A NOTE ON STAR COLORING OF CENTRAL GRAPH OF BIPARTITE GRAPH AND CORONA GRAPH OF COMPLETE GRAPH WITH PATH AND CYCLE

V. J. VERNOLD * AND M. VENKATAChALAM

Communicated by Alireza Abdollahi

Abstract. In this paper, we find the star chromatic number of central graph of complete bipartite graph and corona graph of complete graph with path and cycle.

1. Introduction

The notion of star chromatic number was introduced by Branko Grünbaum in 1973.

A star coloring [1, 2, 3] of a graph $G$ is a proper vertex coloring in which every path on four vertices uses at least three distinct colors. Equivalently, in a star coloring, the induced subgraphs formed by the vertices of any two colors has connected components that are star graphs. The star chromatic number $\chi_s(G)$ of $G$ is the least number of colors needed to star color $G$.

A number of results exist for star colorings of graphs formed by certain graph operations. Guillaume Fertin et al. [3] gave the exact value of the star chromatic number of different families of graphs such as trees, cycles, complete bipartite graphs, outerplanar graphs, and 2-dimensional grids. They also investigated and gave bounds for the star chromatic number of other families of graphs, such as planar graphs, hypercubes, $d$-dimensional grids ($d \geq 3$), $d$-dimensional tori ($d \geq 2$), graphs with bounded treewidth, and cubic graphs.

Albertson et al. [1] showed that it is NP-complete to determine whether $\chi_s(G) \leq 3$, even when $G$ is a graph that is both planar and bipartite. The problems of finding star colorings is NP-hard and remain so even for bipartite graphs [4, 5].

MSC(2010): Primary: 05C15; Secondary: 05C75.
Keywords: central graph, corona graph and star coloring.
Received: 09 December 2011, Accepted: 25 February 2012.
*Corresponding author.
In Section 3, we find the star chromatic number of central graph of complete bipartite graph. In Section 4, we find the star chromatic number of corona graph of complete graph with path and cycle.

2. Preliminaries

All graphs considered are loopless graphs without multiple edges. A path on \( n \) vertices will be denoted by \( P_n \). A cycle on \( n \) vertices will be denoted by \( C_n \).

For a given graph \( G = (V, E) \) we do an operation on \( G \), by subdividing each edge exactly once and joining all the non adjacent vertices of \( G \). The graph obtained by this process is called central graph \([6, 7]\) of \( G \) denoted by \( C(G) \).

The corona \([8]\) of two graphs \( G_1 \) and \( G_2 \) is the graph \( G = G_1 \circ G_2 \) formed from one copy of \( G_1 \) and \( |V(G_1)| \) copies of \( G_2 \) where the \( i \)th vertex of \( G_1 \) is adjacent to every vertex in the \( i \)th copy of \( G_2 \).

3. Star coloring on central graph of complete bipartite graph

**Theorem 3.1.** Let \( K_{m,n} \) be a complete bipartite graph on \( m \) and \( n \) vertices. Then

\[
\chi_s(C(K_{m,n})) = \begin{cases} 
m; & \text{if } m \geq n, \\
n; & \text{otherwise.}
\end{cases}
\]

**Proof.** Let \( \{v_i : 1 \leq i \leq n\} \) and \( \{u_j : 1 \leq j \leq m\} \) be the vertices of \( K_{m,n} \) and by the definition of complete bipartite graph, every vertex from \( \{v_i : 1 \leq i \leq n\} \) is adjacent to every vertex from the set \( \{u_j : 1 \leq j \leq m\} \). Let \( \{e_{ij} : 1 \leq i \leq n; 1 \leq j \leq m\} \) be the set of edges of \( K_{m,n} \).

By the definition of central graph, the edges \( \{e_{ij} : 1 \leq i \leq n; 1 \leq j \leq m\} \) be subdivided by the vertex \( \{w_{ij} : 1 \leq i \leq n; 1 \leq j \leq m\} \) in \( C(K_{m,n}) \), and let \( V = \{v_1, v_2, \ldots, v_n\} \), \( V' = \{u_1, u_2, \ldots, u_m\} \). Clearly \( V(C(K_{m,n})) = V \cup V' \cup \{w_{ij} : 1 \leq i \leq n; 1 \leq j \leq m\} \). Note that in \( C(K_{m,n}) \), the induced subgraphs \( \langle \{v_1, v_2, \ldots, v_n\} \rangle \) and \( \langle \{u_1, u_2, \ldots, u_m\} \rangle \) are complete. Therefore,

\[
\chi_s(C(K_{m,n})) \geq \begin{cases} 
m; & \text{if } m \geq n, \\
n; & \text{if } n \geq m.
\end{cases}
\]

The following coloring for \( C(K_{m,n}) \) is star chromatic: For \( 1 \leq i \leq n \), assign the color \( c_i \) for \( v_i \). For \( 1 \leq j \leq m \), assign the color \( c_j \) for \( u_j \). For \( 1 \leq i \leq n \) and \( 1 \leq j \leq m \), assign to vertex \( \{w_{ij} : 1 \leq i \leq n; 1 \leq j \leq m\} \) one of allowed colors - such color exists, because \( \text{deg}(w_{ij}) = 2 \). Thus we have,

\[
\chi_s(C(K_{m,n})) \leq \begin{cases} 
m; & \text{if } m \geq n, \\
n; & \text{if } n \geq m.
\end{cases}
\]

Hence,

\[
\chi_s(C(K_{m,n})) = \begin{cases} 
m; & \text{if } m \geq n, \\
n; & \text{otherwise.}
\end{cases}
\]

\( \square \)
4. Star coloring on corona graph of complete graph with path and cycle

Theorem 4.1. Let $K_n$ be a complete graph on $n$ vertices. Then

$$\chi_s(K_n \circ P_n) = n, \ \forall \ n \geq 3.$$  

Proof. Let $V(K_n) = \{v_1, v_2, \ldots, v_n\}$ and $V(P_n) = \{u_1, u_2, \ldots, u_n\}$. Let $V(K_n \circ P_n) = \{v_i : 1 \leq i \leq n\} \cup \{u_{ij} : 1 \leq i \leq n; 1 \leq j \leq n\}$. By the definition of corona graph, each vertex of $G$ is adjacent to every vertex of a copy of $P_n$. i.e., every vertex $v_i \in V(K_n)$ is adjacent to every vertex from the set $\{u_{ij} : 1 \leq j \leq n\}$. Note that $\chi_s(K_n \circ P_n) \geq n$, since $K_n$ is $n$ chromatic.

Assign the following star coloring for $K_n \circ P_n$ as star-chromatic:

- For $1 \leq i \leq n$, assign the color $c_i$ to $v_i$.
- For $1 \leq i \leq n$, assign the color $c_i$ to $u_{1i}$, $\forall \ i \neq 1$.
- For $1 \leq i \leq n$, assign the color $c_i$ to $u_{2i}$, $\forall \ i \neq 2$.
- For $1 \leq i \leq n$, assign the color $c_i$ to $u_{3i}$, $\forall \ i \neq 3$.
- For $1 \leq i \leq n$, assign the color $c_i$ to $u_{4i}$, $\forall \ i \neq 4$.

Thus we have, $\chi_s(K_n \circ P_n) \leq n$. Hence, $\chi_s(K_n \circ P_n) = n$, $\forall \ n \geq 3$. \square

Theorem 4.2. Let $K_n$ be a complete graph on $n$ vertices. Then

$$\chi_s(K_n \circ C_n) = n, \ \forall \ n \geq 3.$$  

Proof. The proof is similar to the proof of Theorem 4.1. \square

Acknowledgments

The authors wish to thank the referee and the Editor-in-Chief for their encouragements and helpful suggestions and comments to improve this paper.

References


Vivin J. Vernold
Department of Mathematics, University College of Engineering Nagercoil, Anna University of Technology Tirunelveli, Nagercoil - 629 004, Tamil Nadu, India
Email: vernoldvivin@yahoo.in

M. Venkatachalam
Department of Mathematics, RVS Faculty of Engineering, RVS Educational Trust’s Group of Institutions, Coimbatore - 641 402, Tamil Nadu, India
Email: venkatmaths@gmail.com