ON THE HARMONIC INDEX OF GRAPH OPERATIONS

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Abstract. The harmonic index of a connected graph $G$, denoted by $H(G)$, is defined as
$H(G) = \sum_{uv \in E(G)} \frac{2}{d_u + d_v}$ where $d_v$ is the degree of a vertex $v$ in $G$. In this paper, expressions for the Harary
indices of the join, corona product, Cartesian product, composition and symmetric difference of graphs
are derived.

1. Introduction and Preliminaries

Throughout this paper we consider only simple connected graphs, i.e. connected graphs without
loops and multiple edges. For a graph $G$, $V(G)$ and $E(G)$ denote the set of all vertices and edges,
respectively. For a graph $G$, the degree of a vertex $v$ is the number of edges incident to $v$ and denoted
by $d_v$.

The Harmonic index $H(G)$ is vertex-degree-based topological index. This index first appeared in [3],
and was defined as [13], [16]

$$H(G) = \sum_{uv \in E(G)} \frac{2}{d_u + d_v}$$

The connectivity index introduced in 1975 by Milan Randic [12], who has shown this index to reflect
molecular branching. Randic index was defined as follows;

$$\chi(G) = \sum_{uv \in E(G)} \frac{1}{\sqrt{d_u d_v}}$$

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Ernesto Estrada et al. [3], introduced atom-bond connectivity (ABC) index, which it has been applied up until now to study the stability of alkanes and the strain energy of cycloalkanes. This index is defined as follows:

$$ABC(G) = \sum_{e=uv \in E(G)} \sqrt{\frac{d_u + d_v - 2}{d_ud_v}}$$

For a vertex $v$ in a connected nontrivial graph $G$, The eccentricity $ecc_G(v)$ of a vertex $v$ is the greatest geodesic distance between $v$ and any other vertex. The Diameter $d(G)$ of $G$ is defined as $d(G) = \max\{ecc_G(v) | v \in V(G)\}$.

The Composition (also called lexicographic product [6]) $G = G_1[G_2]$ of graphs $G_1$ and $G_2$ with disjoint vertex sets $V(G_1)$ and $V(G_2)$ and edge sets $E(G_1)$ and $E(G_2)$ is the graph with vertex set $V(G_1) \times V(G_2)$ and $(u_i, v_j)$ is adjacent with $(u_k, v_l)$ whenever $u_i$ is adjacent with $u_k$, or $u_i = u_k$ and $v_j$ is adjacent with $v_l$.

The Cartesian product $G_1 \times G_2$ of graphs $G_1$ and $G_2$ has the vertex set $V(G_1 \times G_2) = V(G_1) \times V(G_2)$ and $(u_i, v_j)(u_k, v_l)$ is an edge of $G_1 \times G_2$ if $u_i = u_k$ and $v_jv_l \in E(G_2)$, or $u_iu_k \in E(G_1)$ and $v_j = v_l$.

For given graphs $G_1$ and $G_2$ we define their Corona product $G_1{\circ}G_2$ as the graph obtained by taking $|V(G_1)|$ copies of $G_2$ and joining each vertex of the $i$-th copy with vertex $v_i \in V(G_1)$. Obviously, $|V(G_1{\circ}G_2)| = |V(G_1)|(1 + |V(G_2)|)$ and $|E(G_1{\circ}G_2)| = |E(G_1)| + |V(G_1)||V(G_2)| + |E(G_2)|$.

A sum $G_1 + G_2$ of two graphs $G_1$ and $G_2$ with disjoint vertex sets $V(G_1)$ and $V(G_2)$ is the graph on the vertex set $V(G_1) \cup V(G_2)$ and the edge set $E(G_1) \cup E(G_2) \cup \{uv | u \in V(G_1), v \in V(G_2)\}$. Hence, the sum of two graphs is obtained by connecting each vertex of one graph to each vertex of the other graph, while keeping all edges of both graphs [10].

The symmetric difference $G_1 \oplus G_2$ of two graphs $G_1$ and $G_2$ is the graph with vertex set $V(G_1) \times V(G_2)$ and $E(G_1 \oplus G_2) = \{(u_1, u_2)(v_1, v_2) | u_1v_1 \in E(G_1) \text{ or } u_2v_2 \in E(G_2) \text{ but not both}\}$. Obviously,$|E(G_1 \oplus G_2)| = |E(G_1)||V(G_2)| + |E(G_2)||V(G_1)| - 2|E(G_1)||E(G_2)|$.

In this paper, expressions for the Harmonic indices of the join, corona product, Cartesian product, composition and symmetric difference of graphs are derived [2, 3, 11, 13, 14, 15].

2. Harmonic index of graph operations

**Theorem 2.1.** Let $G_1$ and $G_2$ be two connected graphs with order $n_1$, $n_2$ and size $m_1$, $m_2$ respectively. Then

$$H(G_1[G_2]) \leq \frac{1}{(1 + n_2)^2} \left\{ \frac{n_1}{2} ABC(G_2) + n_1R(G_2) + n_1n_2m_2 
+ \frac{(n_2)^3}{2} ABC(G_1) + (n_2)^3R(G_1) + (n_2)^2m_1 \right\}.$$
Proof. Let $V(G_1) = \{u_1, u_2, \ldots, u_n\}$ and $V(G_2) = \{v_1, v_2, \ldots, v_{n_2}\}$ be a set of vertex for $G_1$ and $G_2$ respectively. By the definition of the composition of two graph one can see that,

$$|E(G_1[G_2])| = |E(G_1)||V(G_2)|^2 + |E(G_2)||V(G_1)|$$

$$d_{G_1[G_2]}(u, v) = |V(G_2)|d_{G_1}(u) + d_{G_2}(v)$$

$$H(G_1[G_2]) = \sum_{(u,v),(u',v') \in E(G_1[G_2]), (u,v) \neq (u',v')} \frac{2}{d_{G_1[G_2]}(u,v) + d_{G_1[G_2]}(u',v')}$$

$$= \sum_{(u,v),(u',v') \in E(G_1[G_2]), j \neq l} \frac{2}{d_{G_1[G_2]}(u,v) + d_{G_1[G_2]}(u',v')}$$

$$+ \sum_{(u,v),(u',v') \in E(G_1[G_2]), i \neq k} \frac{2}{d_{G_1[G_2]}(u,v) + d_{G_1[G_2]}(u',v')}$$

$$= \sum_{u \in V(G_1)} \sum_{v \in V(G_2)} \frac{2}{|V(G_2)||d_{G_1}(u) + d_{G_2}(v) + |V(G_2)||d_{G_1}(u) + d_{G_2}(v)|}$$

$$= \sum_{u \in V(G_1)} \sum_{v \in V(G_2)} \frac{2}{d_{G_2}(v) + d_{G_2}(v) + 2n_2d_{G_1}(u)}$$

$$+ \sum_{u \in V(G_1)} \sum_{v \in V(G_2)} \sum_{v' \in V(G_2)} \frac{2}{n_2d_{G_1}(u) + n_2d_{G_1}(u) + d_{G_2}(v) + d_{G_2}(v')}$$

(1)

$$= A_1 + A_2$$

where $A_1, A_2$ are the sums of the above terms, in order.

For any vertex $u, v, w$ of a graph $G$ and for any positive integer $n_1, n_2, n_3$

$$\frac{1}{(n_1+n_2+n_3)(n_1d_u + n_2d_v + n_3d_w)} \leq \frac{1}{(n_1+n_2+n_3)(n_1d_u + n_2d_v + n_3d_w)}$$

with equality if and only if $d_u = d_v = d_w$.

Therefore

$$A_1 = \sum_{u \in V(G_1)} \sum_{v \in V(G_2)} \frac{2}{d_{G_2}(v) + d_{G_2}(v) + 2n_2d_{G_1}(u)}$$

$$\leq \frac{2}{2 + 2n_2} \sum_{u \in V(G_1)} \sum_{v \in V(G_2)} \frac{1}{2 + 2n_2} \left( \frac{1}{d_{G_2}(v)} + \frac{1}{d_{G_2}(v)} + \frac{2n_2}{d_{G_1}(u)} \right)$$

$$= \frac{1}{1 + n_2} \left\{ \sum_{u \in V(G_1)} \sum_{v \in V(G_2)} \frac{1}{d_{G_2}(v) d_{G_2}(v)} \right\}$$

$$+ \frac{n_2}{1 + n_2} \sum_{u \in V(G_1)} \sum_{v \in V(G_2)} \frac{1}{d_{G_1}(u)}$$
\[
A_2 = \sum_{u_i, u_k \in E(G_1)} \sum_{v_j \in V(G_2)} \frac{2}{n_2 d_{G_1}(u_i) + n_2 d_{G_1}(u_k) + d_{G_2}(v_j) + d_{G_2}(v_i)}
\]
\[
\leq \frac{1}{1 + n_2} \left\{ \frac{n_2}{2 + 2n_2} \sum_{u_i, u_k \in E(G_1)} \sum_{v_j \in V(G_2)} \left( \frac{1}{d_{G_2}(v_j)} + \frac{1}{d_{G_2}(v_j)} \right) \right\}
\]
\[
= \frac{1}{1 + n_2} \left\{ \frac{n_2}{2 + 2n_2} \sum_{u_i, u_k \in E(G_1)} \sum_{v_j \in V(G_2)} \left( \frac{1}{d_{G_2}(v_j)} + \frac{1}{d_{G_2}(v_j)} \right) \right\}
\]
\[
\leq \frac{1}{1 + n_2} \left\{ \frac{n_2}{2 + 2n_2} \sum_{u_i, u_k \in E(G_1)} \sum_{v_j \in V(G_2)} \frac{d_{G_1}(u_i) + d_{G_1}(u_k) - 2}{d_{G_1}(u_i) d_{G_1}(u_k)} \right\}
\]
\[
+ \frac{n_2}{1 + n_2} \sum_{u_i, u_k \in E(G_1)} \sum_{v_j \in V(G_2)} \sum_{v_j \in V(G_2)} \left( 1 + 1 \right)
\]
\[
(3) = \frac{1}{1 + n_2} \left\{ \frac{n_2}{2 + 2n_2} ABC(G_1) + \frac{n_2}{1 + n_2} R(G_1) + \frac{n_2^2}{1 + n_2} \right\}
\]

From equation (1), (2) and (3) we get
\[
H(G_1 | G_2) \leq \frac{1}{(1 + n_2)^2} \left\{ \frac{n_1}{2} ABC(G_2) + n_1 R(G_2) + n_1 n_2 m_2 \right. 
\]
\[
\left. + \frac{n_2}{2} ABC(G_1) + (n_2)^3 R(G_1) + (n_2)^2 m_1 \right\}.
\]

\[\square\]

**Theorem 2.2.** Let \( G_1 \) and \( G_2 \) be two connected graphs with order \( n_1, n_2 \) and size \( m_1, m_2 \) respectively. Then
\[
H(G_1 \times G_2) \leq \frac{1}{8} \left\{ n_1 ABC(G_2) + 2n_1 R(G_2) + 2n_1 m_2 \right\}
\]
\[ + n_2 ABC(G_1) + 2n_2 R(G_1) + 2n_2 m_1 \].

**Proof.** Let \( V(G_1) = \{ u_1, u_2, \ldots, u_{n_1} \} \) and \( V(G_2) = \{ v_1, v_2, \ldots, v_{n_2} \} \) be a set of vertex for \( G_1 \) and \( G_2 \) respectively. By the Definition of the cartesian product of two graph one can see that,

\[
|E(G_1 \times G_2)| = |E(G_1)||V(G_2)| + |E(G_2)||V(G_1)|
\]

\[
d_{G_1 \times G_2}(u,v) = d_{G_1}(u) + d_{G_2}(v)
\]

\[
H(G_1 \times G_2) = \sum_{(u_i,v_j),(u_k,v_l) \in E(G_1 \times G_2), (u_i,v_j) \neq (u_k,v_l)} \frac{2}{d_{G_1 \times G_2}(u_i,v_j) + d_{G_1 \times G_2}(u_k,v_l)}
\]

\[
= \sum_{(u_i,v_j),(u_i,v_l) \in E(G_1 \times G_2), v_j,v_l \in E(G_2)} \frac{2}{d_{G_1 \times G_2}(u_i,v_j) + d_{G_1 \times G_2}(u_i,v_l)}
\]

\[
+ \sum_{(u_i,v_j),(u_k,v_l) \in E(G_1 \times G_2), u_i,u_k \in E(G_1)} \frac{2}{d_{G_1 \times G_2}(u_i,v_j) + d_{G_1 \times G_2}(u_k,v_l) + 2d_{G_1}(u_i)}
\]

\[
= B_1 + B_2
\]

where \( B_1, B_2 \) are the sums of the above terms, in order.

\[
B_1 = \sum_{u_i \in V(G_1)} \sum_{v_j,v_l \in E(G_2)} \frac{2}{d_{G_2}(v_j) + d_{G_2}(v_l) + 2d_{G_1}(u_i)}
\]

\[
\leq \frac{1}{8} \sum_{u_i \in V(G_1)} \sum_{v_j,v_l \in E(G_2)} \left( \frac{1}{d_{G_2}(v_j)} + \frac{1}{d_{G_2}(v_l)} + \frac{2}{d_{G_1}(u_i)} \right)
\]

\[
= \frac{1}{8} \left\{ \sum_{u_i \in V(G_1)} \sum_{v_j,v_l \in E(G_2)} \left( \frac{1}{d_{G_2}(v_j)} + \frac{1}{d_{G_2}(v_l)} - \frac{2}{d_{G_2}(v_j)d_{G_2}(v_l)} \right) \right\}
\]

\[
+ 2 \sum_{u_i \in V(G_1)} \sum_{v_j,v_l \in E(G_2)} \frac{1}{d_{G_2}(v_j)d_{G_2}(v_l)}
\]

\[
+ 2 \sum_{u_i \in V(G_1)} \sum_{v_j,v_l \in E(G_2)} \frac{1}{d_{G_1}(u_i)}
\]

\[
\leq \frac{1}{8} \left\{ \sum_{u_i \in V(G_1)} \sum_{v_j,v_l \in E(G_2)} \sqrt{\frac{d_{G_2}(v_l) + d_{G_2}(v_j) - 2}{d_{G_2}(v_l)d_{G_2}(v_j)}} \right\}
\]

\[
+ 2 \sum_{u_i \in V(G_1)} \sum_{v_j,v_l \in E(G_2)} \frac{1}{d_{G_2}(v_j)d_{G_2}(v_l)}
\]

\[
+ 2 \sum_{u_i \in V(G_1)} \sum_{v_j,v_l \in E(G_2)} 1
\]
(5) \[ \frac{1}{8} \left\{ n_1 ABC(G_2) + 2n_1 R(G_2) + 2n_1 m_2 \right\} \]

Similarly we get,

(6) \[ B_1 \leq \frac{1}{8} \left\{ n_2 ABC(G_1) + 2n_2 R(G_1) + 2n_2 m_1 \right\} \]

From equation (3), (5) and (6) we get

\[ H(G_1 \times G_2) \leq \frac{1}{8} \left\{ n_1 ABC(G_2) + 2n_1 R(G_2) + 2n_1 m_2 \\ + n_2 ABC(G_1) + 2n_2 R(G_1) + 2n_2 m_1 \right\} . \]

□

**Theorem 2.3.** For \( i \in \{1, 2\} \), let \( G_i \) be a graph of minimum degree \( \delta_i \), maximum degree \( \Delta_i \), order \( n_i \) and size \( m_i \). Then

\[ H(G_1 \circ G_2) \geq \frac{m_1}{\Delta_1 + n_2} + \frac{m_2 n_1}{\Delta_2 + 1} + \frac{2n_1 n_2}{\Delta_1 + \Delta_2 + n_2 + 1} \]

\[ H(G_1 \circ G_2) \leq \frac{m_1}{\delta_1 + n_2} + \frac{m_2 n_1}{\delta_2 + 1} + \frac{2n_1 n_2}{\delta_1 + \delta_2 + n_2 + 1} . \]

**Proof.** The edges of \( G_1 \circ G_2 \) are partitioned into three subsets \( E_1, E_2 \) and \( E_3 \) as follows

\[ E_1 = \{ e \in E(G_1 \circ G_2), e \in E(G_1) \} \]

\[ E_2 = \{ e \in E(G_1 \circ G_2), e \in E(G_2), i = 1, 2, \ldots, |V(G_1)| \} \]

\[ E_3 = \{ e \in E(G_1 \circ G_2), e = uv, u \in V(G_2), i = 1, 2, \ldots, |V(G_1)| \text{ and } v \in V(G_1) \} \]

and if \( u \) is a vertex of \( G_1 \circ G_2 \), then

\[ d_{G_1 \circ G_2}(u) = \begin{cases} d_{G_1}(u) + |V(G_2)| & \text{if } u \in V(G_1) \\ d_{G_2}(u) + 1 & \text{if } u \in V(G_2) \end{cases} \]

Let \( G_i = (V_i, E_i), i \in \{1, 2\} \) and let \( G_1 \circ G_2 = (V, E) \) we have

\[ H(G_1 \circ G_2) = \sum_{uv \in E(G_1 \circ G_2)} \frac{2}{d_{G_1 \circ G_2}(u) + d_{G_1 \circ G_2}(v)} \]

\[ = Q_1 + Q_2 + Q_3 \]

where

\[ Q_1 = \sum_{uv \in E_1} \frac{2}{d_{G_1}(u) + n_2 + d_{G_1}(v) + n_2} \]

\[ \geq \frac{2m_1}{\Delta_1 + n_2 + \Delta_1 + n_2} = \frac{m_1}{\Delta_1 + n_2} \]

\[ Q_2 = n_1 \sum_{uv \in E_2} \frac{2}{d_{G_2}(u) + 1 + d_{G_2}(v) + 1} \]

\[ \geq \frac{2n_1 m_2}{\Delta_2 + 1 + \Delta_2 + 1} = \frac{n_1 m_2}{\Delta_2 + 1} \]
\[ Q_3 = \sum_{uv \in E_3, u \in V_1 \text{and} v \in V_2} \frac{2}{d_{G_1}(u) + n_2 + d_{G_2}(v) + 1} + 1 \]

(10)

From equation (7), (8), (9) and (11) we get

\[ H(G_1 \circ G_2) \geq \frac{m_1}{\Delta_1 + n_2} + \frac{m_2 n_1}{\Delta_2 + 1} + \frac{2 n_1 n_2}{\Delta_1 + \Delta_2 + n_2 + 1} \]

Similarly we deduce the lower bound

\[ H(G_1 \circ G_2) \leq \frac{m_1}{\delta_1 + n_2} + \frac{m_2 n_1}{\delta_2 + 1} + \frac{2 n_1 n_2}{\delta_1 + \delta_2 + n_2 + 1} . \]

\[ \Box \]

**Theorem 2.4.** Let \( G_1 \) and \( G_2 \) be two connected graphs with order \( n_1 \), \( n_2 \) and size \( m_1 \), \( m_2 \) respectively. Then

\[ H(G_1 + G_2) \geq \frac{2}{m_1 + 2n_2 + 1} R(G_1) + \frac{2}{m_2 + 2n_1 + 1} R(G_2) + \frac{n_1 n_2}{n_1 + n_2 - 1} . \]

**Proof.** Let \( V(G_1) = \{u_1, u_2, \ldots, u_{n_1}\} \) and \( V(G_2) = \{v_1, v_2, \ldots, v_{n_2}\} \) be a set of vertex for \( G_1 \) and \( G_2 \) respectively. By the Definition of the join of two graph one can see that, if \( u \) is a vertex of \( G_1 + G_2 \), then

\[ d_{G_1 + G_2}(u) = \begin{cases} d_{G_1}(u) + |V(G_2)| & \text{if } u \in V(G_1) \\ d_{G_2}(u) + |V(G_1)| & \text{if } u \in V(G_2) \end{cases} \]

Therefore,

\[ H(G_1 + G_2) = \sum_{uv \in E(G_1 + G_2)} \frac{2}{d_{G_1}(u) + d_{G_2}(v)} \]

\[ = \sum_{uv \in E(G_1)} \frac{2}{d_{G_1}(u) + n_2 + d_{G_2}(v) + n_2} + \sum_{uv \in E(G_2)} \frac{2}{d_{G_2}(u) + n_1 + d_{G_2}(v) + n_1} + \sum_{u \in V(G_1), v \in V(G_2)} \frac{2}{d_{G_1}(u) + n_2 + d_{G_2}(v) + n_1} \]

(11)

\[ = A_1 + A_2 + A_3 \]

where

\[ A_1 = \sum_{uv \in E(G_1)} \frac{2}{d_{G_1}(u) + d_{G_1}(v) + 2n_2} \]

Since for each edge \( uv \) of \( G \), we have \( d_G(u) + d_G(v) \leq |E(G)| + 1 \) with equality iff every other edge of \( G \) is adjacent to the edge \( uv \). And \( 1 \leq \sqrt{d_u d_v} \) the equality hold iff \( d_u = d_v = 1 \). Hence

\[ A_1 \geq \sum_{uv \in E(G_1)} \frac{2}{|E(G_1)| + 1 + 2n_2} \times \frac{1}{\sqrt{d_u d_v}} \]

(12)

\[ = \frac{2}{m_1 + 1 + 2n_2} R(G_1) \]
Similarly

\begin{equation}
A_2 \geq \frac{2}{m_2 + 1 + 2n_1} R(G_2)
\end{equation}

Since for any graph \( G \) with \( n \) vertices \( d_u \leq n - 1 \), therefore

\begin{equation}
A_3 = \sum_{u \in V(G_1), v \in V(G_2)} \frac{2}{d_G(u) + n_2 + d_G(v) + n_1} \geq \frac{2}{n_1 - 1 + n_2 + n_2 - 1 + n_1} \frac{n_1 n_2}{n_1 + n_2 - 1}
\end{equation}

From equation (13), (14), (15) and (16) we get

\begin{equation}
H(G_1 + G_2) \geq \frac{2}{m_1 + 2n_2 + 1} R(G_1) + \frac{2}{m_2 + 2n_1 + 1} R(G_2) + \frac{n_1 n_2}{n_1 + n_2 - 1}.
\end{equation}

\[ \square \]

**Theorem 2.5.** For \( i \in \{1, 2\} \), let \( G_i \) be a graph of diameter \( d(G_i) \), order \( n_i \) and size \( m_i \). Then

\begin{equation}
H(G_1 \oplus G_2) \geq \frac{n_1^2 m_1 + n_2^2 m_2 - 4m_1 m_2}{n_2(n_1 - d(G_1)) + n_1(n_2 - d(G_2)) - 2(n_1 - d(G_1))(n_2 - d(G_2))}.
\end{equation}

**Proof.** Let \( V(G_1) = \{u_1, u_2, \ldots, u_{n_1}\} \) and \( V(G_2) = \{v_1, v_2, \ldots, v_{n_2}\} \) be a set of vertex for \( G_1 \) and \( G_2 \) respectively.

\begin{equation}
H(G_1 \oplus G_2) = \sum_{(u_i, v_j), (u_k, v_l) \in E(G_1 \oplus G_2)} \frac{2}{d_{G_1 \oplus G_2}(u_i, v_j) + d_{G_1 \oplus G_2}(u_k, v_l)}
= \sum_{v_j \in V(G_2)} \sum_{v_l \in V(G_2)} \sum_{u_i, u_k \in E(G_1)} \frac{2}{d_{G_1 \oplus G_2}(u_i, v_j) + d_{G_1 \oplus G_2}(u_k, v_l)}
+ \sum_{u_i \in V(G_1)} \sum_{u_k \in V(G_1)} \sum_{v_j, v_l \in E(G_2)} \frac{2}{d_{G_1 \oplus G_2}(u_i, v_j) + d_{G_1 \oplus G_2}(u_k, v_l)}
\end{equation}

As per the definition of the Symmetric difference of a graph

\begin{equation}
d_{G_1 \oplus G_2}(u_i, v_j) + d_{G_1 \oplus G_2}(u_k, v_l) = n_2[d_{G_1}(u_i) + d_{G_1}(u_k)] + n_1[d_{G_2}(v_j) + d_{G_2}(v_l)]
- 2[d_{G_1}(u_i)d_{G_2}(v_j) + d_{G_1}(u_k)d_{G_2}(v_l)]
\end{equation}

Since for each vertex \( u \) of a graph \( G \) with \( n \) vertices, \( d_G(u) \leq n - ecc_G(u) \) and the diameter \( d(G) \geq ecc_G(u) \).

Therefore,

\begin{equation}
d_{G_1 \oplus G_2}(u_i, v_j) + d_{G_1 \oplus G_2}(u_k, v_l) \leq n_2[n_1 - ecc_{G_1}(u_i) + n_1 - ecc_{G_1}(u_k)]
+ n_1[n_2 - ecc_{G_2}(v_j) + n_2 - ecc_{G_2}(v_l)]
- 2[(n_1 - ecc_{G_1}(u_i))(n_2 - ecc_{G_2}(v_j))]
\end{equation}
\begin{equation}
\begin{split}
&+ (n_1 - \text{ecc}_{G_1}(u_k))(n_2 - \text{ecc}_{G_2}(v_l)) \\
&\leq n_2[2n_1 - 2d(G_1)] + n_1[2n_2 - 2d(G_2)] \\
&- 2[(n_1 - d(G_1))(n_2 - d(G_2)) + (n_1 - d(G_1))(n_2 - d(G_2))] \\
&= 2[n_2[n_1 - d(G_1)] + n_1[n_2 - d(G_2)] - 2[(n_1 - d(G_1))(n_2 - d(G_2))]]
\end{split}
\end{equation}
(16)

From equation (15) and (16) we get
\[H(G_1 \oplus G_2) \geq \frac{n_2^2 m_1}{n_2[n_1 - d(G_1)] + n_1[n_2 - d(G_2)] - 2[(n_1 - d(G_1))(n_2 - d(G_2))]} \]
\[+ \frac{n_2^2 m_2}{n_2[n_1 - d(G_1)] + n_1[n_2 - d(G_2)] - 2[(n_1 - d(G_1))(n_2 - d(G_2))]} \]
\[= \frac{n_2^2 m_1 + n_1^2 m_2 - 4m_1 m_2}{n_2[n_1 - d(G_1)] + n_1[n_2 - d(G_2)] - 2[(n_1 - d(G_1))(n_2 - d(G_2))]} . \]

\[\square\]

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