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A TYPICAL GRAPH STRUCTURE OF A RING

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Abstract. The zero-divisor graph of a commutative ring R with respect to nilpotent elements is a simple undirected graph $\Gamma_N^*(R)$ with vertex set $\mathcal{Z}_N(R)^*$, and two vertices x and y are adjacent if and only if xy is nilpotent and $xy \neq 0$, where $\mathcal{Z}_N(R) = \{x \in R : xy \text{ is nilpotent, for some } y \in R^*\}$. In this paper, we investigate the basic properties of $\Gamma_N^*(R)$. We discuss when it will be Eulerian and Hamiltonian. We further determine the genus of $\Gamma_N^*(R)$.

1. Introduction

The study of algebraic structures, using the properties of graphs, became an exciting research topic in the past twenty years, leading to many fascinating results and questions. In 1988, I. Beck began to investigate the possibility of coloring a commutative ring R by associating to the ring a *zero-divisor graph*, defined as a simple graph, the vertices of which are the elements of the ring R, with two distinct elements x and y being adjacent if and only if $xy = 0$ [4]. Retaining the original definition, the next decade brought little progress. However, in 1999, D. F. Anderson and P. S. Livingston [2] modified and studied the zero-divisor graph $\Gamma(R)$ whose vertices are the nonzero zero-divisors of the commutative ring R. Note that $\Gamma(R)^c$ is the *complement of the zero-divisor graph* of R. In [7], Chen defined a kind of graph structure of rings. He let all the elements of ring R be the vertices of the graph and two vertices x and y are adjacent if and only if xy is nilpotent. However, in 2010, A. Li and Q. Li modified and studied a kind of new undirected graph $\Gamma_N(R)$ whose vertices are non zero elements of the set $\mathcal{Z}_N(R)$, and two vertices x and y are adjacent if and only if xy is nilpotent, where $\mathcal{Z}_N(R) = \{x \in R : xy \text{ is nilpotent, for some } y \in R^*\}$. For any set X, let X^* denote the nonzero

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elements of X. In this paper, we construct a graph called *zero-divisor graph of a commutative ring R with respect to nilpotent elements* as a simple undirected graph $\Gamma_N^*(R)$ with vertex set $\mathcal{Z}_N(R)^*$, and two vertices x and y are adjacent if and only if xy is nilpotent and $xy \neq 0$. We investigate the interplay between the graph theoretic properties of $\Gamma_N^*(R)$ and the ring theoretic properties of R. We denote the ring of integers modulo n by $\mathbb{Z}_n$, the field with q elements by $\mathbb{F}_q$ and the set of all nilpotent elements in R by $N(R)$. Note that $R^\times$ be the set of all units in R and $J(R)$ be the Jacobson radical of R. For basic definitions on rings, one may refer [3].

Let $G = (V, E)$ be a simple connected graph. The girth of a graph with a cycle is the length of its shortest cycle. A graph with no cycle has infinite girth. The closure of a graph G is the graph obtained from G by recursively joining pairs of nonadjacent vertices whose degree sum is at least $|V(G)|$ until no such pair remains. Note that a connected graph is Hamiltonian if and only if its closure is Hamiltonian. For basic definitions on graphs, one may refer [6]. The following results are listed for ready reference.

Theorem 1.1. [5] *If R is a finite local ring, then $|R| = p^n$ for some prime p and some positive integer $n \geq 1$.*

Theorem 1.2. [3] *If R is a finite commutative ring, then $R \cong R_1 \times \cdots \times R_n$, where each R_i is a local ring.*

Example 1.3. [1, 11] Let $(R, \mathfrak{m})$ be a finite local ring and $\Gamma(R)$ be the zero-divisor graph of R. Then

$\lvert Z(R)^*\rvert$	R	$\lvert R\rvert$	$\Gamma(R)$
1	$\mathbb{Z}_4,\ \frac{\mathbb{Z}_2[x]}{\langle x^2\rangle}$	4	K_1
2	$\mathbb{Z}_9,\ \frac{\mathbb{Z}_3[x]}{\langle x^2\rangle}$	9	K_2
3	$\mathbb{Z}_8,\ \frac{\mathbb{Z}_2[x]}{\langle x^3\rangle},\ \frac{\mathbb{Z}_4[x]}{\langle 2x, x^2-2\rangle}$	8	$K_{1,2}$
	$\frac{\mathbb{Z}_4[x]}{\langle 2,x\rangle^2},\ \frac{\mathbb{Z}_2[x,y]}{\langle x,y\rangle^2}$	8	K_3
	$\frac{\mathbb{F}_4[x]}{\langle x^2\rangle},\ \frac{Z_4[x]}{\langle x^2+x+1\rangle}$	16	K_3
4	$\mathbb{Z}_{25},\ \frac{\mathbb{Z}_5[x]}{\langle x^2\rangle}$	25	K_4

Theorem 1.4. [2] *Let R be a finite commutative ring. If $\Gamma(R)$ is complete, then either $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ or R is a local ring with char $R = p$ or p^2 and $|\Gamma(R)| = p^n - 1$, where p is prime and $n \geq 1$.*

Lemma 1.5. $g(K_n) = \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil$ *if $n \geq 3$. In particular, $g(K_n) = 1$ if $n = 5, 6, 7$.*

Lemma 1.6. $g(K_{m,n}) = \left\lceil \frac{(m-2)(n-2)}{4} \right\rceil$ *if $m, n \geq 2$. In particular, $g(K_{4,4}) = g(K_{3,n}) = 1$ if $n = 3, 4, 5, 6$. Also $g(K_{5,4}) = g(K_{6,4}) = g(K_{m,4}) = 2$ if $m = 7, 8, 9, 10$.*

2. Basic Properties of $\Gamma_N^*(R)$

Remark 2.1. Let R be a reduced ring. Then $\mathcal{Z}_N(R)^* = Z(R)^*$, $\Gamma_N(R) \cong \Gamma(R)$ and by definition, $\Gamma_N^*(R)$ is an empty graph.

Remark 2.2. Let R be a local ring, but not a field. Then by the definition, $\Gamma_N^*(R)$ is connected and $diam(\Gamma_N^*(R)) = 2$.

Theorem 2.3. *Let $R \cong R_1 \times \cdots \times R_n$ be a finite commutative ring with identity, where each R_i is a local ring and $n \geq 2$. Then $\Gamma_N^*(R)$ is connected if and only if R_i is not a field for every i. Further $diam(\Gamma_N^*(R)) = 2$.*

Proof. Suppose R_i is not field for every i. Then $Z(R_i) \neq \{0\}$ for every i. Let $x, y \in \mathcal{Z}_N(R)^*$ and $x \neq y$. Note that $J(R) = N(R)$.

If $N(R)^* = \emptyset$, then $N(R) = 0$ which is a contradiction as $Z(R_i) \neq \{0\}$ for every i. Therefore $N(R)^* \neq \emptyset$.

Case 1. $x, y \in R^\times$

Then there exists $a \in N(R)^*$ such that $xa, ya \in N(R)^*$. Thus $x - a - y$ is a path in $\Gamma_N^*(R)$.

Case 2. $x \in R^\times$ and $y \in N(R)^*$

Then $xy \in N(R)^*$ and so x and y are adjacent in $\Gamma_N^*(R)$.

Case 3. $x \in R^\times$ and $y \in Z(R) - N(R)$

Then there exists $b \in N(R)^*$ such that $xb, yb \in N(R)^*$. Thus $x - b - y$ is a path in $\Gamma_N^*(R)$.

Case 4. $x, y \in Z(R) - N(R)$

Then there exists $c \in N(R)^*$ such that $xc, yc \in N(R)^*$. Thus $x - c - y$ is a path in $\Gamma_N^*(R)$.

Case 5. $x, y \in N(R)^*$

Then there exists $u \in R^\times$ such that $xu, yu \in N(R)^*$ and so $x - u - y$ is a path in $\Gamma_N^*(R)$.

Case 6. $x \in Z(R) - N(R)$ and $y \in N(R)^*$

If $xy \neq \{0\}$, then $xy \in N(R)^*$ and so x and y are adjacent in $\Gamma_N^*(R)$. If $xy = 0$, then there exists $z \in N(R)^*$ such that $xz, yz \in N(R)^*$. Thus $x - z - y$ is a path in $\Gamma_N^*(R)$. Hence $\Gamma_N^*(R)$ is connected.

Conversely, let $\Gamma_N^*(R)$ be a connected graph. Suppose R_i is a field for some i. Then there exists an element $x = (0, \ldots, 0, 1, 0 \ldots, 0) \in \mathcal{Z}_N(R)^*$, with 1 in the i^{th} place of x, such that x is not adjacent to any other vertex of $\Gamma_N^*(R)$, a contradiction. □

Theorem 2.4. *Let R be a finite commutative ring with identity such that $\Gamma_N^*(R)$ is connected. Then $\Gamma(R)^c$ is a subgraph of $\Gamma_N^*(R)$ if and only if R is a local ring.*

Proof. By Theorems 1.2 and 2.3, $R \cong R_1 \times \cdots \times R_n$, where each $(R_i, \mathfrak{m}_i)$ is a local ring. Suppose R is a local ring. Then by definition of $\mathcal{Z}_N(R)^*$, $\mathcal{Z}_N(R)^* = R^*$ and so $Z(R)^* \subset \mathcal{Z}_N(R)^*$. Also $R \cong R_1$ and so x is nilpotent for all $x \in \mathfrak{m}_1^*$. Let $x, y \in \mathfrak{m}_1^*$ with $x \neq y$. If x and y are adjacent in $\Gamma(R)$, then $xy = 0$, and so x and y are non-adjacent in $\Gamma_N^*(R)$. If x and y are non-adjacent in $\Gamma(R)$, then $xy \neq 0$, $xy \in N(R)^*$ and so x and y are adjacent in $\Gamma_N^*(R)$. Hence $\Gamma(R)^c$ is a subgraph of $\Gamma_N^*(R)$.

Conversely, let $\Gamma(R)^c$ be a subgraph of $\Gamma_N^*(R)$. Suppose R is a non-local ring and $n \geq 2$. Let $a = (1, 0, \ldots, 0)$, $b = (u, 0, \ldots, 0) \in Z(R)^* \subset \mathcal{Z}_N(R)$, where u is a unit in R_1. Then $ab \neq 0$ in R and so a and b are non-adjacent in $\Gamma(R)$. Clearly $ab \notin N(R)$ and so a and b are non-adjacent in $\Gamma_N^*(R)$, a contradiction. □

Theorem 2.5. *Let R be a finite commutative ring with identity and $\Gamma_N^*(R)$ be a connected graph. Then $\Gamma_N^*(R)$ is a complete bipartite graph if and only if $\Gamma(R)$ is a complete graph.*

Proof. Suppose $\Gamma(R)$ is complete. Then by Theorems 1.4 and 2.3, R is a local ring with unique maximal ideal $\mathfrak{m}$ and $xy = 0$ for all $x, y \in \mathfrak{m}^*$, $x \neq y$. Note that $\mathcal{Z}_N(R)^* = R^* = \mathfrak{m}^* \cup (R \setminus \mathfrak{m})$. By the definition of $\Gamma_N^*(R)$, $\mathfrak{m}^*$ and $R \setminus \mathfrak{m}$ are independent sets of $\Gamma_N^*(R)$. Also each edge in $\Gamma_N^*(R)$ has one end in $\mathfrak{m}^*$ and other end in $R \setminus \mathfrak{m}$. Hence $\Gamma_N^*(R)$ is a complete bipartite graph with bipartition $(\mathfrak{m}^*, R \setminus \mathfrak{m})$.

Conversely, suppose $\Gamma_N^*(R))$ is complete bipartite graph. Note that $R \cong R_1 \times \cdots \times R_n$, where each $(R_i, \mathfrak{m}_i)$ is a local ring. If R is non-local ring, then $n \geq 2$. Let $a \in \mathfrak{m}_1^*$, $b \in \mathfrak{m}_2^*$. Then $(a, b, \ldots, 0) - (1, 0, \ldots, 0) - (a, 1, 0, \ldots, 0) - (a, b, 0 \ldots, 0)$ is a cycle of length 3 in $\Gamma_N^*(R)$, a contradiction. Hence $R \cong R_1$ is a local ring. Suppose $\Gamma(R)$ is not complete. Then there exists elements $x_1, y_1 \in Z(R)^*$ such that $x_1 y_1 \neq 0$ and $x_1 y_1 \in N(R)^*$. Thus $x_1 - u - y_1 - x_1$ is a cycle of length 3 in $\Gamma_N^*(R)$, where $u \in R^\times$, a contradiction. Hence $\Gamma(R)$ is complete. $\square$

In view of Theorems 1.4 and 2.5, if $\Gamma_N^*(R)$ is connected then we have $\Gamma_N^*(R)$ is complete bipartite if and only if R is a local ring with *char* $R = p$ or p^2.

Theorem 2.6. *Let R be a finite commutative ring with identity and $\Gamma_N^*(R)$ be a connected graph. Then the following are equivalents:*
(i) $\Gamma_N^*(R)$ is a star
(ii) $\Gamma_N^*(R)$ is a tree
(iii) $R \cong \mathbb{Z}_4$ or $\frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle}$.

Proof. $(i) \Rightarrow (ii)$ follows from the definition of tree.
$(ii) \Rightarrow (iii)$ Suppose $\Gamma_N^*(R)$ is tree. Then $\Gamma_N^*(R)$ contains no cycle. Since R is connected, by Remark 2.2 and Theorem 2.3, $R \cong R_1 \times \cdots \times R_n$ where each $(R_i, \mathfrak{m}_i)$ is a local ring, but not a field. If $n \geq 2$, then $(a, b, \ldots, 0) - (1, 0, \ldots, 0) - (a, 1, 0, \ldots, 0) - (a, b, 0 \ldots, 0)$ is a cycle in $\Gamma_N^*(R)$, where $a \in \mathfrak{m}_1^*, b \in \mathfrak{m}_2^*$, a contradiction. Thus $n = 1$ and so R is local. Suppose $|\mathfrak{m}_1^*| \geq 2$. Then $|R^\times| \geq 3$. Let $x, y \in \mathfrak{m}_1^*$ with $xy = 0$ and $u, v \in R^\times$. Then $x - u - y - v - x$ is a cycle in $\Gamma_N^*(R)$, a contradiction. Hence $|\mathfrak{m}_1^*| = 1$ and so $R \cong \mathbb{Z}_4$ or $\frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle}$.
$(iii) \Rightarrow (i)$ If $R \cong \mathbb{Z}_4$ or $\frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle}$, then $\Gamma_N^*(R) \cong K_{1,2}$. $\square$

Theorem 2.7. *Let R be a finite commutative ring with identity and $\Gamma_N^*(R)$ be a connected graph.*
(i) $gr(\Gamma_N^*(R)) = \infty$ if and only if $R \cong \mathbb{Z}_4$ and $\frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle}$.
(ii) $gr(\Gamma_N^*(R)) = 4$ if and only if R is local with $Z(R)^2 = 0$ and $|Z(R)^*| \geq 2$.
(iii) $gr(\Gamma_N^*(R)) = 3$ if and only if $R \ncong \mathbb{Z}_4$, $\frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle}$ and R is not a local ring with $Z(R)^2 = 0$ and $|Z(R)^*| \geq 2$.

Proof. (i) follows from Theorem 2.6.
(ii) Suppose R is local with $Char(R) = p^2$ and $|Z(R)^*| \geq 2$. Then by Theorems 1.4 and 2.5, $\Gamma_N^*(R)$ is a complete bipartite graph and so $gr(\Gamma_N^*(R)) = 4$.

Conversely, let $gr(\Gamma_N^*(R)) = 4$. Then $\Gamma_N^*(R)$ does not contain a cycle of length 3 and $|Z(R)^*| \geq 4$. Since R is finite, $R \cong R_1 \times \cdots \times R_n$, where each $(R_i, \mathfrak{m}_i)$ is a local ring. Since $\Gamma_N^*(R)$ is connected, $\mathfrak{m}_i \neq \{0\}$ for all i. Suppose $n \geq 2$. Let $x_1 \in \mathfrak{m}_1^*$ and $x_2 \in \mathfrak{m}_2^*$. Then $(x_1, 0, \ldots, 0) - (0, x_2, 0, \ldots, 0) - (1, 0, \ldots, 0) - (x_1, 0, \ldots, 0)$ is a cycle in $\Gamma_N^*(R)$, a contradiction. Thus R is local. If $Z(R)^2 \neq 0$, then there exists $x, y \in Z(R)^*$ such that $xy \neq 0$ and so $x - u - y - x$ is a cycle in $\Gamma_N^*(R)$, a contradiction. (iii) follows from (i) and (ii). $\square$

Theorem 2.8. [8, S. Földes, P. L. Hammer] *Let G be a connected graph. Then G is a split graph if and only if G contains no induced subgraph isomorphic to $2K_2$, C_4, C_5, where C_4 is a cycle of length 4.*

Theorem 2.9. *Let R be a finite commutative ring with identity and $\Gamma_N^*(R)$ be a connected graph. Then $\Gamma_N^*(R)$ is split if and only if $R \cong \mathbb{Z}_4$ or $\frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle}$.*

Proof. Suppose $\Gamma_N^*(R)$ is split. Since R is finite, $R \cong R_1 \times \cdots \times R_n$, where each $(R_i, \mathfrak{m}_i)$ is a local ring. If $n \geq 2$, then $(1, \ldots, 1) - (a_1, 0, \ldots, 0) - (u_1, u_2, \ldots, u_n) - (0, a_2, 0, \ldots, 0)$ is cycle of length 4 in $\Gamma_N^*(R)$, where $a_1 \in \mathfrak{m}_1$, $a_2 \in \mathfrak{m}_2$, $u_i \in R_i^\times$. By Theorem 2.8, $\Gamma_N^*(R)$ is not split, a contradiction. Hence R is local. If $|\mathfrak{m}_1^*| \geq 2$, then C_4 is a subgraph of $\Gamma_N^*(R)$, a contradiction. Hence $|\mathfrak{m}_1| = 2$ and so $R \cong \mathbb{Z}_4$ or $\frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle}$.

Conversely, if $R \cong \mathbb{Z}_4$ or $\frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle}$, then by Theorem 2.6(i), $\Gamma_N^*(R)$ is star and so $\Gamma_N^*(R)$ is split. $\square$

3. Eulerian and Hamiltonian Nature of $\Gamma_N^*(R)$

In this section, we are interested in the Eulerian and Hamiltonian nature of $\Gamma_N^*(R)$.

Theorem 3.1. *Let $(R, \mathfrak{m})$ be a finite local ring but not a field and $|R| = p^n$, where p is prime and $n > 1$. If $\Gamma_N^*(R)$ is a connected graph with $|\mathfrak{m}^*| \geq 2$, then $\Gamma_N^*(R)$ is Eulerian if and only if $|R|$ is odd and $x^2 = 0$ for all $x \in \mathfrak{m}^*$.*

Proof. Note that $|\mathfrak{m}| = p^k$ for some $k < n$ and $|R^\times| = p^k(p^{n-k} - 1)$. Suppose $\Gamma_N^*(R)$ is Eulerian. Then $deg_{\Gamma_N^*(R)}(v)$ is even for all $v \in \mathcal{Z}_N(R)^*$. Suppose $|R|$ is even. Then $|\mathfrak{m}^*|$ is odd and so $deg_{\Gamma_N^*(R)}(u) = p^k - 1$ is odd for all $u \in R^\times$, a contradiction. Hence $|R|$ is odd and so $|\mathfrak{m}^*|$, $|R^\times|$ are even. If $x^2 \neq 0$ for some $x \in \mathfrak{m}^*$, then $|ann(x)| = p^\ell$ for some $\ell < n$ and so $deg_{\Gamma_N^*(R)}(x) = |R^\times| + |\mathfrak{m}| - p^\ell - 1$ is odd, a contradiction. Hence $x^2 = 0$ for all $x \in \mathfrak{m}^*$.

Conversely, let $|R|$ be an odd integer such that $x^2 = 0$ for all $x \in \mathfrak{m}^*$. Then $deg_{\Gamma_N^*(R)}(u) = p^k - 1$ is even for all $u \in R^\times$ and $deg_{\Gamma_N^*(R)}(z) = |R^\times|$ is even for all $z \in \mathfrak{m}^*$. Hence $\Gamma_N^*(R)$ is Eulerian. $\square$

Theorem 3.2. *Let R be a finite commutative nonlocal ring with identity and $\Gamma_N^*(R)$ be a connected graph. Then $\Gamma_N^*(R)$ is Eulerian if and only if $|R|$ is odd and $x^2 = 0$ for all $x \in J(R)^*$.*

Proof. Suppose $\Gamma_N^*(R)$ is Eulerian. Then $deg_{\Gamma_N^*(R)}(x)$ is even for all $x \in \mathcal{Z}_N(R)^*$. Since R is finite, $R \cong R_1 \times \cdots \times R_n$ where each $(R_i, \mathfrak{m}_i)$ is a local ring and $n \geq 2$. If $|R|$ is even, then $|R_i|$ is even for some i, $|\mathfrak{m}_i|$ is even and so $deg_{\Gamma_N^*(R)}(u) = (\Pi_{i=1}^n |\mathfrak{m}_i|) - 1$ is odd for all $u \in R^\times$, a contradiction. Hence $|R|$ is odd and so $|\mathfrak{m}_i|$ is odd.

If $x^2 \neq 0$ for some $x \in J(R)^*$, then $deg_{\Gamma_N^*(R)}(x) = |R| - (|ann(x)| + 1)$ is odd, a contradiction. Hence $x^2 = 0$ for all $x \in J(R)^*$.

Conversely, let $|R|$ be an odd integer and $x^2 = 0$ for all $x \in J(R)^*$. Then $|R_i|$ is odd for all i and so $|\mathfrak{m}_i|$ is odd. Let $y = (y_1, \ldots, y_n) \in \mathcal{Z}_N(R)^*$. Then

$$deg_{\Gamma^*_N(R)}(y) = \begin{cases} |R| - |ann(y)| & \text{if } y \in J(R)^* \\ (\Pi_{i=1}^{n} |\mathfrak{m}_i|) - 1 & \text{if } y \in R^\times \end{cases} \tag{3.1}$$

In equation 3.1, $deg_{\Gamma^*_N(R)}(y)$ is even. If $y \in Z(R) \setminus J(R)$, then $deg_{\Gamma^*_N(R)}(y) = |Z(R)| - |ann(y)| - 1$ is even. Hence $deg_{\Gamma^*_N(R)}(a)$ is even for all $a \in \mathcal{Z}_N(R)^*$ and so $\Gamma^*_N(R)$ is Eulerian. □

Theorem 3.3. *Let $(R, \mathfrak{m})$ be a finite local ring but not a field and $|R| = p^n \geq 4$, where p is prime and $n > 1$. Then $\Gamma^*_N(R)$ has a Hamiltonian path if and only if $R/\mathfrak{m} \cong \mathbb{Z}_2$. Hence $|R|$ is even.*

Proof. Note that $|\mathfrak{m}| = p^k$ for some $k < n$ and $|R^\times| = p^k(p^{n-k} - 1) = t$. Clearly $K_{p^k-1,t}$ is a subgraph of $\Gamma^*_N(R)$ and $R^\times$ is an independent subset of $\Gamma^*_N(R)$. Suppose $R/\mathfrak{m} \cong \mathbb{Z}_2$. Then $|\mathfrak{m}| = |R^\times|$ and $|\mathcal{Z}_N(R)| = |\mathfrak{m}^*| + |R^\times|$. Note that K_{p^k-1,p^k} is a subgraph of $\Gamma^*_N(R)$. From this, we get $u_1 - z_1 - u_2 - z_2 - u_3 - \cdots - u_{p^k-1} - z_{p^k-1} - u_{p^k}$ is a Hamiltonian path in $\Gamma^*_N(R)$, where $u_i \in R^\times$ and $z_j \in \mathfrak{m}^*$.

Conversely, suppose $\Gamma^*_N(R)$ has a Hamiltonian path. Suppose $|R/\mathfrak{m}| > 2$. Then $|\mathfrak{m}| < |R^\times|$ and so $|R^\times - \mathfrak{m}| > 2$. Note that the closure of $\Gamma^*_N(R)$ is $K_{p^k-1} + \overline{K}_{p^k(p^{n-k}-1)}$. Clearly $K_{p^k-1} + \overline{K}_{p^k(p^{n-k}-1)}$ is not Hamiltonian and so $\Gamma^*_N(R)$ is not Hamiltonian. Hence $|R/\mathfrak{m}| = 2$ and so $R/\mathfrak{m} \cong \mathbb{Z}_2$. □

From the above theorem, one can observe the following corollary.

Corollary 3.4. *Let $(R, \mathfrak{m})$ be a finite local ring but not a field. Then $\Gamma^*_N(R)$ is nonhamiltonain.*

4. Genus of $\Gamma^*_N(R)$

In this section, we characterizes the class of rings for which $\Gamma^*_N(R)$ is planar. Also we determine all isomorphism classes of finite commutative rings with identity whose $\Gamma^*_N(R)$ has genus one.

Let S_k denote the sphere with k handles, where k is a non-negative integer. The genus of any graph G, denoted $g(G)$, is the minimal integer ℓ such that the graph can be embedded in S_ℓ. A genus 0 graph is called a planar graph and a genus 1 graph is called a toroidal graph. For details on embedding a graph in a surface, see [13].

Theorem 4.1. *Let R be a finite commutative ring with identity and $\Gamma^*_N(R)$ be a connected graph. Then $\Gamma^*_N(R)$ is planar if and only if R is isomorphic to one of the following rings: $\mathbb{Z}_4$, $\frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle}$, $\mathbb{Z}_9$, or $\frac{\mathbb{Z}_3[x]}{\langle x^2 \rangle}$.*

Proof. Suppose $\Gamma^*_N(R)$ is planar. Clearly if R is a non-local ring, then $K_{3,3}$ is a subgraph of $\Gamma^*_N(R)$. Hence $(R, \mathfrak{m})$ is a local ring with $|R| = p^n$ where p is prime and $n > 1$. Note that $|\mathfrak{m}| = p^k$ for some $k < n$ and $|R^\times| = |R \setminus \mathfrak{m}| = p^k(p^{n-k} - 1)$. If $|\mathfrak{m}^*| \geq 3$, then $|R^\times| \geq 4$ and so $K_{3,4}$ is a subgraph of $\Gamma^*_N(R)$, a contradiction. Hence $|\mathfrak{m}^*| \leq 2$ and so $R \cong \mathbb{Z}_4$, $\frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle}$, $\mathbb{Z}_9$, or $\frac{\mathbb{Z}_3[x]}{\langle x^2 \rangle}$. Converse is obvious. □

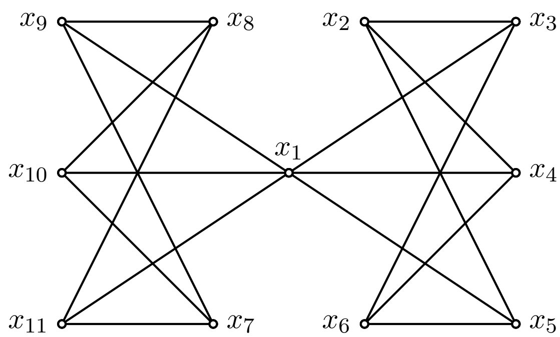

Fig. 2.1: A graph $\mathbb{G}'$ with two blocks, each isomorphic to $K_{3,3}$

Theorem 4.2. *Let R be a finite commutative ring with identity and $\Gamma_N^*(R)$ be a connected graph. Then $g(\Gamma_N^*(R)) = 1$ if and only if R is isomorphic to one of the following rings: $\mathbb{Z}_8$, $\frac{\mathbb{Z}_2[x]}{\langle x^3\rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle 2x, x^2-2\rangle}$, $\frac{\mathbb{Z}_2[x,y]}{\langle x,y\rangle^2}$ or $\frac{\mathbb{Z}_4[x]}{\langle 2,x\rangle^2}$.*

Proof. We only need to prove the necessary part. Suppose $g(\Gamma_N^*(R)) = 1$. Then $\Gamma_N^*(R)$ is non-planar. Since R is finite, $R \cong R_1 \times \cdots \times R_n$, where each $(R_i, \mathfrak{m}_i)$ is a local ring. Since $\Gamma_N^*(R)$ is connected, $\mathfrak{m}_i \neq \{0\}$ for every i. Suppose $n \geq 2$. Let $a_1 \in \mathfrak{m}_1$, $a_2 \in \mathfrak{m}_2$ with $a_1^2 = 0$ and $a_2^2 = 0$.

Consider $x_1 = (0, a_2, 0, \ldots, 0)$, $x_2 = (a_1, 0, \ldots, 0)$, $x_3 = (1, 1, 0, \ldots, 0)$, $x_4 = (1, u_2, 0, \ldots, 0)$, $x_5 = (u_1, 1, 0, \ldots, 0, 0)$, $x_6 = (a_1, a_2, 0, \ldots, 0)$, $x_7 = (u_1, a_2, 0, \ldots, 0)$, $x_8 = (1, a_2, 0, 0, \ldots, 0)$, $x_9 = (0, u_2, 0, \ldots, 0)$, $x_{10} = (a_1, 1, 0, 0, \ldots, 0)$ and $x_{11} = (a_1, u_2, 0, \ldots, 0) \in \mathcal{Z}_N(R)^*$, where $u_1 \in R_1^\times, u_2 \in R_2^\times$. Let $\Omega = \{x_1, \ldots, x_{11}\}$. Then $\mathbb{G}$ is a subgraph of $\langle\Omega\rangle$ in $\Gamma_N^*(R)$(see, Fig. 2.1). Note that $g(\mathbb{G}) = 2$[12, C. Wickham]. Therefore $g(\Gamma_N^*(R)) \geq 2$, a contradiction. Hence $(R, \mathfrak{m})$ is local.

Since $\Gamma_N^*(R)$ is non-planar, by Theorem 4.1, $|\mathfrak{m}^*| \geq 3$. If $|\mathfrak{m}^*| = 4$, then by Example 1.3, $R \cong \mathbb{Z}_{25}$ or $\frac{\mathbb{Z}_5[x]}{\langle x^2\rangle}$, $|R^\times| = 20$ and so $K_{4,20}$ is a subgraph of $\Gamma_N^*(R)$. By Lemma 1.6, $g(\Gamma_N^*(R)) > 1$, a contradiction. If $|\mathfrak{m}^*| \geq 5$, then $|R^\times| \geq 6$ and so $K_{5,6}$ is a subgraph of $\Gamma_N^*(R)$, now $g(\Gamma_N^*(R)) > 1$, a contradiction. Thus $|\mathfrak{m}^*| = 3$ and by Example 1.3, R is isomorphic to one of the following rings: $\mathbb{Z}_8$, $\frac{\mathbb{Z}_2[x]}{\langle x^3\rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle 2x, x^2-2\rangle}$, $\frac{\mathbb{Z}_2[x,y]}{\langle x,y\rangle^2}$, $\frac{\mathbb{Z}_4[x]}{\langle 2,x\rangle^2}$, $\frac{\mathbb{F}_4[x]}{\langle x^2\rangle}$ or $\frac{\mathbb{Z}_4[x]}{\langle x^2+x+1\rangle}$.

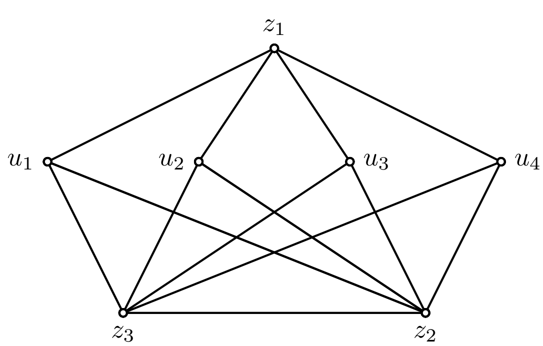

Fig. 2.2(a):$\Gamma_N^*(\mathbb{Z}_8) \cong \Gamma_N^*\left(\frac{\mathbb{Z}_2[x]}{\langle x^3\rangle}\right) \cong \Gamma_N^*\left(\frac{\mathbb{Z}_4[x]}{\langle 2x, x^2-2\rangle}\right)$

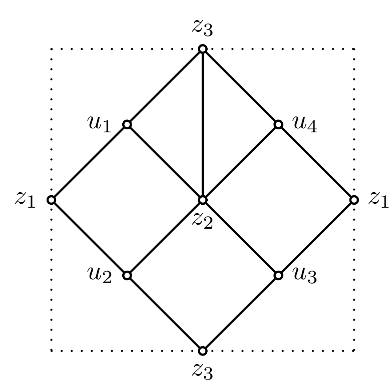

Fig. 2.2(b):Embedding of $\Gamma_N^*(R)$ in S_1

Suppose $R \cong \frac{\mathbb{F}_4[x]}{\langle x^2\rangle}$ or $\frac{\mathbb{Z}_4[x]}{\langle x^2+x+1\rangle}$. Then by Example 1.3, $|R| = 16$, $|R^\times| = 12$ and so $K_{3,12}$ is a subgraph of $\Gamma_N(R)$ and by Lemma 1.6, $g(\Gamma_N^*(R)) \geq 2$, a contradiction.

Since $\Gamma_N^*(\mathbb{Z}_8) \cong \Gamma_N^*\left(\frac{\mathbb{Z}_2[x]}{\langle x^3\rangle}\right) \cong \Gamma_N^*\left(\frac{\mathbb{Z}_4[x]}{\langle 2x, x^2-2\rangle}\right) \cong K_{3,4}$ and Fig. 2.2(b), R is isomorphic to one of the following rings: $\mathbb{Z}_8$, $\frac{\mathbb{Z}_2[x]}{\langle x^3\rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle 2x, x^2-2\rangle}$, $\frac{\mathbb{Z}_2[x,y]}{\langle x,y\rangle^2}$ or $\frac{\mathbb{Z}_4[x]}{\langle 2,x\rangle^2}$. □

Theorem 4.3. *There exists no finite local ring R with $g(\Gamma_N^*(R)) = 2$*

Proof. Let $(R, \mathfrak{m})$ be a finite local ring with $|R| = p^n$, where p is prime and $n > 1$. Then $|\mathfrak{m}| = p^k$ for some $k < n$ and $|R^\times| = p^k(p^{n-k} - 1) > |\mathfrak{m}^*| = p^k - 1$. By Theorems 4.1 and 4.2, $|\mathfrak{m}| \geq 3$.

If $|\mathfrak{m}| = 3$, then by Example 1.3 and Theorem 4.2, $R \cong \frac{\mathbb{F}_4[x]}{\langle x^2\rangle}$ or $\frac{\mathbb{Z}_4[x]}{\langle x^2+x+1\rangle}$. In this case, $\Gamma_N^*(R) \cong K_{3,12}$ and by Lemma 1.6, $g(\Gamma_N^*(R)) = 5$.

If $|\mathfrak{m}^*| \geq 4$, then $H = K_{p^k-1, p^k(p^{n-k}-1)}$ is a subgraph of $\Gamma_N^*(R)$ and so $g(\Gamma_N^*(R)) \geq g(H) > 2$. Hence there exists no finite local ring R with $g(\Gamma_N^*(R)) = 2$. □

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