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STAR-PATH AND STAR-STRIPE BIPARTITE RAMSEY NUMBERS IN MULTICOLORING

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Abstract. For given bipartite graphs $G_1, G_2, \ldots, G_t$, the bipartite Ramsey number $bR(G_1, G_2, \ldots, G_t)$ is the smallest integer n such that if the edges of the complete bipartite graph $K_{n,n}$ are partitioned into t disjoint color classes giving t graphs $H_1, H_2, \ldots, H_t$, then at least one H_i has a subgraph isomorphic to G_i. In this paper, we study the multicolor bipartite Ramsey number $bR(G_1, G_2, \ldots, G_t)$, in the case that $G_1, G_2, \ldots, G_t$ being either stars and stripes or stars and a path.

1. **Introduction**

In this paper, we only concerned with undirected simple finite graphs and we follow [1] for terminology and notations not defined here. For a given graph G, we denote its vertex set, edge set, maximum degree and minimum degree by $V(G)$, $E(G)$, $\Delta(G)$ and $\delta(G)$, respectively, and for a vertex $v \in V(G)$, we use $\deg_G(v)$ (or simply $\deg(v)$) and $N_G(u)$ to denote the degree and neighbors of v in G, respectively. As usual, a cycle and a path on m vertices are denoted by C_m and P_m, respectively. Also the complete bipartite graph with partite set (X, Y) is denoted by $K[X, Y]$ and if $|X| = m$ and $|Y| = n$ briefly we denote it by $K_{m,n}$. Also by a *stripe* mK_2 we mean a graph on $2m$ vertices and m independent edges.

Recall that a *proper edge coloring* of a graph $G = (V, E)$ is assigning colors to the edges so that any two edges having end vertex in common, have different colors. The minimum number of colors required for a proper edge coloring of G is called the *chromatic index* of G and denoted by $\chi'(G)$. For a bipartite graph G, it is well known [1] that $\chi'(G) = \Delta(G)$.

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Ramsey theory explores the question of how big a structure must be to contain a certain substructure or substructures. Since the 1970's, Ramsey theory has grown into one of the most active areas of research within combinatorics, overlapping variously with graph theory, number theory, geometry and logic. For given graphs $G_1, G_2, \ldots, G_t$, the *multicolor Ramsey number* $R(G_1, G_2, \ldots, G_t)$, is defined to be the smallest integer n such that if the edges of the complete graph K_n are colored in any fashion with t colors, then for some i, $1 \leq i \leq t$, the spanning subgraph whose edges are colored with the i-th color, contains a copy of G_i. The existence of such a positive integer is guaranteed by Ramsey's classical theorem. Determining $R(G_1, G_2, \ldots, G_t)$ for general graphs appears to be a difficult problem and a survey including many results on Ramsey theory can be found in [8].

Bipartite Ramsey problems deal with the same questions but the graph explored is the complete bipartite graph instead of the complete graph. Let $G_1, G_2, \ldots, G_t$ be bipartite graphs. The *multicolor bipartite Ramsey number* $bR(G_1, G_2, \ldots, G_t)$ is the smallest positive integer n such that if the edges of the complete bipartite graph $K_{n,n}$ are partitioned into t disjoint color classes giving t graphs $H_1, H_2, \ldots, H_t$, then at least one H_i has a subgraph isomorphic to G_i. The existence of such a positive integer is guaranteed by a result of Erdős and Rado [2]. It is easy to see that for bipartite graphs $G_1, G_2, \ldots, G_t$ we have $R(G_1, G_2, \ldots, G_t) \leq 2bR(G_1, G_2, \ldots, G_t)$. The bipartite case has also been studied extensively. For $n \geq 21$, Irving [7] showed that $bR(K_{n,n}, K_{n,n}) < 2^{n-1}(n-1)$. In addition, the asymptotics for $bR(K_{n,n}, K_{n,n})$ (see [6]) are the same as those of the classical Ramsey number: For all sufficiently large n, $bR(K_{n,n}, K_{n,n}) > \frac{\sqrt{2}}{e} n 2^{n/2}$. An upper bound for $bR(K_{m,m}, K_{n,n})$ is given in [6]:

$$bR(K_{n,n}, K_{n,n}) \leq \binom{m+n}{m} - 1.$$

Exact solutions were given for simpler cases of the problem. The exact value of the bipartite Ramsey number of paths, $bR(P_n, P_m)$, follows from a special case of some results of Faudree and Schelp [3] and Gyárfás and Lehel [4]. Also the bipartite Ramsey number $bR(K_{1,n}, P_m)$ was determined by Hatting and Henning in [5]. In this paper, we study the multicolor bipartite Ramsey number $bR(G_1, G_2, \ldots, G_t)$, in the case that $G_1, G_2, \ldots, G_t$ being either stars and stripes or stars and a path.

2. Main results

In this section, we establish the main results of the paper. Before that, we give some lemmas which help to prove main results of the paper. Through the paper, for a t-edge coloring of a graph H with colors $\alpha_1, \alpha_2, \ldots, \alpha_t$ we denote by H_i, $1 \leq i \leq t$, the subgraph of H induced by the edges of color α_i. Also for given integers $n_1, n_2, \ldots, n_t$, we use Σ to denote $\sum_{i=1}^{t}(n_i - 1)$.

Lemma 2.1. *If G is a bipartite graph with $\delta(G) \geq \delta$ and at least 2δ vertices in each partite set, then G contains a matching with at least 2δ edges.*

Proof. Let $G = (U, W)$ and M be a maximum matching in G. On the contrary, let $|E(M)| < 2\delta$. The maximality of M implies that for any two M-unsaturated vertices $u \in U$ and $w \in W$ we have

$uw \notin E(G)$, which means that $N(u) \subseteq W \cap V(M)$ and $N(w) \subseteq U \cap V(M)$. Since $|U|, |W| \geq 2\delta$, there exist vertices $u \in U$ and $w \in W$ such that u and w are M-unsaturated. But $\delta(G) \geq \delta$, implies that vertices u and w have at least δ neighbors in $W \cap V(M)$ and $U \cap V(M)$, respectively. Therefore there exists an edge $e = xy \in M$ such that $xw, yu \in E(G)$. Now, $M' = (M \setminus \{e\}) \cup \{xw, yu\}$ is a matching in G with $|M'| > |M|$, which contradicts the maximality of M. This observation shows that $|M| \geq 2\delta$ and this completes the proof □

Lemma 2.2. *Let $n_1, n_2, \ldots, n_t$ be positive integers and H be a graph with $\chi'(H) \leq \Sigma$. Then H can be decomposed into the edge-disjoint subgraphs $H_1, H_2, \ldots, H_t$ such that $\Delta(H_i) \leq n_i - 1$.*

Proof. Let c be a proper edge-coloring of H with $\chi'(H)$ colors. Partition these colors into t classes $A_1, A_2, \ldots, A_t$ of sizes at most $n_1 - 1, n_2 - 1, \ldots, n_t - 1$, respectively. Let H_i be the subgraph of H induced by the edges of colors in A_i. Since each A_i, $1 \leq i \leq t$, contains at most $n_i - 1$ colors, each H_i has maximum degree at most $n_i - 1$ and so H_i's are the desired subgraphs which decompose H. □

Now we are ready to compute $bR(K_{1,n_1}, K_{1,n_2}, \ldots, K_{1,n_t}, mK_2)$.

Theorem 2.3. *If $n_1, n_2, \ldots, n_t$ and m are positive integers, then*

$$bR(K_{1,n_1}, K_{1,n_2}, \ldots, K_{1,n_t}, mK_2) = \begin{cases} m & \text{if} \quad \Sigma < \lfloor \frac{m}{2} \rfloor, \\ \Sigma + \lfloor \frac{m-1}{2} \rfloor + 1 & \text{if} \quad \Sigma \geq \lfloor \frac{m}{2} \rfloor. \end{cases}$$

Proof. Set $bR = bR(K_{1,n_1}, K_{1,n_2}, \ldots, K_{1,n_t}, mK_2)$ and let $C = \{\alpha_1, \alpha_2, \ldots, \alpha_{t+1}\}$ be a set of $t+1$ colors. First let $\Sigma < \frac{m}{2}$. Coloring edges of $K_{m-1,m-1}$ by color α_{t+1} yields a coloring of $K_{m-1,m-1}$ with $t+1$ colors which contains no monochromatic K_{1,n_i} in color α_i, $1 \leq i \leq t$, and no monochromatic mK_2 in color α_{t+1}, means that $bR \geq m$. Now let c be any $(t+1)$-edge coloring of $G = K_{m,m}$ with color set C such that for $i = 1, 2, \ldots, t$, G contains no monochromatic K_{1,n_i} in color α_i. We prove that G must contain a monochromatic copy of mK_2 in color α_{t+1}. For each i, $1 \leq i \leq t+1$, let G_i be the subgraph of G induced by the edges of color α_i. Clearly for each vertex v of G we have $\deg_{G_i}(v) \leq n_i - 1$, $1 \leq i \leq t$, and so $\deg_{G_{t+1}}(v) \geq m - \Sigma > \frac{m}{2}$. By Lemma 2.1, G_{t+1} contains a copy of mK_2, which shows that $bR \leq m$ and so $bR = m$.

Let $\Sigma \geq \frac{m}{2}$ and $n = \Sigma + \lfloor \frac{m-1}{2} \rfloor + 1$. First we prove that $bR \geq n$. For this purpose, let $H = K_{n-1,n-1}$ with partite set $(V_1 \cup U_1, V_2 \cup U_2)$, where $|U_i| = \Sigma$ and $|V_i| = \lfloor \frac{m-1}{2} \rfloor$, $i = 1, 2$. Color edges between V_i and U_j, $i \neq j$, by color α_{t+1} and let H_{t+1} be the spanning subgraph of H induced by the edges of color α_{t+1}. Also let $\overline{H}$ be the spanning subgraph of H with edge set $E(\overline{H}) = E(H) \setminus E(H_{t+1})$. Clearly, $\overline{H}$ is a bipartite graph with $\chi(\overline{H}) = \Delta(\overline{H}) = \Sigma$ and so by Lemma 2.2, $\overline{H}$ is the union of edge-disjoint graphs $H_1, H_2, \ldots, H_t$ such that $\Delta(H_i) \leq n_i - 1$, $1 \leq i \leq t$. Coloring edges of H_i, $1 \leq i \leq t$, with color α_i, yields a $(t+1)$-edge coloring of H without monochromatic copy of K_{1,n_i} in color α_i, $1 \leq i \leq t$, and monochromatic copy of mK_2 in color α_{t+1}, which means that $bR \geq n$.

Now let c be any $(t+1)$-edge coloring of $G = K_{n,n}$ with colors $\alpha_1, \alpha_2, \ldots, \alpha_{t+1}$ such that for $i = 1, 2, \ldots, t$, G contains no monochromatic copy of K_{1,n_i} in color α_i. We prove that $K_{n,n}$ must contain a monochromatic copy of mK_2 in color α_{t+1}. Clearly for each vertex v of G we have $\deg_{G_i}(v) \leq n_i - 1$, $1 \leq i \leq t$, and so $\deg_{G_{t+1}}(v) \geq n - \Sigma = \lfloor \frac{m-1}{2} \rfloor + 1$. Using Lemma 2.1, G_{t+1} contains matching M with at least $2(\lfloor \frac{m-1}{2} \rfloor + 1) \geq m$ edges, i.e. $mK_2 \subseteq G_{t+1}$ and so the proof is completed. $\square$

Theorem 2.4. *Let $m_1, m_2, \ldots, m_s$ and $n_1, n_2, \ldots, n_t$ be positive integers with $\Lambda = \Sigma_{i=1}^{s}(m_i - 1)$ and $\Sigma = \Sigma_{i=1}^{t}(n_i - 1)$. Then $bR(K_{1,n_1}, K_{1,n_2}, \ldots, K_{1,n_t}, m_1K_2, \ldots, m_sK_2) = n$, where*

$$n = \begin{cases} \Lambda + 1 & \text{if} \quad \Sigma < \lfloor \frac{\Lambda+1}{2} \rfloor, \\ \Sigma + \left\lfloor \frac{\Lambda}{2} \right\rfloor + 1 & \text{if} \quad \Sigma \geq \lfloor \frac{\Lambda+1}{2} \rfloor. \end{cases}$$

Proof. Consider an arbitrary edge coloring of $K_{n,n}$ with colors $\alpha_1, \alpha_2, \ldots, \alpha_{t+s}$ and let there is no monochromatic copy of K_{1,n_i} in color α_i, $1 \leq i \leq t$. Using Theorem 2.3, there exists a copy of $(\Lambda+1)K_2$ such that its edges are colored by α_{t+j}, $1 \leq j \leq s$, which implies that there is a monochromatic copy of m_jK_2 for some j, $1 \leq j \leq s$. This means that $bR(K_{1,n_1}, K_{1,n_2}, \ldots, K_{1,n_t}, m_1K_2, \ldots, m_sK_2) \leq n$.

To see the reverse inequality, first let $\Sigma \geq \lfloor \frac{\Lambda+1}{2} \rfloor$ and consider $H = K_{n-1,n-1}$ with partite set $(V_1 \cup U_1, V_2 \cup U_2)$ so that $|U_i| = \Sigma$ and $|V_i| = \lfloor \frac{\Lambda}{2} \rfloor$, $i = 1, 2$. Let m_i be even when $1 \leq i \leq l$ and odd otherwise. Partition V_1 (resp. V_2) into sets $X_1, X_2, \ldots, X_s$ (resp. $Y_1, Y_2, \ldots, Y_s$) so that $|X_i| = \lfloor \frac{m_i-1}{2} \rfloor + t$ (resp. $|Y_i| = \lfloor \frac{m_i-1}{2} \rfloor + t'$), where $t = 1$ if $1 \leq i \leq \lfloor \frac{l}{2} \rfloor$ and $t = 0$ otherwise (resp. $t' = 1$ if $\lceil \frac{l}{2} \rceil + 1 \leq i \leq l$ and $t' = 0$ otherwise). Color edges $[X_i, U_2]$ and $[Y_i, U_1]$ by color α_{t+i}, $1 \leq i \leq s$, and let H' be the spanning subgraph of H induced by edges $[X_i, U_2] \cup [Y_i, U_1]$. Also let $\overline{H}$ be the spanning subgraph of H with edge set $E(\overline{H}) = E(H) \setminus E(H')$. Clearly $\overline{H}$ is a bipartite graph with maximum degree and chromatic index Σ and so by Lemma 2.2, edges of $\overline{H}$ can be colored by colors $\alpha_1, \alpha_2, \ldots, \alpha_t$ so that there is no monochromatic copy of K_{1,n_i} in color α_i, $1 \leq i \leq t$. This yields a $(t+s)$-edge coloring of H with no monochromatic copy of K_{1,n_i} in color α_i, $1 \leq i \leq t$, and no copy of m_jK_2 in color α_j, $t+1 \leq j \leq s$.

Now, let $\Sigma < \lfloor \frac{\Lambda+1}{2} \rfloor$ and consider $H = K_{\Lambda,\Lambda}$ with partite set (U, V). Partition U into sets $X_1, X_2, \ldots, X_s$ where $|X_i| = m_i - 1$, $1 \leq i \leq s$, and color edges $[X_i, V]$ by color α_{t+i}, $1 \leq i \leq s$. In this coloring of H three is no monochromatic copy of K_{1,n_i} in color α_i, $1 \leq i \leq t$, and no monochromatic copy of m_jK_2 in color α_j, $t+1 \leq j \leq s$, which completes the proof. $\square$

In the sequel, we extend the result of Hatting and Henning [5] by determining the multicolor bipartite Ramsey number $bR(K_{1,n_1}, K_{1,n_2}, \ldots, K_{1,n_t}, P_m)$. In [5] the authors proved the following.

Theorem 2.5. *For integers $m, n \geq 2$,*

$$bR(P_m, K_{1,n}) = \begin{cases} \frac{m}{2} + n - 1 & \text{if} \quad n \geq \frac{m}{2} + 1, \ m \text{ even}, \\ 2n - 1 & \text{if} \quad \frac{1}{2}\lfloor \frac{m}{2} \rfloor + 1 \leq n < \lfloor \frac{m}{2} \rfloor + 1, \\ \frac{m-1}{2} + n & \text{if} \quad n \geq \frac{m-1}{2} + 1, \ m \text{ odd}, \ n - 1 \equiv 0 \ mod(\frac{m-1}{2}), \\ \frac{m-1}{2} + n - 1 & \text{if} \quad n \geq \frac{m-1}{2}, \ m \text{ odd}, \ n - 1 \not\equiv 0 \ mod(\frac{m-1}{2}), \\ \lfloor \frac{m+1}{2} \rfloor & \text{if} \quad n < \frac{1}{2}\lfloor \frac{m}{2} \rfloor + 1. \end{cases}$$

To determine the multicolor bipartite Ramsey number of paths versus stars and a path, we need the following lemma.

Lemma 2.6. *If H is an arbitrary graph, then $bR(K_{1,n_1}, K_{1,n_2}, \ldots, K_{1,n_t}, H) \leq bR(K_{1,\Sigma+1}, H)$.*

Proof. Let $n = bR(K_{1,\Sigma+1}, H)$ and c be an arbitrary $(t+1)$-edge coloring of $G = K_{n,n}$ with colors $\alpha_1, \alpha_2, \ldots, \alpha_{t+1}$. Recolor edges whit colors $\alpha_1, \alpha_2, \ldots, \alpha_t$ by a new color α and retain the color of the remaining edges. This yields a 2-edge coloring of G by colors α and α_{t+1}. Since $n = bR(K_{1,\Sigma+1}, H)$ so G contains a copy of $K_{1,\Sigma+1}$ of color α or a copy of H of color α_{t+1}. If the first case occurs, return to c, restricted to this set of edges which clearly we have a monochromatic copy of K_{1,n_i} in color α_i for some i, $1 \leq i \leq t$, otherwise we obtain a monochromatic copy of H in color α_{t+1}. This observation completes the proof. $\square$

Theorem 2.7. *Let m be a positive integer and $bR = bR(K_{1,n_1}, K_{1,n_2}, \ldots, K_{1,n_t}, P_m)$. Then $bR = n$, where*

$$n = \begin{cases} \lfloor \frac{m+1}{2} \rfloor & \text{if } \ \Sigma < \frac{1}{2}\lfloor \frac{m}{2} \rfloor, \\ 2\Sigma + 1 & \text{if } \ \frac{1}{2}\lfloor \frac{m}{2} \rfloor \leq \Sigma < \lfloor \frac{m}{2} \rfloor, \\ \Sigma + \frac{m}{2} & \text{if } \ \Sigma \geq \frac{m}{2}, \ m \text{ even}, \\ \Sigma + \frac{m+1}{2} & \text{if } \ \Sigma \geq \frac{m-1}{2}, \ m \text{ odd}, \ \Sigma \equiv 0 \ mod(\frac{m-1}{2}), \\ \Sigma + \frac{m-1}{2} & \text{if } \ \Sigma \geq \frac{m-1}{2}, \ m \text{ odd}, \ \Sigma \not\equiv 0 \ mod(\frac{m-1}{2}). \end{cases}$$

Proof. Using Lemma 2.6 and Theorem 2.5 we have $bR \leq n$. To see $bR \geq n$, we give a decomposition of $H = K_{n-1,n-1}$ into edge-disjoint union graphs $H_1, H_2, \ldots, H_{t+1}$ such that $K_{1,n_i} \nsubseteq H_i$, $1 \leq i \leq t$, and $P_m \nsubseteq H_{t+1}$. If $\Sigma < \frac{1}{2}\lfloor \frac{m}{2} \rfloor$, the assertion holds by assuming H_i, $1 \leq i \leq t$, is trivial and $H_{t+1} \cong H$. Now, consider the following cases.

Case 1. $\frac{1}{2}\lfloor \frac{m}{2} \rfloor \leq \Sigma < \lfloor \frac{m}{2} \rfloor$.

Let $H_{t+1} \cong 2K_{\Sigma,\Sigma}$ and let $\overline{H}$ be the complement of H_{t+1} relative to H. Clearly, $\overline{H}$ is a bipartite graph with $\chi(\overline{H}) = \Delta(\overline{H}) = \Sigma$ and so Lemma 2.2 gives the desired decomposition of H.

Case 2. m is even and $\Sigma \geq \frac{m}{2}$.

Let $\Sigma = p(\frac{m}{2}) + r$, where $p \geq 1$ and $0 \leq r < \frac{m}{2}$. Consider the complete bipartite graph H with partite sets U and V such that $|U| = |V| = \Sigma + \frac{m}{2} - 1$. Partition U and V into sets $U_1, U_2, \ldots, U_{p+2}$ and $V_1, V_2, \ldots, V_{p+2}$, respectively such that for $i = 1, 2, \ldots, p+1$, $|U_i| = |V_i| = \frac{m}{2} - 1$ and $|U_{p+2}| = |V_{p+2}| = p + r$. Suppose that $H_{t+1} \cong \bigcup_{i=1}^{p+1} K[U_i, V_i] \cup K[U_{p+2}, V_{p+1}] \cup K[V_{p+2}, U_p]$ and $\overline{H}$ is the complement of H_{t+1} relative to H. Clearly, $\overline{H}$ is a bipartite graph with $\chi(\overline{H}) = \Delta(\overline{H}) = \Sigma$ and so by Lemma 2.2, $\overline{H}$ can be written as a union of t subgraphs $H_1, H_2, \ldots, H_t$ such that $K_{1,n_i} \nsubseteq H_i$, for $i = 1, 2, \ldots, t$. Furthermore, the longest path in H_{t+1} has order $2(\frac{m}{2} - 1) + 1 = m - 1$, so $P_m \nsubseteq H_{t+1}$.

Case 3. m is odd, $\Sigma \geq \frac{m-1}{2}$ and $\Sigma \equiv 0 \text{ mod}(\frac{m-1}{2})$.

Let $\Sigma = p(\frac{m-1}{2})$ where $p \geq 1$ and also let $H_{t+1} = (p+1)K_{\frac{m-1}{2},\frac{m-1}{2}}$. Clearly each partite set has $\Sigma + \frac{m-1}{2}$ vertices and the longest path in H_{t+1} has $2(\frac{m-1}{2}) = m-1$ vertices. So $P_m \nsubseteq H_{t+1}$. Let $\overline{H}$ be the complement of H_{t+1} relative to H. Clearly $\chi(\overline{H}) = \Delta(\overline{H}) \leq \Sigma$ and by Lemma 2.2, $\overline{H}$ is the edge-disjoint union of graphs H_i, $1 \leq i \leq t$, so that $K_{1,n_i} \nsubseteq H_i$.

Case 4. m is odd, $\Sigma \geq \frac{m-1}{2}$ and $\Sigma \not\equiv 0 \bmod(\frac{m-1}{2})$.

Let $\Sigma = p(\frac{m-1}{2}) + r$ where $p \geq 1$ and $0 < r < \frac{m-1}{2}$. Consider the complete bipartite graph H with partite sets U and V such that $|U| = |V| = \Sigma + \frac{m-1}{2} - 1$. Note that $\Sigma + \frac{m-1}{2} - 1 = (p+1)(\frac{m-1}{2}) + r - 1$. Partition U and V into sets $U_1, U_2, \ldots, U_{p+2}$ and $V_1, V_2, \ldots, V_{p+2}$, respectively such that $|U_i| = |V_i| = \frac{m-1}{2}$ for $i = 1, 2, \ldots, p-1$, $|U_p| = |V_{p+1}| = \frac{m-1}{2} - 1$, $|U_{p+1}| = |V_p| = \frac{m-1}{2}$ and $|U_{p+2}| = |V_{p+2}| = r$. Suppose that $H_{t+1} \cong \bigcup_{i=1}^{p+1} K[U_i, V_i] \cup K[U_p, V_{p+2}] \cup K[U_{p+2}, V_{p+1}]$ and $\overline{H}$ is the complement of H_{t+1} relative to H. Clearly $\Delta(\overline{H}) \leq \Sigma$ and so $\overline{H}$ can be written as an edge-disjoint union of t subgraphs $H_1, H_2, \ldots, H_t$ such that $K_{1,n_i} \nsubseteq H_i$, for $i = 1, 2, \ldots, t$. Furthermore, the longest path in H_{t+1} has $2(\frac{m-1}{2}) = m-1$ vertices and so $P_m \nsubseteq H_{t+1}$. □

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