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TOTAL DOMINATOR CHROMATIC NUMBER OF A GRAPH

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Abstract. Given a graph G, the total dominator coloring problem seeks a proper coloring of G with the additional property that every vertex in the graph is adjacent to all vertices of a color class. We seek to minimize the number of color classes. We initiate to study this problem on several classes of graphs, as well as finding general bounds and characterizations. We also compare the total dominator chromatic number of a graph with the chromatic number and the total domination number of it.

1. Introduction

All graphs considered here are finite, undirected and simple. For standard graph theory terminology not given here we refer to [9]. Let $G = (V, E)$ be a graph with the *vertex set* V of *order* $n(G)$ and the *edge set* E of *size* $m(G)$. The *open neighborhood* and the *closed neighborhood* of a vertex $v \in V$ are $N_G(v) = \{u \in V \mid uv \in E\}$ and $N_G[v] = N_G(v) \cup \{v\}$, respectively. The *degree* of a vertex v is also $deg_G(v) =\mid N_G(v) \mid$. The *minimum* and *maximum degree* of G are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. If $\delta(G) = \Delta(G) = k$, then G is called *k-regular*. We say that a graph is *connected* if there is a path between every two vertices of the graph, and otherwise is called *disconnected*. We write K_n, C_n and P_n for a *complete graph*, a *cycle* and a *path* of order n, respectively, while $G[S]$, W_n and $K_{n_1,n_2,\ldots,n_p}$ denote the *subgraph induced* of G by a vertex set S of G, a *wheel* of order $n+1$, and a *complete p-partite graph*, respectively. The *complement* of a graph G is denoted by $\overline{G}$ and is a graph with the vertex set $V(G)$ and for every two vertices v and w, $vw \in E(\overline{G})$ if and only if $vw \notin E(G)$.

A *dominating set*, briefly DS, S of a graph G is a subset of the vertices in G such that for each vertex v, $N_G[v] \cap S \neq \emptyset$. The *domination number* $\gamma(G)$ of G is the cardinality of a minimum dominating set. The topics has long been of interest to researchers [6, 7].

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Similarly, a *total dominating set*, briefly TDS, S of a graph G is a subset of the vertices in G such that for each vertex v, $N_G(v) \cap S \neq \emptyset$. The *total domination number* $\gamma_t(G)$ of G is the cardinality of a minimum total dominating set. The literature on the subject on total domination in graphs has been surveyed and detailed in the recent book [8]. A vertex v in G *totally dominates* a subset X of vertices in G, written $v \succ_t X$, if $X \subseteq N(v)$; that is, if v is adjacent to every vertex in X. Also $v \not\succ_t X$ means that the vertex v does not totally dominate X.

A *proper coloring* of a graph $G = (V, E)$ is a function from the vertices of the graph to a set of colors such that any two adjacent vertices have different colors. The *chromatic number* $\chi(G)$ of G is the minimum number of colors needed in a proper coloring of a graph. Graph coloring is used as a model for a vast number of practical problems involving allocation of scarce resources (e.g., scheduling problems), and has played a key role in the development of graph theory and, more generally, discrete mathematics and combinatorial optimization. Graph k-colorability is NP-complete in the general case, although the problem is solvable in polynomial time for many classes [2].

In a proper coloring of a graph, a *color class* of the coloring is a set consisting of all those vertices assigned the same color. If f is a proper coloring of G with the coloring classes $V_1, V_2, \ldots, V_\ell$ such that every vertex in V_i has color i, we write simply $f = (V_1, V_2, \ldots, V_\ell)$.

A *dominator coloring*, briefly DC, of a graph G is a proper coloring of G such that every vertex of $V(G)$ dominates all vertices of at least one color class (possibly its own class). The *dominator chromatic number* $\chi_d(G)$ of G is the minimum number of color classes in a dominator coloring of G. As a consequence result we have $\chi(G) \leq \chi_d(G)$. The concept of dominator coloring was introduced and studied by Gera, Horton and Rasmussen [5] and studied further, for example, by Gera [3, 4] and Chellali and Maffray [1].

In this paper, we introduce and study a new graph coloring concept, called total dominator colorings. More exactly, we study the total dominator chromatic number on several classes of graphs, as well as finding general bounds and characterizations. We also compare the total dominator chromatic number of a graph with the chromatic number and the total domination number of it. First, we define it and present some needed definitions and terminologies.

Definition 1.1. A *total dominator coloring* , briefly TDC, of a graph G is a proper coloring of G in which each vertex of the graph is adjacent to every vertex of some color class. The *total dominator chromatic number* $\chi_d^t(G)$ of G is the minimum number of color classes in a TDC of G. A $\chi_d^t(G)$*-coloring* of G is any total dominator coloring of G with $\chi_d^t(G)$ colors.

Definition 1.2. Let $f = (V_1, V_2, \ldots, V_\ell)$ be a TDC of G. A vertex v is called a *common neighbor* of V_i if $v \succ_t V_i$. The set of all common neighbors of V_i is called the *common neighborhood* of V_i in G and denoted by $CN_G(V_i)$ or simply by $CN(V_i)$.

Definition 1.3. Let $f = (V_1, V_2, \ldots, V_\ell)$ be a TDC of G. A vertex v is called the *private neighbor of* V_i *with respect to* f if $v \succ_t V_i$ and $v \not\succ_t V_j$ for all $j \neq i$. The set of all private neighbors of V_i is called the *private neighborhood* of V_i in G and denoted by $pn_G(V_i; f)$ or simply by $pn(V_i; f)$.

The following proposition can easily be proved by Definitions 1.1 and 1.2.

Proposition 1.4. *Let $f = (V_1, V_2, \ldots, V_\ell)$ be a TDC of a graph G, and let $I = \{i \mid |V_i| \leq \Delta(G)\}$. Then $V(G) = \cup_{i \in I} CN_G(V_i)$.*

2. Complexity

In this section, we formally establish the difficulty of finding the total dominator chromatic number of an arbitrary graph. First, we define some relevant decision problems.

CHROMATIC NUMBER Given a graph G and a positive integer k, does there exist a function $f : V(G) \to \{1, 2, \ldots, k\}$ such that $f(u) \neq f(v)$ whenever $uv \in E(G)$?

TOTAL DOMINATOR CHROMATIC NUMBER Given a graph G and a positive integer k, does there exist a function $f : V(G) \to \{1, 2, \ldots, k\}$ such that $f(u) \neq f(v)$ whenever $uv \in E(G)$ and for any vertex $x \in V(G)$ there exists a color class V_i such that $x \succ_t V_i$?

Theorem 2.1. TOTAL DOMINATOR CHROMATIC NUMBER *is NP-complete.*

Proof. TOTAL DOMINATOR CHROMATIC NUMBER is clearly in NP, since we can efficiently verify that an assignment of colors to the vertices of G is both a proper coloring and that every vertex v dominates some color class other than the color class of v.

Now we transform CHROMATIC NUMBER to TOTAL DOMINATOR CHROMATIC NUMBER. Consider an arbitrary instance (G, k) of CHROMATIC NUMBER. Create an instance (G', k') of TOTAL DOMINATOR CHROMATIC NUMBER as follows. Add a vertex v' to G and add an edge from v' to every vertex in G. Set $k' \to k + 1$.

Suppose G has a proper coloring using k colors. Then the coloring of G' that colors v' with a new color is a proper coloring of G'. Since $v' \in N(u)$ for every $u \in V(G)$ and for any color i other than the color of v, $\{u \in V(G) | f(u) = i\} \subseteq N(v')$, this coloring is a TDC of G', and is uses $k' = k + 1$ colors.

Now suppose G' has a TDC using k' colors. Since v' is adjacent to every other vertex in G', it must be the only vertex of its color in the hypothesized coloring. Then the removal of v' leaves a proper coloring of G that uses $k' - 1 = k$ colors. □

3. Some bounds

In this section, we will present some sharp lower and upper bounds for the total dominator chromatic number of a graph. First, we state the following observation.

Observation 3.1. *Let G be a graph of order n and without isolated vertices. Then*

$$\max\{\chi_d(G), \gamma_t(G)\} \leq \chi_d^t(G) \leq n.$$

The next theorem gives some lower and upper bounds for the total dominator chromatic number of a graph in terms of the total dominator chromatic numbers of its connected components.

Theorem 3.2. *Let G be a graph without isolated vertices. If $G_1, G_2, \ldots, G_\omega$ are all connected components of G, then*

$$\max_{1\le i\le \omega} \chi_d^t(G_i) + 2\omega - 2 \le \chi_d^t(G) \le \Sigma_{i=1}^{\omega}\chi_d^t(G_i).$$

Proof. For $1 \le i \le \omega$, let f_i be a χ_d^t-coloring of G_i. Let f be a function on $V(G)$ such that for any vertex $v \in V(G_i)$, $f(v) = (i, f_i(v))$. Then f is a TDC of G, implying that $\chi_d^t(G) \le \Sigma_{i=1}^{\omega}\chi_d^t(G_i)$.

Now let $\chi_d^t(G_j) = \max_{1\le i\le \omega} \chi_d^t(G_i)$, for some $1 \le j \le \omega$. Since for any $i \ne j$ at least two new colors are required to color the vertices of G_i, we obtain

$$\chi_d^t(G) \ge \max_{1\le i\le \omega} \chi_d^t(G_i) + 2\omega - 2.$$

□

If we look carefully at the proof of Theorem 3.2, we obtain following.

Remark 3.3. *For any graph G which has no isolated vertex and $G_1, G_2, \ldots, G_\omega$ are all connected components of it, $\chi_d^t(G) = \max_{1\le i\le \omega} \chi_d^t(G_i) + 2\omega - 2$ if and only if at most one connected component of G is not complete bipartite graph.*

In the remained of our paper, we assume that G is a connected graph. The next theorem present the lower bound 2 and the upper bound n for the total dominator chromatic number of a connected graph of order n which has no isolated vertex.

Theorem 3.4. *If G is a connected graph of order n and without isolated vertices, then $2 \le \chi_d^t(G) \le n$. Furthermore, $\chi_d^t(G)$ is 2 or n if and only if G is a complete bipartite graph, or the complete graph K_n, respectively.*

Proof. Observation 3.1 implies that $\chi_d^t(G) \ge \gamma_t(G)$, and since the total domination number of any graph is at least 2, we obtain $2 \le \chi_d^t(G) \le n$.

If G is a complete bipartite graph or the complete graph K_n, then, obviously, $\chi_d^t(G) = 2$ or $\chi_d^t(G) = n$, respectively. Now let $\chi_d^t(G) = 2$, and let $f = (V_1, V_2)$ be a $\chi_d^t(G)$-coloring, where $V_i = \{v \in V(G) \mid f(v) = i\}$ for $i = 1, 2$. Then G is the complete bipartite graph with the vertex partition $V(G) = V_1 \cup V_2$ to the independent sets V_1 and V_2.

In the second case, we assume that $G \ne K_n$, and $\chi_d^t(G) = n$. Let f be a $\chi_d^t(G)$-coloring. Without loss of generality, we may assume that $n \ge 3$. If $deg_G(x) = 1$ for some vertex x, let α be an arbitrary element in $\{1, 2, 3, \ldots, n\} - \{f(x)\}$. Define a function g on $V(G)$ such that for any vertex v,

$$g(v) = \begin{cases} f(v) & \text{if } v \ne x, \\ \alpha & \text{if } v = x. \end{cases}$$

Then g is a TDC of G with $n-1$ color classes, implying that $\chi_d^t(G) < n$, a contradiction. Therefore $\delta(G) \ge 2$. Now let u and u' be two non-adjacent vertices in G. Define a function h on $V(G)$ such that for any vertex v,

$$h(v) = \begin{cases} f(v) & \text{if } v \ne u, \\ f(u') & \text{if } v = u. \end{cases}$$

Then h is a TDC of G with $n-1$ color classes, implying that $\chi_d^t(G) < n$, a contradiction. Therefore, $G = K_n$. □

Let S be an independent vertex set in a graph $G=(V,E)$ such that the induced subgraph $G[V-S]$ has no isolated vertex or every isolated vertex in it is adjacent to all vertices in S. Let $\alpha_0(G)$ be the maximum cardinality of such a set in G. With this definition and notation we state following.

Theorem 3.5. *Let G be a connected graph of order n and without isolated vertices. Then*

$$\chi_d^t(G) \leq n+1-\alpha_0(G).$$

Proof. Let S be an independent vertex set in G such that the induced subgraph $G[V(G)-S]$ has no isolated vertex or every isolated vertex in it is adjacent to all vertices of S and $\mid S \mid = \alpha_0(G)$. We assign $n-\alpha_0(G)$ colors to $n-\alpha_0(G)$ vertices in $G[V(G)-S]$, and then assign $(n-\alpha_0(G)+1)$-th color to all vertices in S. This is a TDC of G, and so $\chi_d^t(G) \leq n+1-\alpha_0(G)$. □

Corollary 3.6. *Let G be a connected k-regular graph of order n and without isolated vertices. If the independence number of G is k, then*

$$\chi_d^t(G) \leq n+1-k.$$

The next theorem present a sharp upper bound for the total dominator chromatic number of a connected graph in terms of its total domination number and the chromatic number of an induced subgraph of it.

Theorem 3.7. *Let G be a connected graph without isolated vertices. Then*

$$\chi_d^t(G) \leq \gamma_t(G) + \min_S \chi(G[V(G)-S]),$$

where $S \subseteq V(G)$ is a $\gamma_t(G)$-set. Also this upper bound is sharp.

Proof. Let $\ell = \min\{\chi(G[V(G)-S]) \mid\ S \text{ is a } \gamma_t(G)\text{-set}\}$, and let $D=\{v_1,v_2,\ldots,v_m\}$ be a $\gamma_t(G)$-set such that $\chi(G[V(G)-D])=\ell$. Let also $f: V(G)-D \to \{1,2,\ldots,\ell\}$ be a proper coloring of $G[V(G)-D]$. We define $g: V(G) \to \{1,2,3,\ldots,\ell+m\}$ such that for any vertex v,

$$g(v) = \begin{cases} \ell+i & \text{if } v=v_i \in D, \\ f(v) & \text{if } v \notin D. \end{cases}$$

Since D is a TDS in G, g will be a TDC of G. Hence

$$\chi_d^t(G) \leq m+\ell = \gamma_t(G) + \min\{\chi(G[V(G)-S]) \mid\ S \text{ is a } \gamma_t(G)\text{-set}\}.$$

This upper bound is sharp for the complete graph K_n of order $n \geq 3$, the complete p-partite graph $K_{1,1,n_1,\ldots,n_{p-2}}$, where $p \geq 3$, and the wheel W_n of order even $n+1 \geq 4$ (for the last, see Proposition 4.1). □

Corollary 3.8. *If G is a connected p-partite graph without isolated vertices, then*

$$\chi_d^t(G) \leq \gamma_t(G) + p.$$

The next result gives another upper bound on the total dominator chromatic number of a connected p-partite graph.

Theorem 3.9. *Let G be a connected p-partite graph of order n which has the numbers $n_1, n_2, \ldots, n_p$ as the cardinalities of its independent sets. If $\delta(G) \geq n_i$, for some i, then $\chi_d^t(G) \leq n - n' + 1$, where $n' = \max\{n_i | \delta(G) \geq n_i\}$.*

Proof. Let G be a connected p-partite graph of order n that is partitioned to the independent sets $V_1, \ldots, V_p$ of the cardinalities $n_1, \ldots, n_p$, respectively. Let $n' = n_i$, for some i. Then the coloring that assigns colors $1, 2, \ldots, n - n_i$ to the vertices of $V(G) - V_i$, and color $n - n_i + 1$ to the vertices of V_i, is a TDC of G. Hence $\chi_d^t(G) \leq n - n' + 1$. □

Note that if a graph G has a χ_d^t-coloring f without singleton color class, then f is also a dominator coloring of G, and hence $\chi_d^t(G) = \chi_d(G)$. The next proposition shows that this condition is not necessary for $\chi_d^t(G) = \chi_d(G)$.

Proposition 3.10. *Let G be a connected graph of order n and without isolated vertices. If $\Delta(G) = n - 1$, then $\chi_d^t(G) = \chi_d(G) = \chi(G)$.*

Proof. Let $f = (V_1, V_2, \ldots, V_m)$ be a proper coloring of G, where $m = \chi(G)$, and $V_1 = \{v\}$ for some vertex v of degree $n - 1$. Then $w \succ_t V_1$ for each vertex $w \in V(G) - V_1$. Also for each $2 \leq i \leq m$, $v \succ_t V_i$. Therefore f is a TDC of G with $\chi(G)$ color classes, implying that $\chi_d^t(G) \leq \chi(G)$. Now Observation 3.1 implies that $\chi_d^t(G) = \chi_d(G) = \chi(G)$. □

Corollary 3.11. *Let G be a connected graph of order n and without isolated vertices. If $\Delta(G) = n-1$ and $v_1, \ldots, v_\ell$ be all vertices of degree $n - 1$, then*

$$\chi_d^t(G) = \ell + \chi(G[V - \{v_1, \ldots, v_\ell\}]).$$

4. The total dominator chromatic number of some graphs

Obviously, the total dominator chromatic number of every complete p-partite graph is p. In this section, we calculate this number for some other classes of graphs.

Proposition 4.1. *Let W_n be a wheel of order $n + 1 \geq 4$. Then*

$$\chi_d^t(W_n) = \begin{cases} 3 & \textit{if } n \textit{ is even}, \\ 4 & \textit{if } n \textit{ is odd}. \end{cases}$$

Proof. As a consequence of Corollary 3.11, we have

$$\begin{aligned} \chi_d^t(W_n) &= 1 + \chi(C_n) \\ &= \begin{cases} 3 & \text{if } n \text{ is even}, \\ 4 & \text{if } n \text{ is odd}. \end{cases} \end{aligned}$$

□

Note that $\chi_d^t(W_n) = \chi_d(W_n)$, by [3].

Proposition 4.2. *Let C_n be a cycle of order $n \geq 3$. Then*

$$\chi_d^t(C_n) = \begin{cases} 2 & \textit{if } n = 4, \\ 4\lfloor \frac{n}{6} \rfloor + r & \textit{if } n \neq 4 \textit{ and for } r = 0,1,2,4,\ n \equiv r \pmod 6, \\ 4\lfloor \frac{n}{6} \rfloor + r - 1 & \textit{if } \ n \equiv r \pmod 6, \textit{ where } r = 3,5. \end{cases}$$

Proof. Let $V(C_n) = \{v_i \mid 1 \leq i \leq n\}$, and let $v_iv_j \in E(C_n)$ if and only if $j \equiv i+1 \pmod n$. We claim that for every TDC f of C_n, at least four colors are required to color every six consecutive vertices. Let v_i, v_{i+1}, v_{i+2}, v_{i+3}, v_{i+4}, v_{i+5} be six consecutive vertices. Then some color, say a, must be assigned to at least two vertices. We assign colors a, b, a to the vertices v_i, v_{i+1}, v_{i+2}, respectively. We can assign color b to the vertex v_{i+3} or not. In each case, at least two new colors, say c and d, are required to color the remained vertices. Because, by the new colors c and d, we have to assign the color c to the vertex v_{i+4} and the color d to the vertex v_{i+5} in the first case, and the colors c, d, c to the vertices v_{i+3}, v_{i+4}, v_{i+5}, respectively, in the second case. Therefore, our claim is proved. Note that any six consecutive vertices can be colored by four new colors a, b, c, d in

way 1: a,b,a,b,c,d, or *way 2: a,b,a,c,d,c.*

In *way* 1, we have

$$v_{i+1} \in pn(V_a; f), \quad v_{i+2} \in pn(V_b; f), \quad v_{i+3} \in pn(V_c; f), \quad v_{i+4} \in pn(V_d; f),$$

while in *way* 2 we have

$$v_{i+1} \in pn(V_a; f), \quad v_{i+2} \in pn(V_b; f), \quad v_{i+4} \in pn(V_c; f), \quad v_{i+3} \in pn(V_d; f).$$

We complete our proof in the following six cases.

Case 0: $n \equiv 0 \pmod 6$. Let f_0 be a proper coloring that is obtained by mixing each of the ways 1 or 2. Then f_0 is a TDC of C_n with the minimum number $4\lfloor \frac{n}{6} \rfloor$ color classes, as desired.

Case 1: $n \equiv 1 \pmod 6$. Let f_0 be the TDC of $C_n - \{v_n\}$ given in Case 0. Since one new color are required to color the vertex v_n, by assigning a new color to it, we obtain a TDC of C_n with the minimum number $4\lfloor \frac{n}{6} \rfloor + 1$ color classes, as desired.

Case 2: $n \equiv 2 \pmod 6$. Let f_0 be the TDC of $C_n - \{v_{n-1}, v_n\}$ given in Case 0. Since two new colors are required to color the vertices v_{n-1} and v_n, by assigning two new colors to them, we obtain a TDC of C_n with the minimum number $4\lfloor \frac{n}{6} \rfloor + 2$ color classes, as desired.

Case 3: $n \equiv 3 \pmod 6$. Let f_0 be the TDC of $C_n - \{v_{n-2}, v_{n-1}, v_n\}$ given in Case 0. Since two new colors are required to color the vertices v_{n-2}, v_{n-1} and v_n, by assigning new colors ε, θ, ε to the vertices v_{n-2}, v_{n-1}, v_n, respectively, we obtain a TDC of C_n with the minimum number $4\lfloor \frac{n}{6} \rfloor + 2$ color classes, as desired.

Case 4: $n \equiv 4 \pmod 6$. Let f_0 be the TDC of $C_n - \{v_{n-3}, v_{n-2}, v_{n-1}, v_n\}$ given in Case 0. Since four new colors are required to color the vertices v_{n-3}, v_{n-2}, v_{n-1} and v_n, by assigning new four colors to them, we obtain a TDC of C_n with the minimum number $4\lfloor \frac{n}{6} \rfloor + 4$ color classes, as desired.

Case 5: $n \equiv 5 \pmod 6$. Let f_0 be the TDC of $C_n - \{v_{n-4}, v_{n-3}, v_{n-2}, v_{n-1}, v_n\}$ given in Case 0. Since four new colors are required to color the vertices v_{n-4}, v_{n-3}, v_{n-2}, v_{n-1}, v_n, by assigning new colors π, ς, π, θ, ε to the vertices v_{n-4}, v_{n-3}, v_{n-2}, v_{n-1} and v_n, respectively, we obtain a TDC of C_n with the minimum number $4\lfloor \frac{n}{6} \rfloor + 4$ color classes, as desired. □

Proposition 4.3. *Let P_n be a path of order $n \geq 2$. Then*

$$\chi_d^t(P_n) = \begin{cases} 2\lceil \frac{n}{3} \rceil - 1 & \text{if } n \equiv 1 \pmod 3, \\ 2\lceil \frac{n}{3} \rceil & \text{otherwise.} \end{cases}$$

Proof. Let $V(P_n) = \{v_i \mid 1 \leq i \leq n\}$ and let $v_i v_j \in E(C_n)$ if and only if $j = i+1$. Let $f = (V_1, V_2, \ldots, V_\ell)$ be an arbitrary TDC of P_n. We see that at least two, three or four colors are required to color every three, four or five consecutive vertices, respectively. Because every vertex v_i has degree two if $1 < i < n$ and has degree one otherwise. Therefore either $V_j = \{v_{i-1}, v_{i+1}\}$ for some $1 \leq j \leq \ell$, or $v_{i-1} \in V_j$ and $v_{i+1} \in V_k$ for some $1 \leq j < k \leq \ell$ such that $|V_j| = 1$ or $|V_k| = 1$. This implies that $V(P_n)$ has partitioned to the subsets of three consecutive vertices which are assigned colors a, b, a to them, or to the subsets of four consecutive vertices which are assigned colors a, b, c, a to them, or to the subsets of five consecutive vertices which are assigned either colors a, b, a, c, d, or colors a, b, c, d, a to them, respectively. (notice that the colors used in each step are different).

If $n \equiv 0 \pmod 3$, define a function f_0 on $V(P_n)$ such that for any vertex v_i,

$$f_0(v_i) = \begin{cases} 1 + 2k & \text{if } i = 1 + 3k \text{ or } i = 3 + 3k, \\ 2 + 2k & \text{if } i = 2 + 3k, \end{cases}$$

when $0 \leq k \leq \frac{n}{3} - 1$. Then f_0 is a TDC of P_n with the minimum number $2\lceil \frac{n}{3} \rceil$ color classes, as desired.

If $n \equiv 1 \pmod 3$, define a function f_1 on $V(P_n)$ such that for any vertex v_i,

$$f_1(v_i) = \begin{cases} 1 + 2k & \text{if } i = 1 + 3k \text{ or } i = 3 + 3k, \\ 2 + 2k & \text{if } i = 2 + 3k, \end{cases}$$

when $0 \leq k \leq \lfloor \frac{n}{3} \rfloor - 2$, and

$$f_1(v_{n-3}) = f_1(v_n) = 2\lfloor \tfrac{n}{3} \rfloor - 1, \quad f_1(v_{n-2}) = 2\lfloor \tfrac{n}{3} \rfloor, \quad f_1(v_{n-1}) = 2\lfloor \tfrac{n}{3} \rfloor + 1.$$

Then f_1 is a TDC of P_n with the minimum number $2\lceil \frac{n}{3} \rceil - 1$ color classes, as desired.

Now let $n \equiv 2 \pmod 3$. If $n = 2$, then $P_2 = K_2$, and $\chi_d^t(P_2) = 2$. Let $n = 5$. In this case, we can assign four colors a, b, c, d to the vertices v_1, v_2, v_3, v_4, v_5 in one of the ways: a, b, a, c, d, or a, b, c, d, a. Hence $\chi_d^t(P_5) = 4$. Now let $n \geq 8$. Define a function f_2 on $V(P_n)$ such that for any vertex v_i,

$$f_2(v_i) = \begin{cases} 1 + 2k & \text{if } i = 1 + 3k \text{ or } i = 3 + 3k, \\ 2 + 2k & \text{if } i = 2 + 3k, \end{cases}$$

when $0 \leq k \leq \lfloor \frac{n}{3} \rfloor - 2$, and

$$f_2(v_{n-4}) = f_2(v_n) = 2\lfloor \tfrac{n}{3} \rfloor - 1, \quad f_2(v_{n-3}) = 2\lfloor \tfrac{n}{3} \rfloor,$$

$$f_2(v_{n-2}) = 2\lfloor \tfrac{n}{3} \rfloor + 1, \quad f_2(v_{n-1}) = 2\lfloor \tfrac{n}{3} \rfloor + 2,$$

Then f_2 is a TDC of P_n with the minimum number $2\lceil \frac{n}{3} \rceil$ color classes, as desired. □

Proposition 4.4. *Let $\overline{C_n}$ be the complement of a cycle C_n of order $n \geq 4$. Then*

$$\chi_d^t(\overline{C_n}) = \begin{cases} 4 & \text{if } n = 4, 5, \\ \lceil \frac{n}{2} \rceil & \text{if } n \geq 6. \end{cases}$$

Proof. Let $V(\overline{C_n}) = \{v_i | 1 \leq i \leq n\}$ and let v_iv_j be an edge if and only if $j \neq i-1, i+1$. If $n = 4, 5$, then $\overline{C_n}$ is isomorphic to $2K_2$ or C_5, respectively, and thus $\chi_d^t(\overline{C_n}) = 4$. Now let $n \geq 6$. Since the independence number of $\overline{C_n}$ is two, for any TDC $f = (V_1, V_2, \ldots, V_\ell)$ of $\overline{C_n}$, we have $|V_i| \leq 2$ for all i. Hence $\chi_d^t(\overline{C_n}) \geq \lceil \frac{n}{2} \rceil$. Now for $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$ let $V_i = \{v_{2i}, v_{2i-1}\}$. Then for even n, $f = (V_1, V_2, \ldots, V_{\lfloor \frac{n}{2} \rfloor})$ is a TDC of $\overline{C_n}$ with $\lceil \frac{n}{2} \rceil$ color classes, while for odd n, $g = (V_1, V_2, \ldots, V_{\lfloor \frac{n}{2} \rfloor}, \{v_n\})$ is a TDC of $\overline{C_n}$ with $\lceil \frac{n}{2} \rceil$ color classes, implying that $\chi_d^t(\overline{C_n}) = \lceil \frac{n}{2} \rceil$. □

Proposition 4.5. *Let $\overline{P_n}$ be the complement of a path P_n of order $n \geq 4$. Then*

$$\chi_d^t(\overline{P_n}) = \begin{cases} 3 & \text{if } n = 4, \\ \lceil \frac{n}{2} \rceil & \text{if } n \geq 5. \end{cases}$$

Proof. Let $V(\overline{P_n}) = \{v_i | 1 \leq i \leq n\}$ and let v_iv_j be an edge if and only if $\{i, j\} = \{1, n\}$ or $j \neq i-1, i+1$. Since $\overline{P_4} = P_4$, Proposition 4.3 implies that $\chi_d^t(\overline{P_n}) = 3$. Now let $n \geq 5$. Since the independence number of $\overline{P_n}$ is two, we obtain $\chi_d^t(\overline{P_n}) \geq \lceil \frac{n}{2} \rceil$. On the other hand, since all the total dominator colorings given in Proposition 4.4 are also total dominator colorings of $\overline{P_n}$ with $\lceil \frac{n}{2} \rceil$ color classes, we obtain $\chi_d^t(\overline{P_n}) = \lceil \frac{n}{2} \rceil$. □

5. **A remark**

By comparing the propositions given in Section 4, we obtain the following results.

Proposition 5.1. *For any $n \geq 3$,*

$$\chi_d^t(P_n) = \begin{cases} \chi_d^t(C_n) + 1 & \text{if } n = 4, \\ \chi_d^t(C_n) - 1 & \text{if } n \equiv 4 \pmod 6 \text{ and } n > 4, \\ \chi_d^t(C_n) & \text{otherwise.} \end{cases}$$

Proposition 5.2. *For any $n \geq 3$,*

$$\begin{array}{ll} \chi_d^t(C_n) < \chi_d^t(W_n) & \text{if } n = 3, 4, \\ \chi_d^t(C_n) = \chi_d^t(W_n) & \text{if } n = 5, \\ \chi_d^t(C_n) > \chi_d^t(W_n) & \text{otherwise.} \end{array}$$

Propositions 5.1 and 5.2 confirm the truth of the next remark.

Remark 5.3. *If H is a subgraph of a graph G, we can not conclude that*

$$\chi_d^t(H) \leq \chi_d^t(G) \quad \text{or} \quad \chi_d^t(H) \geq \chi_d^t(G).$$

6. Trees

In this section, we discuss on the total dominator chromatic number of a tree. First, we present some needed definitions. In a connected graph G the *distance* between two vertices u and v, written $d_G(u,v)$ or simply $d(u,v)$, is the least length of a u,v-path, and the *diameter* of G, written *diam(G)*, is $\max_{u,v\in V(G)} d(u,v)$.

The *eccentricity* of a vertex u, written $\epsilon(u)$, is $\max_{v\in V(G)} d(u,v)$, while the *radius* of G, written *rad(G)*, is $\min_{v\in V(G)} \epsilon(u)$. The *center* of G is the subgraph induced by the vertices of minimum eccentricity.

The following theorem describes the center of trees.

Theorem 6.1. (**Jordan** [9]) *The center of a tree is a vertex or an edge.*

In a tree, a *leaf* is a vertex of degree one, while a *support vertex* is the neighbor of a leaf with degree more than one. The set of leaves in a tree T is denoted by $\mathcal{L}$, and the set of its support vertices is denoted by $\mathcal{S}$, while their cardinalities are denoted by ℓ and s, respectively. Set $\mathcal{S}=\{v_i|1\le i\le s\}$, and $\mathcal{L}=\{u_i|1\le i\le \ell\}$. Also σ denotes a function on $\{1,2,\ldots,s\}$, the set of indices of the members of S, such that $\sigma(i)=j$ if u_i is adjacent to v_j. Thus $v_{\sigma(i)}$ denotes the support vertex of the leaf u_i.

We start our discussion with the following lemma.

Lemma 6.2. *For any tree T of order $n\ge 3$, $\chi_d^t(T)\ge s+1$.*

Proof. Since $N(u_i)=\{v_{\sigma(i)}\}$, we conclude that in every TDC of T, every support vertex of T must be contained in a color class of cardinality one. On the other hand, at least one new color is required to color the leaves of T. Therefore $\chi_d^t(T)\ge s+1$. □

Proposition 6.3. *Let T be a tree of order $n\ge 3$. If $diam(T)\le 3$, then $\chi_d^t(T)=s+1$.*

Proof. The condition $diam(T)\le 3$ implies that for every two leaves u_i and u_j, there exist one of the (u_i,u_j)-paths: $u_iv_{\sigma(i)}v_{\sigma(j)}u_j$ or $u_iv_{\sigma(i)}u_j$. Now this fact that $(\{v_1\},\{v_2\},\ldots,\{v_s\},V(T)-\mathcal{S})$ is a TDC of T and Lemma 6.2 imply that $\chi_d^t(T)=s+1$. □

Proposition 6.4. *Let T be a tree with the center $\mathcal{C}$. If $diam(T)=4$, then*

$$\chi_d^t(T)=\begin{cases} s+1 & \text{if } V(\mathcal{C})\subseteq\mathcal{S},\\ s+2 & \text{if } V(\mathcal{C})\not\subseteq\mathcal{S}.\end{cases}$$

Proof. Let T be a tree with the center $\mathcal{C}$ and $diam(T)=4$. Then Theorem 6.1 implies that the center of T is a vertex. Let $\mathcal{C}=\{w\}$. If $w\in\mathcal{S}$, then $f=(\{v_1\},\{v_2\},\ldots,\{v_s\},V(T)-\mathcal{S})$ is a TDC of T with $s+1$ color classes, implying that $\chi_d^t(T)=s+1$ (by Lemma 6.2).

Now let $w\notin\mathcal{S}$. Then $d(u_i,w)=2$, for any $u_i\in\mathcal{L}$, and for every two leaves u_i and u_j, there exist one of the (u_i,u_j)-paths: $u_iv_{\sigma(i)}wv_{\sigma(j)}u_j$ or $u_iv_{\sigma(i)}u_j$. If $\chi_d^t(T)=s+1$, then $f=(\{v_1\},\{v_2\},\ldots,\{v_s\},V(T)-\mathcal{S})$ is the only TDC of T. But this is not possible, since no support vertex is adjacent to all vertices of a color class. Therefore $\chi_d^t(T)\ge s+2$. Now since $g=(\{v_1\},\{v_2\},\ldots,\{v_s\},\{w\},\mathcal{L})$ is a TDC of T with $s+2$ color classes, we obtain $\chi_d^t(T)=s+2$. □

Proposition 6.5. *Let T be a tree with the center $\mathcal{C}$. If $diam(T) = 5$, then*

$$\chi_d^t(T) = \begin{cases} s+1 & \text{if } |V(\mathcal{C}) \cap \mathcal{S}| = 2 \text{ and } |\mathcal{S}| \geq 3, \\ s+2 & \text{if either } |V(\mathcal{C}) \cap \mathcal{S}| = 1 \text{ and } |\mathcal{S}| \geq 3, \text{ or } |\mathcal{S}| = 2, \\ s+3 & \text{if } |V(\mathcal{C}) \cap \mathcal{S}| = 0 \text{ and } |\mathcal{S}| \geq 3. \end{cases}$$

Proof. Let T be a tree with the center $\mathcal{C}$ and $diam(T) = 5$. Then Theorem 6.1 implies that the center of T is an edge. Let $\mathcal{C} = \{e_1e_2\}$, and let $e_1 \in N(v_1)$ and $e_2 \in N(v_s)$. Assume that $|\mathcal{S}| = 2$. Then $\mathcal{S} = \{v_1, v_2\}$, and $V(T) = \mathcal{L} \cup \mathcal{S} \cup \{e_1, e_2\}$ is a partition of the vertices of T. Lemma 6.2 implies that $\chi_d^t(T) \geq 3$. If $\chi_d^t(T) = 3$, then $f = (\{v_1\}, \{v_2\}, \mathcal{L} \cup \{e_1, e_2\})$ is the only TDC of T. But this is not possible, because e_1e_2 is an edge. Hence $\chi_d^t(T) \geq 4$. Now, since $g = (\{v_1\}, \{v_2\}, N(v_1), N(v_2))$ is a TDC of T with $s+2$ color classes, we obtain $\chi_d^t(T) = 4$.

Now for $|\mathcal{S}| \geq 3$, we continue our discussion in the following three cases.

Case 1: $|\mathcal{S} \cap V(\mathcal{C})| = 2$. Since $g = (\{v_1\}, \{v_2\}, \ldots, \{v_s\}, \mathcal{L})$ is a TDC of T with $s+1$ color classes, Lemma 6.2 implies that $\chi_d^t(T) = s+1$.

Case 2: $|\mathcal{S} \cap V(\mathcal{C})| = 1$. Let $e_1 \in \mathcal{S}$ and $e_2 \notin \mathcal{S}$. If $\chi_d^t(T) = s+1$, then

$$f = (\{v_1\}, \{v_2\}, \ldots, \{v_s\}, V(T) - \mathcal{S})$$

is the only TDC of T. But this is impossible, because $|S| \geq 3$ implies that there exists a vertex $v_i \in \mathcal{S} - \{v_1\}$ such that for every color class W of f, $v_i \not\succ_t W$. Now, since $g = (\{v_1\}, \{v_2\}, \ldots, \{v_s\}, \{e_2\}, \mathcal{L})$ is a TDC of T with $s+2$ color classes, we obtain $\chi_d^t(T) = s+2$.

Case 3: $|\mathcal{S} \cap V(\mathcal{C})| = 0$. Let $deg(e_2) \geq deg(e_1)$. It can easily be verify that $\chi_d^t(T) \geq s+2$. Since $|S| \geq 3$, we have $deg(e_1) \geq 3$ or $deg(e_2) \geq 3$. If $\chi_d^t(T) = s+2$, then every χ_d^t-coloring f of T will be in the form $f = (\{v_1\}, \{v_2\}, \ldots, \{v_s\}, E_1, E_2)$ for some disjoint subsets E_1 and E_2 such that $e_1 \in E_1$ and $e_2 \in E_2$. Without loss of generality, let $I = \{i \mid v_i \in N(e_1)\}$ and $J = \{i \mid v_i \notin N(e_1)\}$. Then $\cup_{i \in I} N(v_i) \subseteq E_1$ and $\cup_{i \in J} N(v_i) \subseteq E_2$. The condition $|S| \geq 3$ implies that $|I| \geq 2$ or $|J| \geq 2$. Without loss of generality, let $|I| \geq 2$. Then $v_1 \not\succ_t W$ for every color classes W of f, implying that $\chi_d^t(T) \geq s+3$. Now, since $g = (\{v_1\}, \{v_2\}, \ldots, \{v_s\}, \{e_1\}, \{e_2\}, \mathcal{L})$ is a TDC of T with $s+3$ color classes, we obtain $\chi_d^t(T) = s+3$. □

7. Further research

We finish our discussion with some problems for further research.

Problem 7.1. *Find some bounds for $\chi_d^t(G) + \chi_d^t(\overline{G})$ and $\chi_d^t(G) \cdot \chi_d^t(\overline{G})$.*

Problem 7.2. *Find the total dominator chromatic number of a tree with diameter at least six.*

Problem 7.3. *For $k \geq 3$, characterize graphs G satisfy $\chi_d^t(G) = k$.*

Problem 7.4. *Characterize graphs G satisfy*
- $\chi_d^t(G) = \chi_d(G)$,
- $\chi_d^t(G) = \chi(G)$,

- $\chi_d^t(G) = \gamma_t(G)$, *or*
- $\chi_d^t(G) = \gamma_t(G) + \min_S \chi(G[V(G) - S])$, *where* $S \subset V(G)$ *is a* $\gamma_t(G)$*-set.*

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