G) = D(T_{n,m}) + 1$, that is $D(G) = m + 1$. Let P be a diametrical path of G of length $m+1$. So, there exists $n-3$ vertices and $\frac{n(n-1)}{2} - 2$ edges outside P in G. Then each vertex in $V(G) - V(P)$ is connected with at most three vertices of P. If a vertex $v \in V(G) - V(P)$ is connected with more than three vertices of P, then the distance between the end-vertices of P is less than $m+1$, a contradiction with the fact that $D(G) = m + 1$. If a vertex $v \in V(G) - V(P)$ is connected with three vertices of P, then these vertices must be consecutive. Thus, by pigeonhole principle and the number of edges of G, there exists a vertex in $V(G) - V(P)$ connected with more than three vertices of P and so the length of P is equal to m, a contradiction. □

The next theorem, is a complete solution of Conjecture 2.1 in [10].

Theorem 2.2. *Let $d_{n,m} \geqslant 3$. Then $E_{n,m}$ is the graph with maximal eccentric connectivity index among all connected graph with n vertices and m edges.*

Proof. By Proposition 2.1, it is not difficult to see that $D(G) \leqslant D(E_{n,m})$ for each graph G with n vertices and m edges. On the other hand, note that each vertex of high degree in $E_{n,m}$ has high eccentricity, which completes the proof. □

Let Γ_n be isomorphic to $((K_{[\frac{2n}{3}]+1} - e) \cdot P_{n-[\frac{2n}{3}]})(u,v)$ that u and e are a vertex and an edge of $K_{[\frac{2n}{3}]+1}$, respectively, and v is a pendant of $P_{n-[\frac{2n}{3}]}$. So, it is not difficult to check that:

Proposition 2.3. *For every $n \geqslant 7$,*

$$\xi^c(\Gamma_n) = \xi^c(P_{n-[\frac{2n}{3}]+2}) - \big[\frac{2n}{3}\big]^3 + \big[\frac{2n}{3}\big]^2(n-1) + \big[\frac{2n}{3}\big](n+6) - 6n$$

where

$$\xi^c(P_{n-[\frac{2n}{3}]+2}) \quad = \quad \begin{cases} \frac{3}{2}(n-\left[\frac{2n}{3}\right]+2)^2 - 3(n-\left[\frac{2n}{3}\right]+2)+2 & if\ 2 \mid n-\left[\frac{2n}{3}\right] \\ \frac{3}{2}(n-\left[\frac{2n}{3}\right]+1)^2 & if\ 2 \nmid n-\left[\frac{2n}{3}\right] \end{cases}.$$

Theorem 2.4. *Let G be a graph with n vertices. Then $\xi^c(G) \leqslant \xi^c(\Gamma_n)$.*

Proof. Suppose u is a vertex of K_x and K_x^i is a graph obtained from K_x by deleting i edges incident with the vertex u. So, if v is a pendant of P_{n-x+1}, then we have

$$\xi^c(K_x^i \cdot P_{n-x+1})(u,v) = 2n - 4x(n+1) + nx^2 + \frac{3}{2}n^2 + \frac{7}{2}(x^2+1) - x^3 - i(3-3x+2n).$$

Therefore, by above relation, it is clear that $\xi^c(K_x \cdot P_{n-x+1})(u,v) \leqslant \xi^c(K_x^1 \cdot P_{n-x+1})(u,v)$ and $\xi^c(K_x^i \cdot P_{n-x+1})(u,v) \leqslant \xi^c(K_x \cdot P_{n-x+1})(u,v)$ for $i > 1$. Thus, the maximum value of the function $f(x) = -x(4n+1) + nx^2 + \frac{3}{2}n^2 + \frac{7}{2}x^2 - x^3 + \frac{1}{2}$ is equal to the maximal eccentric connectivity index, which completes the proof. □

Let G be a connected graph on n vertices. G is called tricyclic if it has $n+2$ edges, and tetracyclic if G has exactly $n+3$ edges. Suppose $\mathcal{C}_n$ and $\mathcal{D}_n$ denote the set of all tricyclic and tetracyclic $n-$vertex graphs, respectively. This part of our paper is a continuation of [10] that the authors computed the maximal eccentric connectivity index in the classes of tricyclic and tetracyclic $n-$vertex graphs. In what follows, we compute the minimal eccentric connectivity index in these classes. We recall that the complement of a graph G is a graph $\bar{G}$ on the same vertices such that two vertices of $\bar{G}$ are adjacent if and only if they are not adjacent in G. We will use $S_n + 3e$ and $S_n + 4e$ to denote the graphs obtained by inserting three and four arbitrary edges of $\bar{S_n}$ to S_n, respectively.

Theorem 2.5. *Suppose $G \in \mathcal{C}_n$, $n \geq 6$. Then $\xi^c(S_n + 3e) \leqslant \xi^c(G)$, with equality if and only if $G \cong S_n + 3e$.*

Proof. Let x be the number of vertices of degree $n-1$ in G. Then $\xi^c(G) \geqslant 4n+8-x(n-1)$, with equality if and only if each vertex of degree less than $n-1$ has eccentric connectivity 2. On the other hand, x is equal to 0 or 1. If $x = 0$, then $\xi^c(G) \geqslant 4n+8$ and if $x = 1$, then $\xi^c(G) \geqslant 3n+9$. So, a graph with this property that one vertex has degree $n-1$ and all other vertices has eccentricity 2, has minimum eccentric connectivity index. In other words, $S_n + 3e$ has minimum eccentric connectivity. □

Using similar arguments as Theorem 2.5, one can prove the following result:

Theorem 2.6. *Let $G \in \mathcal{D}_n$, $n \geq 6$. Then $\xi^c(S_n+4e) \leqslant \xi^c(G)$, with equality if and only if $G \cong S_n+4e$.*

Acknowledgments

The research of the third author is partially supported by the University of Kashan under grant no 159020/11.

References

[1] D. B. West, *Introduction to Graph Theory*, Prentice Hall, Inc., Upper Saddle River, NJ, 1996.

[2] V. Sharma, R. Goswami and A. K. Madan, Eccentric connectivity index: a novel highly discriminating topological descriptor for structure property and structure activity studies, *J. Chem. Inf. Comput. Sci.*, **37** (1997) 273–282.

[3] A. R. Ashrafi, T. Došlić and M. Saheli, The eccentric connectivity index of $TUC_4C_8(R)$ nanotubes, *MATCH Commun. Math. Comput. Chem.*, **65** no. 1 (2011) 221–230.

[4] A. R. Ashrafi, M. Saheli and M. Ghorbani, The eccentric connectivity index of nanotubes and nanotori, *J. Comput. Appl. Math.*, **235** (2011) 4561–4566.

[5] G. Yu, L. Feng and A. Ilić, On the eccentric distance sum of trees and unicyclic graphs, *J. Math. Anal. Appl.*, **375** (2011) 99–107.

[6] B. Zhou and Z. Du, On eccentric connectivity index, *MATCH Commun. Math. Comput. Chem.*, **63** (2010) 181–198.

[7] A. Ilić and I. Gutman, Eccentric connectivity index of chemical trees, *MATCH Commun. Math. Comput. Chem.*, **65** (2011) 731–744.

[8] M. J. Morgan, S. Mukwembi and H. C. Swart, On the eccentric connectivity index of a graph, *Discrete Math.*, **311** (2011) 1229–1234.

[9] T. Došlić, M. Saheli and D. Vukičević, Eccentric Connectivity Index: Extremal Graphs and Values, *Iranian J. Math. Chem.*, **1** no. 2 (2010) 45–56.

[10] J. Zhang, Z. Liu and B. Zhou, On the maximal eccentric connectivity indices of graphs, *Appl. Math. J. Chinese Univ.*, in press.

M. Tavakoli

Department of Mathematics, Ferdowsi University of Mashhad, P. O. Box 1159, Mashhad 91775, I. R. Iran

Email: Mostafa.Tavakoli@stu-mail.um.ac.ir

F. Rahbarnia

Department of Mathematics, Ferdowsi University of Mashhad, P. O. Box 1159, Mashhad 91775, I. R. Iran

M. Mirzavaziri

Department of Mathematics, Ferdowsi University of Mashhad, P. O. Box 1159, Mashhad 91775, I. R. Iran

A. R. Ashrafi

Department of Pure Mathematics, Faculty of Mathematical Sciences, University of Kashan, P. O. Box 51167, Kashan 87317, I. R. Iran

Email: ashrafi@kashanu.ac.ir