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NOTE ON THE SKEW ENERGY OF ORIENTED GRAPHS

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Abstract. The skew energy of oriented graphs is defined as the sum of the norms of all the eigenvalues of the skew adjacency matrix. In this note, we obtain some upper bounds for the skew energy of any oriented graphs, which improve the known upper bound obtained by Adiga et al.

1. **Introduction**

Let G^σ be an oriented graph of G with the orientation σ, which assigns to each edge of G a direction so that the resultant graph G^σ becomes an oriented graph or a directed graph. Then G is called the underlying graph of G^σ. The skew-adjacency matrix of G^σ is the $n \times n$ matrix $S(G^\sigma) = [s_{ij}]$, where $s_{ij} = 1$ and $s_{ij} = -1$ if $\langle v_i, v_j \rangle$ is an arc of G^σ, otherwise $s_{ij} = s_{ji} = 0$. It is easy to see that $S(G^\sigma)$ is a skew-symmetric matrix. Since $S(G^\sigma)$ is skew-symmetric, every eigenvalue of $S(G^\sigma)$ is a pure imaginary number or 0. The skew spectral radius of $S(G^\sigma)$, denoted by $\rho_s(G^\sigma)$, is defined as the spectral radius of $S(G^\sigma)$. Recently, the spectral radii of skew adjacency matrices of oriented graphs have been studied in [2, 3, 5, 6, 7].

The energy of a simple graph G is defined as the sum of the absolute values of all eigenvalues of G. The energy of G has been extensively investigated for a long time [9, 10, 11, 12]. The skew energy of oriented graphs is a generalization for the energy of graphs. The skew energy of G^σ, denoted by $E_s(G^\sigma)$, is defined as the sum of the norms of all the eigenvalues of $S(G^\sigma)$. This concept was introduced by Adiga et al. in [1]. For more details about the skew energy of oriented graphs, see [4, 8, 14] and the references therein.

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In this note, we obtain an upper bound for the skew energy of any oriented graphs in terms of its determinant, the number of vertices and arcs. Then, We derive an upper bound of the skew energy for nonsingular oriented graphs in terms of the order n, arcs m and the maximum degree Δ.

2. Results

In order to obtain bounds for the skew energy of oriented graphs, we need some lemmas.

Lemma 2.1. [8] *Let $\{i\lambda_1, i\lambda_2, \ldots, i\lambda_n\}$ be the skew spectrum of G^σ, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Then (1) $\lambda_j = -\lambda_{n+1-j}$ for all $1 \leq j \leq n$; (2) when n is odd, $\lambda_{(n+1)/2} = 0$ and when n is even, $\lambda_{n/2} \geq 0$; and (3) $\sum_{j=1}^{n} \lambda_j^2 = 2m$.*

Lemma 2.2. [13] *Let $a_1, a_2, \ldots, a_n$ be non-negative numbers. Then*

$$n\left[\frac{1}{n}\sum_{j=1}^{n} a_j - \left(\prod_{j=1}^{n} a_j\right)^{1/n}\right] \leq n\sum_{j=1}^{n} a_j - \left(\sum_{j=1}^{n}\sqrt{a_j}\right)^2$$

$$\leq n(n-1)\left[\frac{1}{n}\sum_{j=1}^{n} a_j - \left(\prod_{j=1}^{n} a_j\right)^{1/n}\right]. \tag{2.1}$$

In [1] the following results for $E_S(G^\sigma)$ were obtained:

$$\sqrt{2m + n(n-1)\left(\det(S)\right)^{2/n}} \leq E_S(G^\sigma) \leq \sqrt{2mn}. \tag{2.2}$$

Theorem 2.3. *Let G^σ be an oriented graph with n vertices, m arcs and skew adjacency matrix S. Then*

$$\sqrt{2m + n(n-1)\left(\det(S)\right)^{2/n}} \leq E_S(G^\sigma) \leq \sqrt{2m(n-1) + n\left(\det(S)\right)^{2/n}}. \tag{2.3}$$

Proof. Let $a_j = \lambda_j^2$, $j = 1, 2, \ldots, n$. Then by Lemma 2.1 and Lemma 2.2 we obtain

$$K \leq n\sum_{j=1}^{n}\lambda_j^2 - \left(\sum_{j=1}^{n}|\lambda_j|\right)^2 \leq (n-1)K,$$

that is,

$$K \leq 2mn - E_S(G^\sigma)^2 \leq (n-1)K,$$

where

$$K = n\left[\frac{1}{n}\sum_{j=1}^{n}\lambda_j^2 - \left(\prod_{j=1}^{n}\lambda_j^2\right)^{1/n}\right]$$

$$= n\left[\frac{2m}{n} - \left(\prod_{j=1}^{n}|\lambda_j|\right)^{2/n}\right]$$

$$= 2m - n\left(\det(S)\right)^{2/n}. \tag{2.4}$$

Hence we get the result. □

Remark 2.4. *The lower bound in (3) coincides the lower bound in (2). The upper bound in (3) is always better than the upper bound in (2). This is because by using arithmetic-geometric mean inequality, we have*

$$n\left(\det(S)\right)^{2/n} \leq 2m,$$

Therefore,

$$\sqrt{2m(n-1) + n\left(\det(S)\right)^{2/n}} \leq \sqrt{2mn}.$$

Based on Lemma 2.1, we establish a new upper bound of $E_S(G^\sigma)$.

Theorem 2.5. *Let G^σ be an oriented graph with n vertices, m arcs and skew adjacency matrix S. Then*

$$\sqrt{4m + n(n-2)\left(\det(S)\right)^{2/n}} \leq E_S(G^\sigma) \leq \sqrt{2m(n-2) + 2n\left(\det(S)\right)^{2/n}}. \tag{2.5}$$

Proof. By Lemma 2.1, we have

$$E_S(G^\sigma) = 2\left(\sum_{j=1}^{\lfloor n/2 \rfloor} |\lambda_j|\right).$$

If n is odd, $\det(S) = 0$ and $E_S(G^\sigma) \leq \sqrt{2m(n-2)}$.

If n is even, and let $a_j = \lambda_j^2$, $j = 1, 2, \ldots, n$. Then by Lemma 2.1 and Lemma 2.2 we obtain

$$K \leq \frac{n}{2}\sum_{j=1}^{n/2}\lambda_j^2 - \left(\sum_{j=1}^{n/2}|\lambda_j|\right)^2 \leq (\frac{n}{2}-1)K,$$

that is,

$$K \leq \frac{nm}{2} - \frac{1}{4}E_S(G^\sigma)^2 \leq (\frac{n}{2}-1)K,$$

where

$$\begin{aligned} K &= n/2\left[\frac{1}{n/2}\sum_{j=1}^{n/2}\lambda_j^2 - \left(\prod_{j=1}^{n/2}\lambda_j^2\right)^{2/(n/2)}\right] \\ &= n\left[\frac{m}{n} - \left(\prod_{j=1}^{n/2}|\lambda_j\right)^{2/(n/2)}\right] \\ &= m - n/2\left(\det(S)\right)^{2/n}. \qquad (2.6) \end{aligned}$$

Hence we get the result. □

Remark 2.6. *The lower bound in (5) coincides the lower bound in Theorem 4.1* [8]. *The upper bound in (5) is always better than the upper bound in (3). This is because by using arithmetic-geometric mean inequality, we have*

$$n\left(\det(S)\right)^{2/n} \leq 2m,$$

Therefore,

$$\sqrt{2m(n-2)+2n\left(\det(S)\right)^{2/n}} \leq \sqrt{2m(n-1)+n\left(\det(S)\right)^{2/n}}.$$

We next derive an upper bound of the skew energy for nonsingular oriented graphs in terms of the order n, arcs m and the maximum degree Δ.

Theorem 2.7. *Let G^σ be an oriented graph with n vertices, m arcs and skew adjacency matrix S. Then*

$$E_S(G^\sigma) \leq 2\sqrt{\Delta} + \sqrt{(n-2)(2m-2\Delta)}, \tag{2.7}$$

and equality holds if and only if $\lambda_1 = \sqrt{\Delta}$ and $\lambda_2 = \cdots = \lambda_{n/2}$.

Proof. Using Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} E_S(G^\sigma) &= \sum_{j=1}^{n} |\lambda_j| = 2\lambda_1 + \sum_{j=2}^{n-1} |\lambda_j| \leq 2\lambda_1 + \sqrt{(n-2)\sum_{j=2}^{n-1}\lambda_j^2} \\ &\leq 2\lambda_1 + \sqrt{(n-2)(2m-2\lambda_1)}. \end{aligned} \tag{2.8}$$

Define a function $f(x) = 2x + \sqrt{(n-2)(a-2x^2)}$, it is easy to see that the function $f(x)$ is monotonously decreasing in $x \geq \sqrt{\frac{2m}{n}}$. By Corollary 2.3 in [8], we have $\lambda_1 \geq \sqrt{\Delta} \geq \sqrt{\frac{2m}{n}}$, then

$$f(\lambda_1) \leq f(\sqrt{\Delta}).$$

which implies that the inequality (7) holds, and equality holds if and only if $\lambda_1 = \sqrt{\Delta}$ and $\lambda_2 = \cdots = \lambda_{n/2}$. Now the proof is complete. □

It is known [5] that all oriented trees T^σ of any undirected tree T have the same skew spectra which are equal to i times the spectrum of T. Therefore, for any oriented tree T^σ of T, $\rho_s(T^\sigma) = \rho(T) \geq \overline{d}$, where where $\overline{d}$ is the average degree of T.

Theorem 2.8. *Let T^σ be an oriented tree with n vertices, m arcs and skew adjacency matrix S. Then*

$$E_S(T^\sigma) \leq 2\overline{d} + \sqrt{(n-2)(2m-2\overline{d}^2)}, \tag{2.9}$$

and equality holds if and only if $\lambda_1 = \overline{d}$ and $\lambda_2 = \cdots = \lambda_{n/2}$.

Proof. The proof is similar to the proof in Theorem 2.3. □

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