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UNICYCLIC GRAPHS WITH STRONG EQUALITY BETWEEN THE 2-RAINBOW DOMINATION AND INDEPENDENT 2-RAINBOW DOMINATION NUMBERS

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Abstract. A 2-*rainbow dominating function* (2RDF) on a graph $G = (V, E)$ is a function f from the vertex set V to the set of all subsets of the set $\{1, 2\}$ such that for any vertex $v \in V$ with $f(v) = \emptyset$ the condition $\bigcup_{u \in N(v)} f(u) = \{1, 2\}$ is fulfilled. A 2RDF f is independent (I2RDF) if no two vertices assigned nonempty sets are adjacent. The *weight* of a 2RDF f is the value $\omega(f) = \sum_{v \in V} |f(v)|$. The 2-*rainbow domination number* $\gamma_{r2}(G)$ (respectively, the *independent* 2-*rainbow domination number* $i_{r2}(G)$) is the minimum weight of a 2RDF (respectively, I2RDF) on G. We say that $\gamma_{r2}(G)$ is strongly equal to $i_{r2}(G)$ and denote by $\gamma_{r2}(G) \equiv i_{r2}(G)$, if every 2RDF on G of minimum weight is an I2RDF. In this paper we characterize all unicyclic graphs G with $\gamma_{r2}(G) \equiv i_{r2}(G)$.

1. Introduction

Throughout this paper, all graphs considered are finite, undirected, loopless and without multiple edges. We refer the reader to [8, 13] for terminology and notation in graph theory. Specifically, let G be a simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. For every vertex $v \in V$, the *open neighborhood* $N(v)$ is the set $\{u \in V \mid uv \in E\}$ and the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. The *degree* of a vertex $v \in V$ is $\deg_G(v) = \deg(v) = |N(v)|$. A *unicyclic graph* is a connected graph containing exactly one cycle. A vertex of degree one is called a $leaf$, and its neighbor is called a *support vertex*. If v is a support vertex, then L_v will denote the set of all leaves adjacent to v. A support vertex v is called a *strong support vertex* if $|L_v| > 1$. For $r, s \geq 1$, a double

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star $S(r, s)$ is a tree with exactly two vertices that are not leaves, with one adjacent to r leaves and the other to s leaves.

For a positive integer k, a *k-rainbow dominating function* (kRDF) of a graph G is a function f from the vertex set $V(G)$ to the set of all subsets of the set $\{1, 2, \ldots, k\}$ such that for any vertex $v \in V(G)$ with $f(v) = \emptyset$ the condition $\bigcup_{u \in N(v)} f(u) = \{1, 2, \ldots, k\}$ is fulfilled. The *weight* of a kRDF f is the value $\omega(f) = \sum_{v \in V} |f(v)|$. The *$k$-rainbow domination number* of a graph G, denoted by $\gamma_{rk}(G)$, is the minimum weight of a kRDF of G. A *$\gamma_{rk}(G)$-function* is a k-rainbow dominating function of G with weight $\gamma_{rk}(G)$. Note that 1-rainbow domination number is the classical domination number $\gamma(G)$. The k-rainbow domination was introduced by Brešar, Henning, and Rall [2] and has been studied by several authors (see for example [3, 6, 7, 12]).

A k-rainbow dominating function f is called an *independent k-rainbow dominating function* (abbreviated IkRDF) on G if the set $\{v \in V \mid f(v) \neq \emptyset\}$ is independent. The *independent k-rainbow domination number*, denoted by $i_{rk}(G)$, is the minimum weight of an IkRDF on G. An independent k-rainbow dominating function with weight $i_{rk}(G)$ is called an *$i_{rk}(G)$-function*. The independent k-rainbow domination number is investigated in [1, 4].

Obviously each independent k-rainbow dominating function is a k-rainbow dominating function, and so $\gamma_{rk}(G) \leq i_{rk}(G)$. If $\gamma_{rk}(G) = i_{rk}(G)$, then every $i_{rk}(G)$-function is also a $\gamma_{rk}(G)$-function. However not every $\gamma_{rk}(G)$-function is an $i_{rk}(G)$-function, even when $\gamma_{rk}(G) = i_{rk}(G)$. We say that $\gamma_{rk}(G)$ and $i_{rk}(G)$ are *strongly equal* and denote by $\gamma_{rk}(G) \equiv i_{rk}(G)$, if every $\gamma_{rk}(G)$-function is an $i_{rk}(G)$-function.

The strong equality between two parameters was introduced by Haynes and Slater in [11] in the first. Also in [9] and [10], Haynes, Henning and Slater gave constructive characterizations of trees with strong equality between some domination parameters.

In this paper, we characterize all unicyclic graphs G with $\gamma_{r2}(G) \equiv i_{r2}(G)$. For this aim, we use the constructive characterization of trees T with $\gamma_{r2}(T) \equiv i_{r2}(T)$ provided recently by Amjadi et al. [1]. Below we present the procedure given in [1] to built such trees. Let $\mathcal{F}_1$ be the family of trees that can be obtained from $k \geq 1$ disjoint stars $K_{1,2}$ by adding either a new vertex v or a path uv and joining the centers of stars to v. Also let $\mathcal{F}_2$ be the family including P_5 and all trees obtained from $k \geq 2$ disjoint P_3 by adding either a new vertex v or a path uv and joining v to a leaf of each P_3. If T belongs to $\mathcal{F}_1 \cup \mathcal{F}_2 - \{P_5\}$ then we call the vertex v, the *special vertex* of T and if $T = P_5$, then its support vertices are special vertices of T. We now define recursively a collection $\mathcal{F}$ of trees such that $K_{1,2} \in \mathcal{F}$, and if T is any tree in $\mathcal{F}$, then we put in $\mathcal{F}$ any tree T' that can be obtained from T by any of the following seven operations:

- Operation $\mathcal{O}_1$: If z is a strong support vertex of $T \in \mathcal{F}$, then $\mathcal{O}_1$ adds a new vertex x and an edge xz.
- Operation $\mathcal{O}_2$: If z is a vertex of $T \in \mathcal{F}$, then $\mathcal{O}_2$ adds a new tree $T_1 \in \mathcal{F}_1$ with special vertex x and an edge xz provided that if x is a support vertex, then $\gamma_{r2}(T - z) \geq \gamma_{r2}(T)$.
- Operation $\mathcal{O}_3$: If z is a strong support vertex of $T \in \mathcal{F}$, then $\mathcal{O}_3$ adds a path zxy.

- Operation $\mathcal{O}_4$: If z is a vertex of $T \in \mathcal{F}$ which is adjacent to a support vertex of degree 2, then $\mathcal{O}_4$ adds a path zxy.
- Operation $\mathcal{O}_5$: If z is a vertex of $T \in \mathcal{F}$ which is adjacent to a strong support vertex, then $\mathcal{O}_5$ adds a path $zxyw$.
- Operation $\mathcal{O}_6$: If z is a vertex of $T \in \mathcal{F}$, then $\mathcal{O}_6$ adds new tree $T_2 \in \mathcal{F}_2$ with special vertex x and an edge xz provided that if x is a support vertex, then $\gamma_{r2}(T - z) \geq \gamma_{r2}(T)$.
- Operation $\mathcal{O}_7$: If z is a vertex of $T \in \mathcal{F}$ such that every $\gamma_{r2}(T)$-function assigns $\emptyset$ to z, then $\mathcal{O}_7$ adds the double star $S(1,2)$ and an edge zx where x is a leaf of $S(1,2)$ whose support vertex has degree 3.

Theorem A (Amjadi et al. [1])**.** Let T be a tree. Then $i_{r2}(T) \equiv \gamma_{r2}(T)$ if and only if $T \in \mathcal{F} \cup \{K_1\}$.

2. Unicyclic graphs G with $\gamma_{r2}(G) \equiv i_{r2}(G)$

We begin by giving some definitions and results that will be useful in our characterization. A vertex $v \in V(G)$ is called an *empty vertex* if $f(v) = \emptyset$ for every $\gamma_{rk}(G)$-function f.

Observation 2.1. If $v \in V(G)$ is an empty vertex of G, then $\gamma_{r2}(G) = \gamma_{r2}(G - v)$.

Proof. Since v is an empty vertex of G, each $\gamma_{r2}(G)$-function is clearly a 2RDF of $G - v$ implying that $\gamma_{r2}(G - v) \leq \gamma_{r2}(G)$. If $\gamma_{r2}(G - v) < \gamma_{r2}(G)$, then let f be a $\gamma_{r2}(G - v)$-function and define the function h by $h(v) = \{1\}$ and $h(x) = f(x)$ for $x \in V(G) - \{v\}$. Clearly h is a 2RDF of G of weight $\gamma_{r2}(G)$ that leads to a contradiction because v is an empty vertex of G. Thus $\gamma_{r2}(G) = \gamma_{r2}(G-v)$. $\square$

A pair (x, y) of vertices of a tree $T \in \mathcal{F}$ is called a *complementary pair,* or*c-pair* for short*,* if there exists a $\gamma_{r2}(T)$-function f such that $\{f(x), f(y)\} = \{\{1\}, \{2\}\}$. For the pair (x, y), let $G_{T_{x,y}}$ be the graph obtained from T by adding a new vertex z and edges xz and zy. Further, if z is an empty vertex of $G_{T_{x,y}}$, then (x, y) is called a c-pair of type 1, and if there is a $\gamma_{r2}(G_{T_{x,y}})$-function that assigns $\{1, 2\}$ to z, then (x, y) is called a c-pair of type 2. Besides these two cases, (x, y) is called a c-pair of type 3. The following observation is straightforward.

Observation 2.2. If $T \in \mathcal{F}$ and $x, y \in V(T)$ are a c-pair, then $\gamma_{r2}(T) = \gamma_{r2}(G_{T_{x,y}})$.

A vertex v of a tree T is said to be an *independent vertex* if (i) every $\gamma_{r2}(T)$-function f with $|f(v)| = 1$ is independent and there is at least one $\gamma_{r2}(T)$-function with this property, (ii) there is no $\gamma_{r2}(T)$-function f such that $|f(v)| = 2$. We denote by $\mathcal{H}$ the set of all trees in $\mathcal{F}$ having an independent vertex.

In the next, we provide a constructive characterization of unicyclic graphs G with $\gamma_{r2}(G) \equiv i_{r2}(G)$. Let $\mathcal{U}$ be the family of graphs G such that G is obtained from some trees in $\mathcal{F}$ by one of the operations $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3$, or $\mathcal{T}_4$ or obtained from even number of trees in $\mathcal{H}$ by operation $\mathcal{T}_5$.

- Operation $\mathcal{T}_1$: Let x and y be two non-adjacent vertices of $T \in \mathcal{F}$. The Operation $\mathcal{T}_1$ adds the edge xy if the following two conditions hold:

(1) every $\gamma_{r2}(T)$-function assigns $\emptyset$ to at least one of the x and y, and
(2) $T-u$, $(u \in \{x,y\})$ has no 2RDF f of weight $\gamma_{r2}(T)$ such that f is not independent and $\bigcup_{v\in(N_T(u)\cup(\{x,y\}-\{u\}))} f(v)=\{1,2\}$.

- Operation $\mathcal{T}_2$: Assume $T_0,T_1,\ldots,T_t \in \mathcal{F}$, $x,y \in V(T_0)$ and $v_i \in V(T_i)$ for $i \in \{1,\ldots,t\}$ and let
  (1) (x,y) is a c-pair of type 1,
  (2) for each $i \in \{1,\ldots,t\}$, v_i is an empty vertex of T_i.
  Then Operation $\mathcal{T}_2$ adds a new vertex z and edges zx, zy and zv_i for each $i \in \{1,\ldots,t\}$.
- Operation $\mathcal{T}_3$: Assume $T_0,T_1,\ldots,T_t \in \mathcal{F}$, $x,y \in V(T_0)$ and $v_i \in V(T_i)$ for $i \in \{1,\ldots,t\}$ and let
  (1) (x,y) is a c-pair of type 2,
  (2) $T_0-\{x,y\}$ has no 2RDF f of weight $\gamma_{r2}(T_0)-1$ that is not independent and $j \in (\cup_{u\in N_{T_0}(x)-\{y\}} f(u)) \cap (\cup_{u\in N_{T_0}(y)-\{x\}} f(u))$ for some $j \in \{1,2\}$,
  (3) for each $i \in \{1,\ldots,t\}$, v_i is an empty vertex of T_i,
  (4) $T_i - v_i \in \mathcal{F}$ for each $i \in \{1,\ldots,t\}$.
  Then Operation $\mathcal{T}_3$ adds a new vertex z and edges zx, zy and zv_i for $i \in \{1,\ldots,t\}$.
- Operation $\mathcal{T}_4$: Assume $T_0,T_1,\ldots,T_t \in \mathcal{F}$, $x,y \in V(T_0)$ and $v_i \in V(T_i)$ for $i \in \{1,\ldots,t\}$ such that
  (1) (x,y) is a c-pair of type 3,
  (2) $T_0-\{x,y\}$ has no 2RDF f of weight $\gamma_{r2}(T_0)-1$ that is not independent and $j \in (\cup_{u\in N_{T_0}(x)-\{y\}} f(u)) \cap (\cup_{u\in N_{T_0}(y)-\{x\}} f(u))$ for $j \in \{1,2\}$,
  (3) v_i is an empty vertex of T_i for each $i \in \{1,\ldots,t\}$,
  (4) for each $i \in \{1,\ldots,t\}$, $T_i - v_i$ has no 2RDF g of weight $\gamma_{r2}(T_i)$ such that g is not independent and $1 \in \cup_{u\in N_{T_i}(v_i)} g(u)$ or $2 \in \cup_{u\in N_{T_i}(v_i)} g(u)$.
  Then Operation $\mathcal{T}_4$ adds a new vertex z and edges zx, zy and zv_i for $i \in \{1,\ldots,t\}$.
- Operation $\mathcal{T}_5$: Let $t \neq 0$ be an even integer, $T_1,T_2,\ldots,T_t \in \mathcal{H}$, and $v_i \in V(T_i)$ be an independent vertex for each $i \in \{1,\ldots,t\}$. Then Operation $\mathcal{T}_5$ adds new vertices $u_1,u_2,\ldots,u_t$ and edges u_iv_i, for each $i \in \{1,\ldots,t\}$, v_iu_{i+1} for each $i \in \{1,\ldots,t-1\}$ and the edge v_tu_1.

Lemma 2.3. If T is a tree with $\gamma_{r2}(T) \equiv i_{r2}(T)$ and G is a unicyclic graph obtained from T by Operation $\mathcal{T}_1$, then $\gamma_{r2}(G) \equiv i_{r2}(G)$.

Proof. Let $T \in \mathcal{F}$ and let x and y be non-adjacent vertices of T jointed by Operation $\mathcal{T}_1$. Clearly every $\gamma_{r2}(T)$-function is a 2RDF of G and hence $\gamma_{r2}(G) \le i_{r2}(G) \le \gamma_{r2}(T) = i_{r2}(T)$. Now we show that $\gamma_{r2}(G) = \gamma_{r2}(T)$. Suppose to the contrary that $\gamma_{r2}(G) < \gamma_{r2}(T)$ and let f be a $\gamma_{r2}(G)$-function. If $f(x)=f(y)=\emptyset$, or $\emptyset \notin \{f(x), f(y)\}$, then f is a 2RDF of T with weight less than $\gamma_{r2}(T)$, a contradiction. If $f(x) \neq \emptyset$ and $f(y)=\emptyset$ (the case $f(x)=\emptyset$ and $f(y)\neq\emptyset$ is similar), then the function g defined by $g(y)=\{1\}, g(w)=f(w)$ for $w \in V(G)-\{y\}$ is a $\gamma_{r2}(T)$-function which contradicts (i). Hence $\gamma_{r2}(G) = i_{r2}(G) = \gamma_{r2}(T)$. It will now be shown that $\gamma_{r2}(G) \equiv i_{r2}(G)$. Suppose g is a $\gamma_{r2}(G)$-function that is not independent. If $g(x) \neq \emptyset$ and $g(y) \neq \emptyset$ then g is a $\gamma_{r2}(T)$-function, that contradicts (i). If $g(x)=g(y)=\emptyset$, then g is a $\gamma_{r2}(T)$-function that is not independent, a contradiction with $T \in \mathcal{F}$. If $g(x) \neq \emptyset$ and $g(y)=\emptyset$ (the case $g(x)=\emptyset$ and $g(y)\neq\emptyset$ is similar), then g is a 2DRF of

$T-y$ of weight $\gamma_{r2}(T)$ that is not independent and $\bigcup_{v\in(N(y)\cup\{x\})} g(v)=\{1,2\}$, a contradiction with (ii). This completes the proof. $\square$

Lemma 2.4. Let $T_0,T_1,\ldots,T_t\in\mathcal{F}$, $x,y\in V(T_0)$ be a c-pair of type 1 and $v_i\in V(T_i)$ is an empty vertex of T_i for every $i\in\{1,\ldots,t\}$. If G is a unicyclic graph obtained from $T_0,T_1,\ldots,T_t$ by Operation $\mathcal{T}_2$, then $\gamma_{r2}(G)\equiv i_{r2}(G)$.

Proof. Assume z is the new vertex added by Operation $\mathcal{T}_2$ for obtaining G. First we show that $\gamma_{r2}(G)=i_{r2}(G)$. Let f_i be a $\gamma_{r2}(T_i)$-function for every $i\in\{0,\ldots,t\}$ and let $\{f_0(x),f_0(y)\}=\{\{1\},\{2\}\}$. Define $h:V(G)\to\mathcal{P}(\{1,2\})$ by $h(z)=\emptyset$ and $h(u)=f_i(u)$ for $u\in V(T_i)$ and $0\le i\le t$. Since $T_i\in\mathcal{F}$, we deduce that h is an I2RDF of G with weight $\sum_{i=0}^{t}\gamma_{r2}(T_i)$ implying that

$$\gamma_{r2}(G)\le i_{r2}(G)\le\sum_{i=0}^{t}\gamma_{r2}(T_i)=\sum_{i=0}^{t}i_{r2}(T_i). \tag{2.1}$$

Assume g is a $\gamma_{r2}(G)$-function. We claim that $g(z)=\emptyset$. Suppose to the contrary that $g(z)\ne\emptyset$. If $\omega(g|_{V(T_j)})<\gamma_{r2}(T_j)$ for some $1\le j\le t$, then the function f defined by $f(v_j)=\{1\}$ and $f(u)=g(u)$ for $u\in V(T_j)-\{v_j\}$ is a 2RDF of T_j with weight at most $\gamma_{r2}(T_i)$ which leads to a contradiction because v_j is an empty vertex of T_j. Thus $\omega(g|_{V(T_j)})\ge\gamma_{r2}(T_j)$ for each $j\in\{1,\ldots,t\}$. It follows from (2.1) that $\omega(g|_{V(T_0)\cup\{z\}})\le\gamma_{r2}(T_0)$. This implies that $g|_{V(G_{T_{0_{xy}}})}$ is a $\gamma_{r2}(G_{T_{0_{xy}}})$-function with $g(z)\ne\emptyset$, a contradiction. Hence $g(z)=\emptyset$. Thus $g|_{V(T_i)}$ is a 2RDF for each $i\in\{0,\ldots,t\}$. Therefore $\omega(g|_{V(T_i)})\ge\gamma_{r2}(T_i)=i_{r2}(T_i)$ for every $i\in\{0,\ldots,t\}$. Hence $\gamma_{r2}(G)=\omega(g)\ge\sum_{i=0}^{t}\gamma_{r2}(T_i)$ and by (2.1) we have $\gamma_{r2}(G)=i_{r2}(G)=\sum_{i=0}^{t}\gamma_{r2}(T_i)$. Now it will be shown that $\gamma_{r2}(G)\equiv i_{r2}(G)$. Assume h is a $\gamma_{r2}(G)$-function that is not independent. Using an argument similar to that described above, shows that $h(z)=\emptyset$ and the function $h\big|_{V(T_i)}$ is a $\gamma_{r2}(T_i)$-function for each i. Since h is not independent, we deduce that $h\big|_{V(T_i)}$ is not independent for some i, a contradiction with $T_i\in\mathcal{F}$. This completes the proof. $\square$

Lemma 2.5. Let $T_0,T_1,\ldots,T_t\in\mathcal{F}$, $x,y\in V(T_0)$ and $v_i\in V(T_i)$ for $1\le i\le t$ satisfy in the condition of Operation $\mathcal{T}_3$. If G is a unicyclic graph obtained from $T_0,T_1,\ldots,T_t$ by Operation $\mathcal{T}_3$, then $\gamma_{r2}(G)\equiv i_{r2}(G)$.

Proof. Suppose z is the vertex added by Operation $\mathcal{T}_3$ for obtaining G. We first show that $\gamma_{r2}(G)=i_{r2}(G)=\sum_{i=0}^{t}\gamma_{r2}(T_i)$. As in the proof of Lemma 2.4, we can see that

$$\gamma_{r2}(G)\le i_{r2}(G)\le\sum_{i=0}^{t}\gamma_{r2}(T_i)=\sum_{i=0}^{t}i_{r2}(T_i). \tag{2.2}$$

Assume g is a $\gamma_{r2}(G)$-function. As in the proof of Lemma 2.4, we have $\omega(g|_{V(T_j)})\ge\gamma_{r2}(T_j)$ for each $j\in\{1,\ldots,t\}$. On the other hand, the function g, restricted to $G_{T_{0_{x,y}}}$ is a 2RDF of $G_{T_{0_{x,y}}}$ implying that $\omega(g|_{G_{T_{0_{x,y}}}})\ge\gamma_{r2}(T_{0_{x,y}})=\gamma_{r2}(T_0)$ by Observation 2.2. Thus $\gamma_{r2}(G)\ge\sum_{i=0}^{t}\gamma_{r2}(T_i)$. It follows from (2.2) that $\gamma_{r2}(G)=i_{r2}(G)=\sum_{i=0}^{t}\gamma_{r2}(T_i)=\sum_{i=0}^{t}i_{r2}(T_i)$. Now we show that $\gamma_{r2}(G)\equiv i_{r2}(G)$. Assume h is a $\gamma_{r2}(G)$-function that is not independent. By above argument, $h|_{V(T_i)}$ is a $\gamma_{r2}(T_i)$-function for each $i\in\{1,\ldots,t\}$. We consider two cases.

Case 1. $h(z) = \emptyset$.
Since h is not independent, $h\big|_{V(T_i)}$ for some i, is a $\gamma_{r2}(T_i)$-function that is not independent, a contradiction with $T_i \in \mathcal{F}$.

Case 2. $h(z) \neq \emptyset$.
If $h(v_i) \neq \emptyset$ for some $1 \leq i \leq t$, then $h\big|_{V(T_i)}$ is a $\gamma_{r2}(T_i)$-function and v_i is not an empty vertex of T_i, a contradiction. Thus $h(v_i) = \emptyset$ for every $i \in \{1, \dots, t\}$. This implies that $h\big|_{V(T_i)-\{v_i\}}$ is a $\gamma_{r2}(T_i - v_i)$-function, by Observation 1. If $h|_{V(T_i)-\{v_i\}}$ is not independent for some i, we obtain a contradiction with $T_i - v_i \in \mathcal{F}$. Henceforth, we let $h|_{V(T_i)-\{v_i\}}$ is independent for each i. Thus $h|_{G_{T_{0x,y}}}$ is a γ_{r2}-function that is not independent. Consider two subcases.

Subcase 2.1. $h(z) = \{1, 2\}$.
If $h(x) \neq \emptyset$ (the case $h(y) \neq \emptyset$ is similar), then the function g defined by $g(y) = \{1\}, g(u) = h(u)$ for $u \in V(T_0) - \{x\}$ is a 2RDF of T_0 of weight less than $\gamma_{r2}(T_0)$, a contradiction. Therefore, we assume $h(x) = h(y) = \emptyset$. Then the function g defined by $g(x) = g(y) = \{1\}, g(u) = h(u)$ for $u \in V(T_0) - \{x, y\}$ is a $\gamma_{r2}(T_0)$-function that is not independent, a contradiction.

Subcase 2.2. $|h(z)| = 1$.
We may assume without loss of generality that $h(z) = \{1\}$. If $\emptyset \notin \{h(x), h(y)\}$, then $h|_{V(T_0)}$ is a 2RDF of weight less than $\gamma_{r2}(T_0)$, a contradiction. If $h(x) \neq \emptyset$ and $h(y) = \emptyset$ (the case $h(x) = \emptyset$ and $h(y) \neq \emptyset$ is similar), then $2 \in \cup_{u \in N_{T_0}(y)} h(u) \neq \emptyset$ and the function g defined by $g(y) = \{1\}$ and $g(u) = h(u)$ for $u \in V(T_0) - \{y\}$ is a 2RDF of weight $\gamma_{r2}(T_0)$ that is not independent, a contradiction with $T_0 \in \mathcal{F}$. Hence, let $h(x) = h(y) = \emptyset$. Then $2 \in \bigcup_{u \in N_{T_0}(v)} h(u)$ for $v \in \{x, y\}$. If $\cup_{u \in N_{T_0}(x)} h(u) = \{1, 2\}$ (the case $\cup_{u \in N_{T_0}(y)} h(u) = \{1, 2\}$ is similar), then the function g defined by $g(y) = \{1\}$ and $g(u) = h(u)$ for $u \in V(T_0) - \{y\}$ is a 2RDF of weight $\gamma_{r2}(T_0)$ that is not independent which is a contradiction again. Thus $\cup_{u \in N_{T_0}(x)} h(u) = \cup_{u \in N_{T_0}(y)} h(u) = \{2\}$. But then $h|_{V(T_0)-\{x,y\}}$ is a 2RDF on $T_0 - \{x, y\}$ of weight $\gamma_{r2}(T_0) - 1$ that is not independent contradicting the assumption. This completes the proof. $\square$

Lemma 2.6. Let $T_0, T_1, \dots, T_t \in \mathcal{F}$, $x, y \in V(T_0)$ and $v_i \in V(T_i)$ for each $i \in \{1, \dots, t\}$ satisfy in the conditions of Operation 4. If G is the unicyclic graph obtain from $T_0, T_1, \dots, T_t$ by Operation $\mathcal{T}_4$, then $\gamma_{r2}(G) \equiv i_{r2}(G)$.

Proof. Suppose z is the vertex added by Operation $\mathcal{T}_4$ to obtain G. As in the proof of Lemma 2.4, we obtain

$$\gamma_{r2}(G) \leq i_{r2}(G) \leq \sum_{i=0}^{t} \gamma_{r2}(T_i). \tag{2.3}$$

Assume g is a $\gamma_{r2}(G)$-function. If $g(z) = \emptyset$ then $g|_{V(T_i)}$ is a 2RDF of T_i for every $i \in \{0, \dots, t\}$ and hence $\omega(g|_{V(T_i)}) \geq \gamma_{r2}(T_i)$ for each $i \in \{0, \dots, t\}$. Therefore $\gamma_{r2}(G) = \omega(g) \geq \sum_{i=0}^{t} \gamma_{r2}(T_i)$. Assume $g(z) \neq \emptyset$. Then $g|_{V(T_0)\cup\{z\}}$ is a 2RDF of $G_{T_{0xy}}$ and for each i either $g|_{V(T_i)}$ is a 2RDF of T_i or $g|_{V(T_i)-\{v_i\}}$ is a 2RDF of $T_i - v_i$. By Observations 2.1 and 2.2, we deduce that $\gamma_{r2}(G) \geq \sum_{i=0}^{t} \gamma_{r2}(T_i)$. It follows from (2.3) that $\gamma_{r2}(G) = i_{r2}(G) = \sum_{i=0}^{t} \gamma_{r2}(T_i)$. Now we show that $\gamma_{r2}(G) \equiv i_{r2}(G)$. Suppose to the contrary that g is a $\gamma_{r2}(G)$-function that is not independent. If $g(z) = \emptyset$, then as in the proof of Case

1 in Lemma 2.5 we obtain a contradiction. Let $g(z) \neq \emptyset$. Since x and y are a c-pair of type 3, we have $|g(z)| = 1$. Assume, without loss of generality, that $g(z) = \{1\}$. Since g is not independent, there are two vertices $u, v \in V(G)$ such that $\emptyset \notin \{f(u), f(v)\}$. Since $\omega(g|_{V(T_i)}) = \gamma_{r2}(T_i)$ and v_i is an empty vertex of T_i, we deduce that $g(v_i) = \emptyset$ for each $1 \leq i \leq t$. It follows that $2 \in \cup_{w\in N_{T_i}(v_i)} g(w)$ for each $1 \leq i \leq t$, and $u, v \in V(G_{T_{0x,y}})$ or $u, v \in V(T_i - v_i)$ for some $1 \leq i \leq t$. By condition 3., we deduce that $u, v \in V(G_{T_{0x,y}})$. If $g(x) = g(y) = \emptyset$, then we must have $2 \in (\cup_{w\in N_{T_0}(x)} f(w)) \cap (\cup_{w\in N_{T_0}(y)} f(w))$ and $u, v \in V(T_0) \setminus \{x, y\}$ contradicting condition 2. Let without loss of generality that $f(x) \neq \emptyset$. Then the function $h : V(G) \to \mathcal{P}(\{1, 2\})$ defined by $h(y) = \{1\}$ and $h(w) = g(w)$ for $w \in V(T_0) - \{y\}$, is a $\gamma_{r2}(T_0)$-function that is not independent, a contradiction. Thus $\gamma_{r2}(G) \equiv i_{r2}(G)$ and the proof is complete. $\square$

Lemma 2.7. Assume t is an even integer, $T_1, T_2, \ldots, T_t \in \mathcal{H}$, and $v_i \in V(T_i)$ is an independent vertex for each $i \in \{1, \ldots, t\}$. If G is the unicyclic graph obtained from $T_1, \ldots, T_t$ by Operation $\mathcal{T}_5$, then $\gamma_{r2}(G) \equiv i_{r2}(G)$.

Proof. We use the same notation as defined in Operation $\mathcal{T}_5$. Suppose f_i is an independent $\gamma_{r2}(T_i)$-function such that $|f(v_i)| = 1$ for each i. We may assume that $f(v_i) = \{1\}$ if i is odd and $f(v_i) = \{2\}$ if i is even. Define the function $f : V(G) \to \mathcal{P}(\{1, 2\})$ by $f(u) = f_i(u)$ for $u \in V(T_i)$ and $f(u_i) = \emptyset$ for each $i \in \{1, \ldots, t\}$. Clearly f is an I2RDF of G of weight $\sum_{i=1}^{t} \gamma_{r2}(T_i)$ implying that

$$\gamma_{r2}(G) \leq i_{r2}(G) \leq \sum_{i=1}^{t} \gamma_{r2}(T_i). \tag{2.4}$$

It will now be shown that $\gamma_{r2}(G) = \sum_{i=1}^{t} \gamma_{r2}(T_i)$. Assume to the contrary that $\gamma_{r2}(G) < \sum_{i=1}^{t} \gamma_{r2}(T_i)$ and let f be a $\gamma_{r2}(G)$-function such that $\sum_{i=1}^{t} |f(u_i)|$ is as small as possible. We claim that $f(u_i) \neq \{1, 2\}$ for each i. Suppose to the contrary that $f(u_i) = \{1, 2\}$ for some i, say $i = 2$. Then the function $g : V(G) \to \mathcal{P}(\{1, 2\})$ defined by $g(u_2) = \emptyset, g(v_1) = f(v_1) \cup \{1\}, g(v_2) = f(v_2) \cup \{2\}$ and $g(x) = f(x)$ otherwise, is a 2RDF of G such that $\sum_{i=1}^{t} |g(u_i)| < \sum_{i=1}^{t} |f(u_i)|$, a contradiction. Hence $f(u_i) \neq \{1, 2\}$ for each i. Since $\gamma_{r2}(G) < \sum_{i=1}^{t} \gamma_{r2}(T_i)$, we deduce that $\omega(f|_{V(T_i)\cup\{u_{i+1}\}}) < \gamma_{r2}(T_i)$ for some $1 \leq i \leq t$, say $i = 1$. If $f|_{V(T_1)}$ is a 2RDF of T_1, then its weight is less than $\gamma_{r2}(T_1)$ which is a contradiction. Let $f|_{V(T_1)}$ is not a 2RDF of T_1. This implies that $f(v_1) = \emptyset$. If $f(u_2) \neq \emptyset$, then $|f(u_2)| = 1$ and the function $h : V(T_1) \to \mathcal{P}(\{1, 2\})$ defined by $h(v_1) = f(u_2), h(x) = f(x)$ for $x \in V(T_1) - \{v_1\}$ is a 2RDF of T_1 of weight less than $\gamma_{r2}(T_1)$ which is a contradiction again. Thus $f(u_2) = \emptyset$. Now to rainbowly dominate v_1, we must have $f(u_1) \neq \emptyset$. Since $f(u_1) \neq \{1, 2\}$, we may assume without loss of generality that $f(u_1) = \{1\}$. It follows that $2 \in \cup_{u\in N_{T_1}(v_1)} f(u)$. But then the function $h_1 : V(T_1) \to \mathcal{P}(\{1, 2\})$ defined by $h_1(v_1) = \{1\}, h_1(x) = f(x)$ for $x \in V(T_1) - \{v_1\}$ is a $\gamma_{r2}(T_1)$-function such that h_1 is not independent and $|h_1(v_1)| = 1$, a contradiction by $T_1 \in \mathcal{H}$. Thus $\omega(f|_{V(T_i)\cup\{u_{i+1}\}}) \geq \gamma_{r2}(T_i)$ for each $i \in \{1, \ldots, t\}$ and hence $\gamma_{r2}(G) \geq \sum_{i=1}^{t} \gamma_{r2}(T_i)$. It follows from (2.4) that $\gamma_{r2}(G) = \sum_{i=1}^{t} \gamma_{r2}(T_i)$ implying that $\gamma_{r2}(G) = i_{r2}(G)$. Now we show that $\gamma_{r2}(G) \equiv i_{r2}(G)$. Suppose to the contrary that g is a $\gamma_{r2}(G)$-function that is not independent and let u and v be two vertices such that $uv \in E(G)$ and $\emptyset \notin \{g(u), g(v)\}$. We consider two cases.

Case 1. $u, v \in V(T_i)$ for some $1 \leq i \leq t$.
If $|g(v_i)| = 1$ then $g|_{V(T_i)}$ is a $\gamma_{r2}(T_i)$-function that is not independent, contradicting (i) of the definition of independent vertex. If $|g(v_i)| = 2$, then obviously $g|_{V(T_i)}$ is a 2RDF of T_i. Since $T_i \in \mathcal{H}$, it follows from the property (2) that $g|_{V(T_i)}$ is not a $\gamma_{r2}(T_i)$-function which implies $\omega(g|_{V(T_i)\cup\{u_{i+1}\}}) \geq \omega(g|_{V(T_i)}) > \gamma_{r2}(T_i)$. Since $\omega(g|_{V(T_i)\cup\{u_{i+1}\}}) \geq \gamma_{r2}(T_i)$ for each $1 \leq i \leq t$, we obtain $\gamma_{r2}(G) > \sum_{i=1}^{t}\gamma_{r2}(T_i)$, a contradiction. Henceforth, we assume $|g(v_i)| = 0$. If $\omega(g|_{V(T_i)}) < \gamma_{r2}(T_i)$ then we define $h(v_i) = \{1\}$ and $h(w) = g(w)$ for $w \in V(T_i) - \{v_i\}$ to obtain a $\gamma_{r2}(T_i)$-function that is not independent, a contradiction with the property (1). Let $\omega(g|_{V(T_i)}) = \gamma_{r2}(T_i)$. Then we must have $g(u_{i+1}) = \emptyset$, otherwise $\omega(g|_{V(T_i)\cup\{v_{i+1}\}}) \geq \omega(g|_{V(T_i)}) > \gamma_{r2}(T_i)$ and we obtain a contradiction as above. It follows from $g(u_{i+1}) = \emptyset$ and $g(v_i) = \emptyset$ that $g(v_{i+1}) = \{1, 2\}$. By the property (2), we have $\omega(g|_{V(T_{i+1})\cup\{u_{i+2}\}}) > \gamma_{r2}(T_{i+1})$ which leads to a contradiction as above.

Case 2. $\{v, u\} = \{v_i, u_{i+1}\}$ for some i. (the case $\{v, u\} = \{v_i, u_i\}$ is similar).
Assume, without loss of generality, that $v = v_i$ and $u = u_{i+1}$. Since $g|_{V(T_i)}$ is a $\gamma_{r2}(T_i)$-function and since $g(u_{i+1}) \neq \emptyset$, we deduce that $\omega(g|_{V(T_i)\cup\{u_i\}}) > \gamma_{r2}(T_i)$. Since $\omega(g|_{V(T_j)\cup\{u_{j+1}\}}) \geq \gamma_{r2}(T_j)$ for each $1 \leq j \leq t$, we obtain $\gamma_{r2}(G) > \sum_{i=1}^{t}\gamma_{r2}(T_i)$, a contradiction. Thus g is independent and hence $\gamma_{r2}(G) \equiv i_{r2}(G)$. This completes the proof. □

In the next, we show that if G is a unicyclic graph with $i_{r2}(G) \equiv \gamma_{r2}(G)$, then $G \in \mathcal{U}$.

Theorem 2.8. If G is a unicyclic graph of order $n \geq 3$, with $i_{r2}(G) \equiv \gamma_{r2}(G)$, then $G \in \mathcal{U}$.

Proof. Let $C = (u_1, u_2, \ldots, u_k)$ be the unique cycle of G and let

$$C_0 = \{u \in V(C) \mid \text{there is a } \gamma_{r2}(G) - \text{function that assigns } \emptyset \text{ to } u\}.$$

Clearly $C_0 \neq \emptyset$ because $\gamma_{r2}(G) \equiv i_{r2}(G)$. Also let $C_1 = \{u_i \in C_0 \mid$ there is a $\gamma_{r2}(G) -$ function f with $f(u_i) = \emptyset$ such that $f(u_{i-1}) \subseteq f(u_{i+1})$ or $f(u_{i+1}) \subseteq f(u_{i-1})\}$. We consider two cases.

Case 1. $C_1 \neq \emptyset$.
Assume, without loss of generality, that $u_2 \in C_1$. By assumption, G has an $i_{r2}(G)$-function f such that $f(u_2) = \emptyset$ and $f(u_1) \subseteq f(u_3)$ or $f(u_3) \subseteq f(u_1)$. Suppose, without loss of generality, that $f(u_1) \subseteq f(u_3)$. Let $T = G - u_1u_2$. Clearly T is a tree. We show that $T \in \mathcal{F}$ and G can be obtained from T by Operation $\mathcal{T}_1$. Obviously f is an I2RDF of T that yields $\gamma_{r2}(T) \leq i_{r2}(T) \leq \gamma_{r2}(G)$. On the other hand, every $\gamma_{r2}(T)$-function is a 2RDF of G implying that $\gamma_{r2}(T) \geq \gamma_{r2}(G)$. Therefore $\gamma_{r2}(T) = i_{r2}(T) = \gamma_{r2}(G)$. If T has a non independent $\gamma_{r2}(T)$-function f, then f is $\gamma_{r2}(G)$-function that is not independent, a contradiction with $i_{r2}(G) \equiv \gamma_{r2}(G)$. This yields $i_{r2}(T) \equiv \gamma_{r2}(T)$. If T has a $\gamma_{r2}(T)$-function g such that $\emptyset \notin \{g(u_1), g(u_2)\}$, then g is a $\gamma_{r2}(G)$-function that is not independent, a contradiction again. Hence T satisfies (i) of Operation $\mathcal{T}_1$. Suppose $u \in \{u_1, u_2\}$. If $T - u$ has a 2RDF g of weight $\gamma_{r2}(T)$ such that g is not independent and $\cup_{v\in N_T(u)\cup(\{u_1,u_2\}-\{u\})} g(v) = \{1, 2\}$, then the function $h : V(G) \to \mathcal{P}(\{1, 2\})$ defined by $h(u) = \emptyset$ and $h(w) = g(w)$ for $w \in V(G) - \{u\}$ is a 2RDF of G of weight $\gamma_{r2}(G)$ that is not independent, a contradiction with $i_{r2}(G) \equiv \gamma_{r2}(G)$. Therefore T satisfies (ii) of Operation $\mathcal{T}_1$. Thus G can be obtained from T by applying Operation $\mathcal{T}_1$.

Case 2. $C_1 = \emptyset$.
Assume, without loss of generality, that $u_2 \in C_0$ and $\deg(u_2) \geq \deg(x)$ for every $x \in C_0$. Since $C_1 = \emptyset$, for every $\gamma_{r2}(G)$-function g with $g(u_2) = \emptyset$, we have $g(u_1) \nsubseteq g(u_3)$ and $g(u_3) \nsubseteq g(u_1)$. This implies $\{g(u_1), g(u_3)\} = \{\{1\}, \{2\}\}$. Let f be a $\gamma_{r2}(G)$-function with $f(u_2) = \emptyset, f(u_1) = \{1\}$ and $f(u_3) = \{2\}$. We consider the following subcases.

Subcase 2.1. $\deg(u_2) \geq 3$.
Let $N(u_2) - \{u_1, u_3\} = \{v_1, v_2, \ldots, v_t\}$ and let $T_0, T_1, \ldots, T_t$ be the component of $G - u_2$ such that $u_1, u_3 \in V(T_0)$ and $v_i \in V(T_i)$ for each $i \in \{1, \ldots, t\}$. If $f(v_j) \neq \emptyset$ for some $1 \leq j \leq t$, then $f(u_1) \subseteq f(v_j)$ or $f(u_3) \subseteq f(v_j)$, say $f(u_1) \subseteq f(v_j)$, and an argument similar to that described in Case 1, shows that G can be obtained from $T = G - u_1u_2$ by Operation $\mathcal{T}_1$. Henceforth, we assume that $f(v_i) = \emptyset$ for each $i \in \{1, \ldots, t\}$. Therefore we may assume for any $\gamma_{r2}(G)$-function g with $g(u_2) = \emptyset$, $g(v_i) = \emptyset$ for each $i \in \{1, \ldots, t\}$. We claim that $T_i \in \mathcal{F}$, $\omega(f|_{V(T_i)}) = \gamma_{r2}(T_i)$ and v_i is an empty vertex of T_i for each $i \in \{1, \ldots, t\}$. Since $i_{r2}(G) \equiv \gamma_{r2}(G)$ and since $f(u_2) = \emptyset$, the function $f|_{V(T_i)}$ is an I2RDF of T_i and hence $\gamma_{r2}(T_i) \leq i_{r2}(T_i) \leq \omega(f|_{V(T_i)})$ for each $0 \leq i \leq t$. If $\gamma_{r2}(T_i) < \omega(f|_{V(T_i)})$ for some i, then let g_1 be a $\gamma_{r2}(T_i)$-function and define $h : V(G) \to \mathcal{P}(\{1, 2\})$ by $h(u) = f(u)$ for $u \in V(G) - V(T_i)$ and $h(u) = g_1(u)$ for $u \in V(T_i)$. Obviously h is a 2RDF of G with weight less than $\gamma_{r2}(G)$, a contradiction. So $\gamma_{r2}(T_i) = \omega(f|_{V(T_i)})$ for each $i \in \{0, \ldots, t\}$. If $T_i \notin \mathcal{F}$ for some i, then let g_2 be a $\gamma_{r2}(T_i)$-function that is not independent and define $h_1 : V(G) \to \mathcal{P}(\{1, 2\})$ by $h_1(u) = f(u)$ for $u \in V(G) - V(T_i)$ and $h_1(u) = g_2(u)$ for $u \in V(T_i)$. Clearly h_1 is a $\gamma_{r2}(G)$-function that is not independent, a contradiction with $\gamma_{r2}(G) \equiv i_{r2}(G)$. Similarly, we can see that v_i is an empty vertex of T_i for each $i \in \{1, \ldots, t\}$. If u_2 is an empty vertex of G then u_1 and u_3 are a c-pair of type 1 of T_0 and G can be obtained by using Operation $\mathcal{T}_2$. Now suppose there is a $\gamma_{r2}(G)$-function h that assigns $\{1, 2\}$ to u_2. Then u_1 and u_3 are a c-pair of type 2. If $T_i - v_i$, for some $1 \leq i \leq t$, has a 2RDF g of weight $\gamma_{r2}(T_i)$ that is not independent, then define $h_1(u) = h(u)$ for $u \in V(G) - V(T_i)$ and $h_1(u) = g(u)$ for $u \in V(T_i - v_i)$ and $h_1(v_i) = \emptyset$ to obtain a non independent $\gamma_{r2}(G)$-function, a contradiction with $\gamma_{r2}(G) \equiv i_{r2}(G)$. Thus $T_i - v_i \in \mathcal{F}$ for each $i \in \{1, \ldots, t\}$. If $T_0 - \{u_1, u_3\}$ has a 2RDF g of weight $\gamma_{r2}(T_0) - 1$ such that g is not independent and $j \in (\cup_{u \in N_{T_0}(u_1)} g(u)) \cap (\cup_{u \in N_{T_0}(u_3)} g(u))$ for some $j \in \{1, 2\}$, then define $h_2(u) = g(u)$ for $u \in V(T_0) - \{u_1, u_3\}$, $h_2(u_1) = h_2(u_3) = \emptyset$, $h_2(u_2) = \{1, 2\} - \{j\}$ and $h_2(u) = g_i(u)$ for $u \in V(T_i)$, $1 \leq i \leq t$, where g_i is arbitrary $\gamma_{r2}(T_i)$-function to obtain a non independent $\gamma_{r2}(G)$-function, a contradiction again. Thus the conditions of Operation $\mathcal{T}_3$ holds and G can be obtain from $T_0, T_1, \ldots, T_t$ by Operation $\mathcal{T}_3$. Finally let, there is a $\gamma_{r2}(G)$-function h that assign $\{1\}$ (the case $\{2\}$ is similar) to u_2. Then u_1 and u_3 are a c-pair of type 3. Since $\gamma_{r2}(G) \equiv i_{r2}(G)$, $h(v_i) = \emptyset$ for each $1 \leq i \leq t$. It follows that $2 \in N_{T_i}(v_i)$ for each $i \in \{1, \ldots, t\}$. If $\omega(h|_{V(T_i)}) < \gamma_{r2}(T_i)$ for some i, then the function $k : V(T_i) \to \mathcal{P}(\{1, 2\})$ defined by $k(v_i) = \{1\}$ and $k(w) = h(w)$ for $w \in V(T_i) - \{v_1\}$, is a $\gamma_{r2}(T_i)$-function that is not independent, a contradiction. Hence, $\omega(h|_{V(T_i)}) = \gamma_{r2}(T_i)$ for each i. We claim that for each i, $T_i - v_i$ has no 2RDF g of weight $\gamma_{r2}(T_i)$ that it is not independent and $j \in \cup_{u \in N_{T_i}(v_i)} g(u)$ for some $j \in \{1, 2\}$. Assume to the contrary that $T_i - v_i$ has an 2RDF g of weight $\gamma_{r2}(T_i)$ such that $j \in \cup_{u \in N_{T_i}(v_i)} g(u)$ $(j \in \{1, 2\})$ and g is not independent for some i. If $j = 2$, then the

function h_3 defined by $h_3(v_i) = \emptyset, h_3(u) = h(u)$ for $u \in V(G) - V(T_i)$ and $h_3(u) = g(u)$ for $u \in V(T_i)$, is a $\gamma_{r2}(G)$-function that is not independent, a contradiction with $\gamma_{r2}(G) \equiv i_{r2}(G)$. If $j = 1$, the defined h_4 by $h_4(v_i) = \emptyset, h_4(u) = h(u)$ for $u \in V(G) - V(T_i)$, $h_4(u) = \{1, 2\} \setminus g(u)$ if $u \in V(T_i)$ and $|g(u)| = 1$, and $h_4(u) = g(u)$ when $u \in V(T_i)$ and $|g(v_i)| \neq 1$, to obtain a $\gamma_{r2}(G)$-function that is not independent, a contradiction again. This proves the claim. As above we can see that $T_0 - \{u_1, u_3\}$ has no 2RDF g of weight $\gamma_{r2}(T_0) - 1$ that it is not independent and $j \in (\cup_{u \in N_{T_0}(u_1)} f(u)) \cap (\cup_{u \in N_{T_0}(u_2)} f(u))$ for some $j \in \{1, 2\}$. Therefore G can be obtained from $T_0, T_1, \ldots, T_t$ by Operation $\mathcal{T}_4$.

Subcase 2.2. $\deg(u_2) = 2$.
Assume, without loss of generality, that $f(u_1) = \{1\}$ and $f(u_3) = \{2\}$. Since $\gamma_{r2}(G) \equiv i_{r2}(G)$, f is independent and hence $f(u_4) = \emptyset$. By the choice of u_2 and the fact that $C_1 = \emptyset$, we have $\deg(u_4) = 2$ and $f(u_5) = \{1\}$. Repeating this process, we deduce that $f(u_{2m}) = \emptyset, f(u_{2m-1}) = \{1\}$ and $f(u_{2m+1}) = \{2\}$ for each integer $m \geq 1$. It follows that $k = 4s$ for some s and and every vertex of C with even index belongs to C_0. By the choice of u_2, we deduce that $\deg(u_2) = \cdots = \deg(u_k) = 2$. First let $C_0 - \{u_2, u_4, \ldots, u_k\} \neq \emptyset$. Assume, without loss of generality, that $u_3 \in C_0 - \{u_2, \ldots, u_k\}$. By the choice of u_2, $\deg(u_3) = 2$. Using an argument similar to that described in this subcase, we obtain $\{u_1, u_3, \ldots, u_{k-1}\} \subseteq C_0$ and $\deg(u_1) = \cdots = \deg(u_{k-1}) = 2$. Since G is connected, we deduce that $G = C_{4s}$. Now we can obtain G from $2s$ single vertices by Operation $\mathcal{T}_5$. Now let $C_0 = \{u_2, u_4, \ldots, u_k\}$. Assume $T_1, T_3, \ldots, T_{4s-1}$ are the components of $G - \{u_2, \ldots, u_k\}$ and $u_{2m-1} \in V(T_{2m-1})$ for $1 \leq m \leq 2s$. First we show that $\gamma_{r2}(T_{2m-1}) = \omega(f|_{V(T_{2m-1})})$ for each m. Clearly $f|_{V(T_{2m-1})}$ is an independent 2RDF of T_{2m-1} and hence $\gamma_{r2}(T_{2m-1}) \leq \omega(f|_{V(T_{2m-1})})$ for each m. Assume to the contrary that $\gamma_{r2}(T_{2m-1}) < \omega(f|_{V(T_{2m-1})})$ for some m and let g be a $\gamma_{r2}(T_{2m-1})$-function. If $|g(v_{2m-1})| \geq 1$, then we can assume that $f(v_{2m-1}) \subseteq g(v_{2m-1})$. Define the function h by $h(u) = g(u)$ if $u \in V(T_{2m-1})$ and $h(u) = f(u)$ otherwise, to obtain a 2RDF of G of weight less than $\gamma_{r2}(G)$, a contradiction. If $|g(v_{2m-1})| = 0$, then $|\cup_{u \in N_{T_{2m-1}}(v_{2m-1})} g(u)| = 2$. Define the function h by $h(u) = f(u)$ for $u \in V(G) - V(T_{2m-1} - v_{2m-1})$ and $h(u) = g(u)$ otherwise to obtain a $\gamma_{r2}(G)$-function that is not independent, a contradiction with $\gamma_{r2}(G) \equiv i_{r2}(G)$. Thus $\gamma_{r2}(T_{2m-1}) = \omega(f|_{V(T_{2m-1})})$ for each i. Now we show that $T_1, T_3, \ldots, T_{4s-1} \in \mathcal{H}$. Note that $f|_{V(T_{2m-1})}$ is an $i_{r2}(T_{2m-1})$-function with $|f(v_{2m-1})| = 1$ for $1 \leq m \leq 2s$. Let g be an $i_{r2}(T_{2m-1})$-function such that $|g(v_{2m-1})| = 1$. Let without loss of generality that $g(v_{2m-1}) = f(v_{2m-1})$. Then the function h defined by $h(u) = g(u)$ if $u \in V(T_{2m-1})$ and $h(u) = f(u)$ otherwise, is an $i_{r2}(G)$-function. Since $\gamma_{r2}(G) \equiv i_{r2}(G)$, h is independent and hence g is independent. Thus T_{2m-1} satisfies (i) in the definition of independent vertex. If there is a $\gamma_{r2}(T_{2m-1})$-function g with $|g(v_{2m-1})| = 2$, then $v_{2m-2}, v_{2m} \in C_1$ a contradiction with $C_1 = \emptyset$. Thus T_{2m-1} satisfies (ii) in the definition of independent vertex, for each m. Thus v_{2m-1} is an independent vertex of T_{2m-1} and hence $T_1, T_3, \ldots, T_{4s-1} \in \mathcal{H}$. Thus G can be obtain from $T_1, T_3, \ldots, T_{4s-1}$ by Operation $\mathcal{T}_5$. This completes the proof. □

Now we are ready to state the main theorem of this paper.

Theorem 2.9. Let G be a unicyclic graph. Then $i_{r2}(G) \equiv \gamma_{r2}(G)$ if and only if $G \in \mathcal{U}$.

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