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ON LAPLACIAN-ENERGY-LIKE INVARIANT AND INCIDENCE ENERGY

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ABSTRACT. For a simple connected graph G with n-vertices having Laplacian eigenvalues μ_1, μ_2, $\dots$, μ_{n-1}, $\mu_n = 0$, and signless Laplacian eigenvalues $q_1, q_2, \dots, q_n$, the Laplacian-energy-like invariant(LEL) and the incidence energy (IE) of a graph G are respectively defined as $LEL(G) = \sum_{i=1}^{n-1} \sqrt{\mu_i}$ and $IE(G) = \sum_{i=1}^{n} \sqrt{q_i}$. In this paper, we obtain some sharp lower and upper bounds for the Laplacian-energy-like invariant and incidence energy of a graph.

1. Introduction

Let G be a finite, undirected, simple graph with n vertices and m edges having vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$. The adjacency matrix $A = (a_{ij})$ of G is a $(0,1)$-square matrix of order n whose (i,j)-entry is equal to 1 if v_i is adjacent to v_j and equal to 0, otherwise. Let $D(G) = diag(d_1, d_2, \dots, d_n)$ be the diagonal matrix associated to G, where d_i is the degree of vertex v_i. The matrices $L(G) = D(G) - A(G)$ and $Q(G) = D(G) + A(G)$ are respectively called Laplacian and signless Laplacian matrices and their spectrum are respectively called Laplacian spectrum (L-spectrum) and signless Laplacian spectrum (Q-spectrum) of the graph G. Being real symmetric, positive semi-definite matrices, we let $0 = \mu_n \leq \mu_{n-1} \leq \cdots \leq \mu_1$ and $0 \leq q_n \leq q_{n-1} \leq \cdots \leq q_1$ be the L-spectrum and Q-spectrum of G, respectively. It is well known that μ_n=0 with multiplicity equal to the number of connected components of G and $\mu_{n-1} > 0$ if and only if G is connected [9]. Moreover, for all $i = 1, 2, \dots, n$, $\mu_i = q_i$ if and only if G is bipartite. For notations and definitions in graphs, we refer to [23].

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Let $I(G)$ be the (vertex-edge) incidence matrix of the graph G. For a graph G with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \ldots, e_m\}$, the (i,j)-entry of $I(G)$ is 1 if e_j is incident with v_i, and 0 otherwise. As given by Jooyandeh et al. [17] the incidence energy of the incidence matrix $I(G)$ of the graph G is defined as

$$IE = IE(G) = \sum_{k=1}^{n} \sqrt{\sigma_k},$$

where $\sigma_1, \sigma_2, \ldots, \sigma_n$ are the singular values of $I(G)$. Recall that the singular values of a (real) matrix M are equal to the positive square roots of the eigenvalues of MM^t. Various properties and bounds for the incidence energy were recently established in [2, 12, 13, 17, 19, 24, 26, 27, 28]. As is well-known in spectral graph theory that $I(G)I(G)^t = Q(G)$, we can define incidence energy as

$$IE = IE(G) = \sum_{k=1}^{n} \sqrt{q_k},$$

where $q_1, q_2, \ldots, q_n$ are the eigenvalues of $Q(G)$.

Further, Laplacian-spectrum-based graph invariant was put forward by Liu and Liu [20] as

$$LEL = LEL(G) = \sum_{k=1}^{n-1} \sqrt{\mu_k}$$

and was named Laplacian-energy-like invariant. The motivation for introducing LEL was in its analogy to the earlier studied graph energy by Gutman [11] and Laplacian energy by Gutman and Zhou [14]; for more details we refer to [3, 15, 16] and the references cited therein. Recently, several mathematical investigations of LEL were communicated [12, 21, 29]. So several researchers established many lower and upper bounds to estimate the invariant for some classes of graphs [25]. In this paper, we obtain sharp lower bounds for the incidence energy and Laplacian-energy-like invariant which improves some previously known lower bounds for some cases (especially in case of trees, unicyclic, bicyclic, tricyclic, tetracyclic graphs). We also obtain upper bounds for these invariants.

2. Bounds on Laplacian-energy-like invariant

In this section, we obtain upper and lower bounds for the Laplacian-energy-like invariant of a graph. First we start with the following observation by Das [8].

Lemma 2.1. Let G be a graph on $n > 3$ vertices whose distinct Laplacian eigenvalues are $0 < \alpha < \beta$. Then the following hold.
(i) The multiplicity of α is $n-2$ if and only if G is one of the graphs $K_{\frac{n}{2},\frac{n}{2}}$ or $K_{n-1,1}$.
(ii) The multiplicity of β is $n-2$ if and only if G is the graph $K_n - e$.

The following Lemma can be found in [9, 10].

Lemma 2.2. Let G be a connected graph of order n and let Δ be its maximum degree. Then $\Delta+1 \leq \mu_1 \leq n$. Equality holds on the left if $\Delta = n-1$ and on the right if and only if G is the join of two graphs.

The following observation can be seen in [18].

Lemma 2.3. Let $G \neq K_n$ be a connected graph of order n and let δ be the smallest vertex degree of G. Then $\mu_{n-1} \leq \delta$, with equality if and only if G is a join of a graph on $\delta(G)$ vertices with another graph.

The next result can be found in [7].

Lemma 2.4.(Pólya-Szegö inequality) Let $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ be two sequences of positive real numbers such that there exist positive numbers A, a, B, b satisfying

$$0 < a \leq a_i \leq A < \infty,\ 0 < b \leq b_i \leq B < \infty,$$

for all $i = 1, 2, \ldots, n$. Then

$$\frac{\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2}{(\sum_{i=1}^{n} a_i b_i)^2} \leq \frac{(ab+AB)^2}{4abAB}. \tag{2.1}$$

The equality holds in (2.1) if and only if $p = \frac{n.\frac{A}{a}}{(\frac{A}{a}+\frac{B}{b})}, q = \frac{n.\frac{B}{b}}{(\frac{A}{a}+\frac{B}{b})}$ are integers and if p of the numbers $a_1, a_2, \ldots, a_n$ are equal to a and q of these numbers are equal to A, and if the corresponding numbers b_i are equal to b and B, respectively.

We first mention some well known lower bounds for LEL. Gutman et al. [13] obtained the following lower bound.

$$LEL(G) \geq \frac{2m}{\sqrt{n}}, \tag{2.2}$$

with equality if and only if $G \cong \overline{K_n}$ or $G \cong K_n$.

Wang et al. [25] obtained the following lower bound.

$$LEL(G) \geq \sqrt{\frac{(2m)^3}{2m+n\Delta^2}}, \tag{2.3}$$

where Δ is the maximum vertex degree. Equality occurs if and only if $G \cong K_n$.

It is shown in [25] that the lower bounds (2.2) and (2.3) are incomparable.

We now give a lower bound for LEL in terms of n, m, k.

Theorem 2.5. Let G be a connected graph with n vertices and m edges having algebraic connectivity $\mu_{n-1} \geq k$. Then

$$LEL(G) \geq \sqrt{\frac{8m(n-1)\sqrt{kn}}{(\sqrt{n}+\sqrt{k})^2}}, \tag{2.4}$$

with equality if and only if $G \cong K_n$.

Proof. Setting in (2.1) $n = n-1$, $a_i = \sqrt{\mu_i}$, $b_i = 1$, for $i = 1, 2, \ldots, n-1$ and $a = \sqrt{\mu_{n-1}}$, $A = \sqrt{\mu_1}$, $b = 1$, $B = 1$, we get

$$\frac{\sum_{i=1}^{n-1} \mu_i \sum_{i=1}^{n-1} 1}{(\sum_{i=1}^{n-1} \sqrt{\mu_i})^2} \leq \frac{(\sqrt{\mu_{n-1}} + \sqrt{\mu_1})^2}{4\sqrt{\mu_1 \mu_{n-1}}}.$$

This gives,

$$LEL(G) \geq \sqrt{\frac{8m(n-1)\sqrt{\mu_1 \mu_{n-1}}}{(\sqrt{\mu_1} + \sqrt{\mu_{n-1}})^2}}.$$

Since,

$$\sqrt{\frac{8m(n-1)\sqrt{\mu_1 \mu_{n-1}}}{(\sqrt{\mu_1} + \sqrt{\mu_{n-1}})^2}} \geq \sqrt{\frac{8m(n-1)\sqrt{k\mu_1}}{(\sqrt{\mu_1} + \sqrt{k})^2}},$$

it follows that

$$LEL(G) \geq \sqrt{\frac{8m(n-1)\sqrt{k\mu_1}}{(\sqrt{\mu_1} + \sqrt{k})^2}}.$$

For $x \leq n$, consider the function $f(x) = \dfrac{8m(n-1)\sqrt{kx}}{(\sqrt{x}+\sqrt{k})^2}$.

For this function, we have

$$f'(x) = \frac{4m(n-1)\sqrt{k}(\sqrt{k}-\sqrt{x})}{\sqrt{x}(\sqrt{x}+\sqrt{k})^3} \leq 0.$$

That is, $f(x)$ is a decreasing function for $x \leq n$. So

$$f(x) \geq f(n) = \frac{8m(n-1)\sqrt{kn}}{(\sqrt{n}+\sqrt{k})^2}.$$

This gives,

$$LEL(G) \geq \sqrt{\frac{8m(n-1)\sqrt{kn}}{(\sqrt{n}+\sqrt{k})^2}}.$$

Equality occurs in (2.4) if and only if equality occurs in (2.1) and $\mu_1 = n$. That is, by Lemma 2.2 and Lemma 2.4, if and only if G is a join of two graphs and p, q are integers, where $p + q = n - 1$ with p of the numbers in $\mu_1, \mu_2, \ldots, \mu_{n-1}$ equal to μ_1 and q of them equal to μ_{n-1}. For p, q integers there are $n-1$ solutions of the equation $p + q = n - 1$ and for any of these integral solutions it follows from Lemma 2.4, that equality occurs if and only if G has two distinct Laplacian eigenvalues. That is, if and only if $G \cong K_n$ [4].

Conversely if $G \cong K_n$, then it is easy to see that equality holds in (2.4). □

Remark 2.6. (i). Let T be a tree of order n, $n \geq 6$ with algebraic connectivity $\mu_{n-1} \geq 0.07$. Then we will show that our lower bound in (2.4) is better than the lower bound in (2.2) for the tree T. For this we have to show that

$$\sqrt{\frac{8m(n-1)\sqrt{(0.07)n}}{(\sqrt{n}+\sqrt{0.07})^2}} \geq \frac{2m}{\sqrt{n}},$$

that is,

$$\frac{80(n-1)^2\sqrt{7n}}{(10\sqrt{n}+\sqrt{7})^2} \geq \frac{4(n-1)^2}{n}, \quad as \quad m = n-1$$

that is,

$$20n\sqrt{7n} \geq (10\sqrt{n}+\sqrt{7})^2,$$

which is true for $n \geq 6$. Since for almost all trees algebraic connectivity $\mu_{n-1} \geq 0.07$, it follows that bound (2.4) is better than bound (2.2) for almost all trees.
(ii). Let G be graph of order n having $m \leq \frac{2n(n-1)\sqrt{n}}{(\sqrt{n}+1)^2}$ edges and algebraic connectivity $\mu_{n-1} \geq 1$. Then the lower bound (2.4) is better than the lower bound in (2.2) for G. We have

$$\sqrt{\frac{8m(n-1)\sqrt{kn}}{(\sqrt{n}+\sqrt{k})^2}} \geq \frac{2m}{\sqrt{n}},$$

that is,

$$\frac{8m(n-1)\sqrt{n}}{(\sqrt{n}+1)^2} \geq \frac{4m^2}{n},$$

that is,

$$2n(n-1)\sqrt{n} \geq m(\sqrt{n}+1)^2, \tag{2.5}$$

which is true. In particular if G is unicyclic, bicyclic, tricyclic, tetracyclic graph, then $m = n, n+1, n+2, n+3$ (respectively). It is easy to see that (2.5) holds for $n \geq 5$.

Remark 2.7. (i). Let T be a tree of order n, $n \geq 3$ with maximum degree $\Delta \geq \frac{n}{2}$ and algebraic connectivity $\mu_{n-1} \geq 0.07$. Then the lower bound (2.4) is better than the lower bound (2.3) for T. We have

$$\sqrt{\frac{8m(n-1)\sqrt{kn}}{(\sqrt{n}+\sqrt{k})^2}} \geq \sqrt{\frac{(2m)^3}{2m+n\Delta^2}}$$

that is,

$$\frac{80(n-1)^2\sqrt{7n}}{(10\sqrt{n}+\sqrt{7})^2} \geq \frac{8(n-1)^3}{2(n-1)+n\Delta^2}, \quad as \quad m = n-1$$

that is,

$$n\Delta^2 \geq \frac{(n-1)(10\sqrt{n}+\sqrt{7})^2}{10\sqrt{7n}} - 2(n-1)$$

which is true for $\Delta \geq \frac{n}{2}$, $n \geq 3$.

(ii). Let G be a graph of order n with maximum degree $\Delta \geq \sqrt{\frac{m^2(\sqrt{n}+1)^2}{n(n-1)\sqrt{n}} - \frac{2m}{n}}$ and algebraic connectivity $\mu_{n-1} \geq 1$. Then the bound (2.4) is better than the bound (2.3) for G. We have

$$\frac{8m(n-1)\sqrt{n}}{(\sqrt{n}+1)^2} \geq \frac{8m^3}{2m+n\Delta^2}$$

that is,

$$n\Delta^2 \geq \frac{m^2(\sqrt{n}+1)^2}{(n-1)\sqrt{n}} - 2m \tag{2.6}$$

which is true for $\Delta \geq \sqrt{\frac{m^2(\sqrt{n}+1)^2}{n(n-1)\sqrt{n}} - \frac{2m}{n}}$. In particular if G is unicyclic, bicyclic, tricyclic, then $m = n, n+1, n+2$. It is easy to see that (2.6) holds for $\Delta \geq \frac{n}{2}$, $n \geq 6$.

We now obtain an upper bound for LEL in terms n, m, Δ, k.

Theorem 2.8. Let G be a connected graph with n vertices and m edges having maximum degree Δ and algebraic connectivity $\mu_{n-1} \geq k$. Then

$$LEL(G) \leq \frac{2\sqrt{nk}+k(n-2)+2m-(\Delta+1)}{2\sqrt{k}}, \tag{2.7}$$

with equality if and only if $k = n$ and $G \cong K_n$ or $k = 1$ and $G \cong K_{n-1,1}$.

Proof. Let $0 = \mu_n < \mu_{n-1} \leq \mu_{n-2} \leq \cdots \leq \mu_2 \leq \mu_1$ be the Laplacian spectrum of G with $\mu_{n-1} \geq k$. Since $\Delta + 1 \leq \mu_1 \leq n$, we have

$$\begin{aligned}
LEL(G) &= \sum_{i=1}^{n-1}\sqrt{\mu_i} = \sqrt{\mu_1} + (n-2)\sqrt{\mu_{n-1}} + \sum_{i=2}^{n-2}\left(\frac{\mu_i - \mu_{n-1}}{\sqrt{\mu_i}+\sqrt{\mu_{n-1}}}\right) \\
&\leq \sqrt{\mu_1} + (n-2)\sqrt{\mu_{n-1}} + \sum_{i=2}^{n-2}\left(\frac{\mu_i - \mu_{n-1}}{2\sqrt{\mu_{n-1}}}\right) \\
&= \frac{2\sqrt{\mu_1\mu_{n-1}} + (n-2)\mu_{n-1} + 2m - \mu_1}{2\sqrt{\mu_{n-1}}} \\
&\leq \frac{2\sqrt{n\mu_{n-1}} + (n-2)\mu_{n-1} + 2m - (\Delta+1)}{2\sqrt{\mu_{n-1}}}.
\end{aligned}$$

For $x \geq k$, consider the function

$$f(x) = \frac{2\sqrt{nx} + (n-2)x + 2m - (\Delta+1)}{2\sqrt{x}}.$$

For this function, we have

$$f'(x) = \frac{x(n-2) + \Delta + 1 - 2m}{4x\sqrt{x}},$$

for all $x \geq k$. Since $(\Delta+1)+(n-2)\mu_{n-1} \leq \mu_1+\mu_2+\cdots+\mu_{n-1}=2m$, it follows that $f(x)$ is a decreasing function for $x \geq k$. So

$$f(x) \leq f(k) = \frac{2\sqrt{nk}+(n-2)k+2m-(\Delta+1)}{2\sqrt{k}},$$

which gives

$$LEL(G) \leq \frac{2\sqrt{nk}+(n-2)k+2m-(\Delta+1)}{2\sqrt{k}}.$$

Equality occurs in (2.7) if and only if $n=\mu_1=\Delta+1$, $\mu_2=\mu_3=\cdots=\mu_{n-2}=\mu_{n-1}=k$. That is by Lemma 2.2, G is a join of two graphs having at most three distinct Laplacian eigenvalues with $\Delta+1=\mu_1$. If G has one distinct Laplacian eigenvalue, then $G \cong \overline{K_n}$, which is not possible as G is connected. If G is a join of two graphs with $\Delta+1=\mu_1$ having two distinct Laplacian eigenvalues, then $G \cong K_n$ [4]. If G is join of two graphs having three distinct Laplacian eigenvalues with $\Delta+1=\mu_1$, then by Lemma 2.1, $G \cong K_{n-1,1}$.

Conversely if G is one of these graphs, then it is easy to see that equality occurs in (2.7). $\square$

3. Bounds for incidence energy

In order to obtain the bounds for the incidence energy, we need the following Lemmas. Lemma 3.1 can be found in [5].

Lemma 3.1. Let G be a graph of order n having maximum degree Δ, minimum degree δ and largest Q-eigenvalue q_1. Then $2\delta \leq q_1 \leq 2\Delta$. For a connected graph G, equality holds in either of these inequalities if and only if G is regular.

The next observation is a well known fact [6].

Lemma 3.2. A graph G has two distinct signless Laplacian (Laplacian) eigenvalues if and only if $G \cong K_n$.

The following observation can be found in [5].

Lemma 3.3. For $i=1,2,\ldots,n$, let μ_i and q_i be the L-spectrum and $Q-$spectrum of the graph G. Then $\mu_i=q_i$, for all $i=1,2,\ldots,n$ if and only if G is bipartite.

By Lemma 3.3, it is clear that $IE(G)=LEL(G)$, if and only if G is bipartite. So, if G is bipartite, then any result that holds for $LEL(G)$ also holds for $IE(G)$. Therefore in this section, we consider non-bipartite graphs only.

We first obtain a lower bound for IE in terms of n, m, Δ, k.

Theorem 3.4. Let G be a connected non-bipartite graph with $n > 2$ vertices and m edges having smallest Q-eigenvalue $q_n \geq k$. Then

$$IE(G) \geq \sqrt{\frac{8mn\sqrt{2k\Delta}}{(\sqrt{2\Delta}+\sqrt{k})^2}}. \tag{3.1}$$

Proof. Setting in equation (2.1) $a_i = \sqrt{q_i}$, $b_i = 1$, for $i = 1, 2, \ldots, n$ and $a = \sqrt{q_n}$, $A = \sqrt{q_1}$, $b = 1$, $B = 1$, we get

$$\frac{\sum_{i=1}^{n} q_i \sum_{i=1}^{n} 1}{(\sum_{i=1}^{n} \sqrt{q_i})^2} \leq \frac{(\sqrt{q_n}+\sqrt{q_1})^2}{4\sqrt{q_1 q_n}}.$$

This gives,

$$IE(G) \geq \sqrt{\frac{8mn\sqrt{q_1 q_n}}{(\sqrt{q_1}+\sqrt{q_n})^2}}.$$

Since,

$$\sqrt{\frac{8mn\sqrt{q_1 q_n}}{(\sqrt{q_1}+\sqrt{q_n})^2}} \geq \sqrt{\frac{8mn\sqrt{kq_1}}{(\sqrt{q_1}+\sqrt{k})^2}},$$

it follows that

$$IE(G) \geq \sqrt{\frac{8mn\sqrt{kq_1}}{(\sqrt{q_1}+\sqrt{k})^2}}.$$

For $x \leq 2\Delta$, consider the function

$$f(x) = \frac{8mn\sqrt{kx}}{(\sqrt{x}+\sqrt{k})^2}.$$

For this function, we have

$$f'(x) = \frac{4mn\sqrt{k}(\sqrt{k}-\sqrt{x})}{\sqrt{x}(\sqrt{x}+\sqrt{k})^3} \leq 0.$$

That is $f(x)$ is a decreasing function for $x \leq 2\Delta$. So

$$f(x) \geq f(2\Delta) = \frac{8mn\sqrt{2k\Delta}}{(\sqrt{2\Delta}+\sqrt{k})^2}.$$

This gives

$$IE(G) \geq \sqrt{\frac{8mn\sqrt{2k\Delta}}{(\sqrt{2\Delta}+\sqrt{k})^2}}.$$

Equality occurs in (3.1) if and only if equality occurs in (2.1), $q_1 = 2\Delta$ and $q_n = k$. That is, by Lemma 2.2 and Lemma 3.1, if and only if G is regular and p, q are integers, where $p + q = n$ with p of the numbers in $q_1, q_2, \ldots, q_n$ equal to $A = q_1$ and q of them equal to $a = q_n$. For p, q integers there are n solutions of the equation $p + q = n$ and for any of these integral solutions it follows from Lemma 2.4, that equality occurs if and only if G has two distinct Q-eigenvalues. By Lemma 3.2, a graph with two distinct Q-eigenvalues is K_n, having $A = q_1 = 2(n-1)$, $p = 1$ and $a = q_n = n-2$, $q = n-1$. For this choice of A, a, p, q, we have from Lemma 2.2, $1 = \frac{n\sqrt{\frac{2(n-1)}{n-2}}}{1+\sqrt{\frac{2(n-1)}{n-2}}}$. That is, $2(n-2)(n-1) = 1$, which gives, $n = \frac{3\pm\sqrt{3}}{2}$, which is not possible. So equality does not occur in (3.1) for any graph G. □

Since $q_1 \leq 2\Delta \leq 2(n-1)$, it follows from Theorem 3.4,

$$f(q_1) \geq f(2(n-1)) = \sqrt{\frac{8mn\sqrt{2k(n-1)}}{(\sqrt{2(n-1)}+\sqrt{k})^2}}.$$

Therefore we have the following observation.

Corollary 3.5. Let G be a connected non-bipartite graph with n vertices and m edges having smallest Q-eigenvalue $q_n \geq k$. Then

$$IE(G) \geq \sqrt{\frac{8mn\sqrt{2k(n-1)}}{(\sqrt{2(n-1)}+\sqrt{k})^2}}. \tag{3.2}$$

Recall from [13] that a lower bound for IE was given as follows.

Lemma 3.6. Let G be graph with n vertices and m edges. Then

$$IE(G) \geq \frac{2m}{\sqrt{n}}, \tag{3.3}$$

with equality if and only if $G \cong \overline{K_n}$ or $G \cong K_2$.

Remark 3.7. Let G be a non-bipartite graph with n vertices, smallest Q-eigenvalue $q_n \geq k = 0.5$ and $m \leq \frac{2n^2\sqrt{n-1}}{(\sqrt{2n-2}+\sqrt{0.5})^2}$. Then the bound (3.2) is better than the bound (3.3) for G. For this we need to show that

$$\frac{\sqrt{8mn\sqrt{2(0.5)(n-1)}}}{\sqrt{2(n-1)}+\sqrt{0.5}} \geq \frac{2m}{\sqrt{n}},$$

that is,

$$2n^2\sqrt{n-1} \geq m(\sqrt{2n-2}+\sqrt{0.5})^2, \tag{3.4}$$

which is true. In particular if G is a unicyclic, bicyclic, tricyclic, tetracyclic graph, then $m = n, n+1, n+2, n+3$ (respectively). It is easy to see that (3.4) holds for $n \geq 5$.

We now have an upper bound for IE in terms of n, m, Δ, δ, k.

Theorem 3.8. Let G be a non-bipartite connected graph with n vertices, m edges having maximum degree Δ, minimum degree δ and smallest Q-eigenvalue $q_n \geq k$. Then

$$IE(G) \leq \frac{2\sqrt{2k\Delta}+(n-1)k+2m-2\delta}{2\sqrt{k}},$$

with equality if and only if $G \cong K_n$.

Proof. This follows by proceeding similarly as in Theorem 2.4.

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