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CHROMATIC AND CLIQUE NUMBERS OF A CLASS OF PERFECT GRAPHS

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ABSTRACT. Let p be a prime number and n be a positive integer. The graph $G_p(n)$ is a graph with vertex set $[n] = \{1, 2, \ldots, n\}$, in which there is an arc from u to v if and only if $u \neq v$ and $p \nmid u + v$. In this paper it is shown that $G_p(n)$ is a perfect graph. In addition, an explicit formula for the chromatic number of such graph is given.

1. Introduction

Let $G = (V, E)$ be a graph on the vertex set $V = [n] := \{1, 2, \ldots, n\}$ and the edge set E. The *clique number* $\omega(G)$ of G is the greatest integer r such that the complete graph on r vertices is an induced subgraph of G. A *d-coloring* of G is a map $c : V \to \{1, 2, \ldots, d\}$ such that $c(v) \neq c(u)$ whenever vu is an edge. The smallest integer k such that G has a k-coloring is the *chromatic number* of G and denoted by $\chi(G)$. A graph G is called *weakly perfect* if $\chi(G) = \omega(G)$. A graph is *perfect* if every induced subgraph is weakly perfect. Hence, every perfect graph is weakly perfect and there are several classes which show the converse does not hold in general. For instance, let C be an odd cycle and $uw \in E_C$ and let H denote the graph obtained from adding a new vertex v together with two new edges vu and vw to C. It is easy to see that $\chi(H) = 3 = \omega(H)$ and $\chi(C) = 3 > 2 = \omega(C)$. So, H is weakly perfect, but it is not perfect because C is an induced subgraph of H. In [2] one can find 120 classes of perfect graphs.

Let p be a prime number and n be a positive integer. The graph $G_p(n)$ is a graph with vertex set $[n] = \{1, 2, \ldots, n\}$, in which there is an arc from u to v if and only if $u \neq v$ and $p \nmid u + v$. In this paper it is shown that $G_p(n)$ is a perfect graph. In addition, we will give an explicit formula for chromatic numbers.

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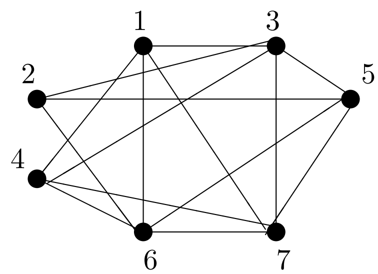
2. Main Results

Let p be a prime number and let n be an arbitrary positive integer. We define the graph $G_p(n)$ as follows

$$V(G_p(n)) = [n], and$$
$$E(G_p(n)) = \{uv|u, v \in [n], \quad u \neq v, and \quad p \nmid u + v\}.$$

Remark 2.1. *Let n be a power of p, say $n = p^k$ for some positive integer k. Then $G_p(n)$ is the graph $G(p^k)$ introduced in* [3].

The graph $G_3(7)$ demonstrate in the following figure.



$\chi(G_3(7)) = \omega(G_3(7)) = 3$

Let m be a positive integer. A graph G is called m-chordal if it does not have any induced cycle of length more than m.

Theorem 2.2. *Let p be a prime number and let n be an arbitrary positive integer. Then $G_p(n)$ is a perfect graph.*

Proof. Let $H_p(n) = \overline{G_p(n)}$ be the complement of the graph $G_p(n)$ which is defined as the graph whose vertices are $1, 2, \ldots, n$ and two vertices i, j are adjacent if and only if $i + j$ is a multiple of p.

For $p = 2$ it means that the graph is the disjoint union of two complete graphs, those on the odd numbers and on the even numbers. For $p > 2$ the prime number condition implies that p is odd. Therefore the numbers divisible by p induce a complete component, and for $0 < i < p$ the numbers congruent with i together with the numbers congruent with $p - i$ induce a complete bipartite graph component.

It is immediate that complete graphs and complete bipartite graphs are perfect. It is also immediate that the vertex-disjoint union of perfect graphs is perfect. Therefore $H_p(n)$ and so $G_p(n)$ are perfect. □

In the rest of this paper we give an explicit formula for chromatic number (equivalently clique number) of $G_p(n)$. First we state some notations. Let n be a positive integer and let p be an odd prime number. We denote by $[n]_p$, the unique remainder of n modulo p. Let $[n]'_p$ define as follows

$$[n]'_p = \begin{cases} [n]_p & \text{if } [n]_p < \frac{p-1}{2}, \\ \\ \frac{p-1}{2} & \text{if } [n]_p \geq \frac{p-1}{2}. \end{cases}$$

Theorem 2.3. *Let p be a prime number and let n be a positive integer. Then*

$$\chi(G_p(n)) = \omega(G_p(n)) = \begin{cases} 2 & \text{if } p = 2, \\ \lfloor \frac{n}{p} \rfloor (\frac{p-1}{2}) + [n]'_p + 1 & \text{if } p > 2. \end{cases}$$

Proof. If $p = 2$, then there is nothing to prove. Consider that $p > 2$ and let $w = \lfloor \frac{n}{p} \rfloor (\frac{p-1}{2}) + [n]'_p + 1$. Define

$$W = \{p\} \cup \{x \in \{1, \ldots, n\} | [x]_p \in \{1, \ldots, \frac{p-1}{2}\}\}.$$

It is easy to see that W is a clique. Now lets compute the cardinality of W. We have the following partition $W = \bigcup W_i$ where for each i, $W_i = \{x \in W | [x]_p = i\}$. Hence

$$\begin{aligned} |W| &= \sum_{0}^{(p-1)/2} |W_i| \\ &= 1 + \underbrace{\lceil \frac{n}{p} \rceil + \ldots + \lceil \frac{n}{p} \rceil}_{[n]'_p \text{ times}} + \lfloor \frac{n}{p} \rfloor + \ldots + \lfloor \frac{n}{p} \rfloor = w. \end{aligned}$$

So, we have $\omega(G_p(n)) \geq w$.

Now we will show the converse. Consider the following map

$$\gamma : V((G_p(n)) \longmapsto \{0, \ldots, p-1\}$$

such that $\gamma(x) = [x]_p$ and assume that W is a clique in $G_p(n)$. For all x and y in W we have $\gamma(x) \neq \gamma(y)$, so the pigeonhole principle yields that

$$|\gamma(W) \cap \{1, \ldots, p-1\}| \leq (\frac{p-1}{2})$$

on the other hand we have

$$\gamma^{-1}(i) = \begin{cases} \lceil \frac{n}{p} \rceil & \text{if } [n]_p \geq i, \\ \\ \lfloor \frac{n}{p} \rfloor & \text{if } [n]_p < i. \end{cases}$$

therefore

$$\begin{aligned} |W| &\leq |\gamma^{-1}(\gamma(W))| \\ &\leq 1 + \underbrace{\lceil \frac{n}{p} \rceil + \ldots + \lceil \frac{n}{p} \rceil}_{[n]'_p \text{ times}} + \underbrace{\lfloor \frac{n}{p} \rfloor + \ldots + \lfloor \frac{n}{p} \rfloor}_{(\frac{p-1}{2}) - [n]'_p \text{ times}} = w \end{aligned}$$

as desired. □

Corollary 2.4. [3, Theorem 2.1.] *Let p be a prime number and let k be a positive integer. Then*

$$\chi(G_p(p^k)) = \omega(G_p(p^k)) = \begin{cases} 2 & \text{if } p = 2, \\ p^{k-1}(\frac{p-1}{2}) + 1 & \text{if } p > 2. \end{cases}$$

Proof. It is easy to see that $\lfloor \frac{p^k}{p} \rfloor = p^{k-1}$ and $[p^k]'_p = 0$. □

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