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ON THE TOTAL DOMATIC NUMBER OF REGULAR GRAPHS

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Abstract. A set S of vertices of a graph $G = (V, E)$ without isolated vertex is a *total dominating set* if every vertex of $V(G)$ is adjacent to some vertex in S. The *total domatic number* of a graph G is the maximum number of total dominating sets into which the vertex set of G can be partitioned. We show that the total domatic number of a random r-regular graph is almost surely at most $r - 1$, and that for 3-regular random graphs, the total domatic number is almost surely equal to 2. We also give a lower bound on the total domatic number of a graph in terms of order, minimum degree and maximum degree. As a corollary, we obtain the result that the total domatic number of an r-regular graph is at least $r/(3\ln(r))$.

1. Introduction

Let $G = (V(G), E(G)) = (V, E)$ be a simple graph of order n with minimum degree $\delta(G) \geq 1$. The neighborhood of a vertex u is denoted by $N_G(u)$ and its degree $|N_G(u)|$ by $d_G(u)$ (briefly $N(u)$ and $d(u)$ when no ambiguity on the graph is possible). The *minimum* and *maximum degree* of a graph G are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. The *open neighborhood* of a set $S \subseteq V$ is the set $N(S) = \cup_{v\in S} N(v)$, and the *closed neighborhood* of S is the set $N[S] = N(S) \cup S$. A *matching* is a set of edges with no shared endvertices. A perfect matching M of G is a matching with $V(M) = V(G)$. The maximum number of edges of a matching in G is denoted by $\alpha'(G)$ (α' for short). If $C = (v_1, v_2, \ldots, v_n)$ is a cycle and v_i, v_k are distinct vertices of C, then the segment $[v_i, v_k]$ of C is defined as the set $\{v_i, v_{i+1}, v_{i+2}, \ldots, v_k\}$, where the subscripts are taken modulo n. If $f(n)$ and $g(n)$ are real valued functions of an integer variable n, then we write $f(n) = O(g(n))$ (or $f(n) = \Omega(g(n))$)

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if there exist constants $C > 0$ and n_0 such that $f(n) \leq Cg(n)$ (or $f(n) \geq Cg(n)$) for $n \geq n_0$. We also write $f(n) \sim g(n)$ if $\lim_{n\to\infty} f(n)/g(n) = 1$. We use [9] for terminology and notation which are not defined here.

A set S of vertices of a graph G with minimum degree $\delta(G) > 0$ is a *total dominating set* if $N(S) = V(G)$. The minimum cardinality of a total dominating set, denoted by $\gamma_t(G)$, is called the *total domination number* of G. A $\gamma_t(G)$-*set* is a total dominating set of G of cardinality $\gamma_t(G)$.

A partition of $V(G)$, all of whose classes are total dominating sets in G, is called a *total domatic partition* of G. The maximum number of classes of a total domatic partition of G is called the *total domatic number* of G and is denoted by $d_t(G)$. The total domatic number was introduced by Cockayne, Dawes and Hedetniemi in [5] and has been studied by several authors (see for example, [2, 4, 11, 12]). More information on the total domination number and the total domatic number can be found in the monographs [7, 8] by Haynes, Hedetniemi and Slater.

We use the following standard model $\mathcal{G}_{n,r}$ to generate r-regular graphs on n vertices uniformly: to construct a random r-regular graph on the vertex set $\{v_1, v_2, \ldots, v_n\}$, take a random matching on the vertex set $\{v_{1,1}, v_{1,2}, \ldots, v_{1,r}, v_{2,1}, \ldots, v_{2,r}, \ldots, v_{n,r}\}$ and collapse each set $\{v_{i,1}, v_{i,2}, \ldots, v_{i,r}\}$ into a single vertex v_i. If the resulting graph contains any loops or multiple edges, discard it. All r-regular graphs are generated uniformly with this method. Wormald [10] has shown that 3-regular graphs are almost surely Hamiltonian, and that the model $\mathcal{G}_{n,r}$ and $\mathcal{H}_n \oplus \mathcal{G}_{n,r-2}$ are contiguous, meaning roughly that events that are almost sure in one model are almost sure in the other. Thus if an event is almost surely true in a random graph constructed from a random Hamilton cycle plus a random matching, then it is almost surely true in a random 3-regular graph. For more details the reader is referred to [10].

We make use of the following results.

Theorem A. ([10]) If G is a 3-regular random graph, then a.a. G consists of a Hamilton cycle plus a random matching.

Theorem B. ([5]) For every graph G of order n without isolated vertices,

$$d_t(G) \leq \min\left\{\delta(G), \frac{n}{\gamma_t(G)}\right\}.$$

2. A lower bound on the total domatic number

In this section we will show that the total domatic number of a random 3-regular graph is at least 2.

Definition 2.1. Let G be a 3-regular graph obtained from a cycle $C = (v_1, v_2, \cdots, v_n)$ by adding a perfect matching M. An edge v_iv_{i+1} of C (the indices are taken modulo n) is a 4-edge if v_i and v_{i+1} have matching partners v_j and v_k respectively, such that the cycle segments $[v_j, v_i]$ and $[v_{i+1}, v_k]$ are disjoint and have cardinality 0 (mod 4).

Lemma 2.2. Let $G = C \cup M$ as above. If C has a 4-edge then $d_t(G) \geq 2$.

Proof. Let v_iv_{i+1} be a 4-edge of C, and let v_j, v_k be their matching partners, respectively. Without loss of generality, we may assume that $i = 1$. If $n \equiv 0 \pmod 4$, then obviously $S_1 = \{v_{4i+1}, v_{4i+2} \mid 0 \leq i \leq \frac{n}{4} - 1\}$ and $S_2 = V(G) - S_1$ are two disjoint total dominating sets and hence $d_t(G) \geq 2$.

Now let $n \equiv 2 \pmod 4$. Then $n = 4s + 2$ for some positive integer s. If $n = 6$, then the result is immediate. Assume that $n \geq 10$. Then $k \equiv 1 \pmod 4$ and $j \equiv 0 \pmod 4$. Let $k = 4\ell + 1$, $j = 4r$ and define $S_1 = \{v_{4i+1}, v_{4i+2} \mid 0 \leq i \leq \frac{k-1}{4} - 1\} \cup \{v_{4i+3}, v_{4i} \mid r \leq i \leq s\}$ if $j = k + 3$, or $S_1 = \{v_{4i+1}, v_{4i+2} \mid 0 \leq i \leq \frac{k-1}{4} - 1\} \cup \{v_{4i+3}, v_{4i} \mid r \leq i \leq s\} \cup \{v_{4i}, v_{4i+1} \mid \ell + 1 \leq i \leq r - 1\}$ when $j > k + 3$ and $S_2 = V(G) - S_1$. Obviously, S_1 and S_2 are two disjoint total dominating sets. Thus $d_t(G) \geq 2$ and the proof is complete. □

The proof of the following lemma is essentially similar to the proof of Lemma 2 of [6].

Lemma 2.3. *Let G be a graph obtained from a cycle $C = (v_1, v_2, \cdots, v_n)$ of even order by adding a random matching M. Then G has a 4-edge a.a.*

Proof. Define random variables X_i for $i = 1, 2, \cdots, n$ by

$$X_i = \begin{cases} 1 & \text{if } v_iv_{i+1} \in E(C) \text{ is a } 4-\text{edge} \\ 0 & \text{otherwise.} \end{cases}$$

and let $X = \sum_{i=1}^{n} X_i$. Then each X_i has expectation

$$E(X_i) = P(X_i = 1) = \frac{1}{2}(\frac{1}{4} + \frac{1}{n})^2 = \frac{1}{32} + O(1/n),$$

and variance

$$\text{var}(X_i) = E(X_i^2) - E(X_i)^2 = E(X_i) - E(X_i)^2 = \frac{31}{1024} + O(1/n).$$

The covariance of X_i and X_j for $i < j$ equals

$$\begin{aligned} \text{cov}(X_i, X_j) &= E(X_iX_j) - E(X_i)E(X_j) \\ &= \begin{cases} 1/1024 - (1/32)^2 + O(1/n) & \text{if } \quad i < j - 1 \\ \frac{8}{3}(\frac{1}{1024}) - (1/32)^2 + O(1/n) & \text{if } \quad i = j - 1 \textit{ and } n \equiv 2 \pmod 4 \\ 0 - (1/32)^2 + O(1/n) & \text{if } \quad i = j - 1 \textit{ and } n \equiv 0, 1, 3 \pmod 4 \end{cases} \\ &= \begin{cases} O(1/n) & \text{if } \quad i < j - 1 \\ O(1) & \text{if } \quad i = j - 1. \end{cases} \end{aligned}$$

Note that $X_iX_{i+1} = 1$ implies that $n \equiv 2 \pmod 4$. To see this, let v_k be the matching partner of v_{i+1}. If $X_iX_{i+1} = 1$, then v_iv_{i+1} and $v_{i+1}v_{i+2}$ are 4-edges and thus $n + 2 \equiv |[v_{i+1}, v_k]| + |[v_k, v_{i+1}]| \equiv 0 + 0 \equiv 0 \pmod 4$, i.e., $n \equiv 2 \pmod 4$.

Hence the random variable X has expectation

$$E(X) = \sum_{i=1}^{n} E(X_i) = n/32 + O(1) = O(n)$$

and variance

$$\begin{aligned}\mathrm{var}(X) &= \textstyle\sum_{i=1}^{n} \mathrm{var}(X_i) + 2\sum_{i<j-1} \mathrm{cov}(X_i, X_j) + 2\sum_{i=1}^{n} \mathrm{cov}(X_i, X_{i+1}) \\ &= \tfrac{31}{1024}n + 2\textstyle\sum_{i<j-1} O(1/n) + 2\sum_{i=1}^{n} O(1) \\ &= O(n)\end{aligned}$$

By Chebyschev's inequality, we have

$$\mathrm{prob}(X = 0) \leq \frac{\mathrm{var}(X)}{E(X)^2} = \frac{O(n)}{(O(n))^2} = O(1/n).$$

Hence $X > 0$ a.a., i.e., G has a 4-edge. □

An immediate consequence of Theorem A and Lemmas 2.2 and 2.3 now follows.

Theorem 2.4. If G is a random 3-regular graph, then $d_t(G) \geq 2$ a.a.

3. An upper bound for the total domatic number

The proof of the following theorem is essentially similar to the proof of Theorem 2 of [6].

Theorem 3.1. Let $r \geq 3$ and let G be a random r-regular graph. Then $d_t(G) \leq r - 1$.

Proof. Suppose to the contrary that G is an r-regular graph with $d_t(G) > r - 1$. It follows from Theorem B that $d_t(G) = r$. Let $V_1, V_2, \cdots, V_r$ be a total domatic partition for G. Then each vertex has a neighbor in every V_i. Since every vertex has precisely r neighbors, we deduce that every vertex in V_i has precisely one neighbor in V_j for each j. Hence,

$$|N(v) \cap V_i| = 1 \quad \text{for all } v \in V(G) \text{ and } i \in \{1, 2, \ldots, r\}.$$

For $i \neq j$ we deduce that

$$E_{ij} := \{uv \in E(G) | u \in V_i, v \in V_j\} \text{ is a perfect } V_i - V_j \text{ matching} \tag{3.1}$$

and so

$$|V_1| = |V_2| = \cdots = |V_r| = \frac{n}{r}. \tag{3.2}$$

It follows from the above argument that every r-regular graph with $d_t = r$ on the vertex set $V(G)$ can be obtained by first partitioning $V(G)$ into r sets, all of equal cardinality, and then adding a perfect matching between the vertices of every partition, implying that n/r is even, and finally adding a perfect matching between all pairs of partition sets. Suppose n is a multiple of r. Since the sets are not distinguishable, the first step can be done in

$$\begin{pmatrix} n \\ n/r, n/r, \cdots, n/r \end{pmatrix} \frac{1}{r!}$$

ways, the second step can be done in

$$[(\frac{n}{r} - 1)(\frac{n}{r} - 3) \cdots 1]^r = \frac{((\frac{n}{r})!)^r}{2^{n/2}((\frac{n}{2r})!)^r}$$

ways, and the last step can be done in

$$((\frac{n}{r})!)^{\binom{r}{2}}$$

ways, since there are $\binom{r}{2}$ different pairs of sets V_i, V_j, and between each pair a matching can be added in $(\frac{n}{r})!$ ways. Hence, an upper bound on the number of labeled r-regular graphs of order n with $d_t = r$, is

$$\binom{n}{n/r, n/r, \cdots,, n/r} \frac{1}{r!}((\frac{n}{r})!)^{\binom{r}{2}} \frac{((\frac{n}{r})!)^r}{2^{n/2}((\frac{n}{2r})!)^r}$$
$$= \frac{n!}{((\frac{n}{r})!)^r r!}((\frac{n}{r})!)^{\frac{r(r-1)}{2}} \frac{((\frac{n}{r})!)^r}{2^{n/2}((\frac{n}{2r})!)^r},$$

and hence, by Stirling's formula ($n! \sim (\frac{n}{e})^n\sqrt{2\pi n}(1+\frac{1}{12n}+O(\frac{1}{n^2}))$) the upper bound is, for large n and constant r,
$(\frac{n}{e})^n\sqrt{2\pi n}(1+\frac{1}{12n}+O(\frac{1}{n^2})).\frac{1}{r!2^{n/2}}.(\frac{n}{re})^{\frac{r(r-1)}{2}}$

$.[\sqrt{2\pi n/r}(1+\frac{1}{12n}+O(\frac{1}{n^2}))]^{\frac{r(r-1)}{2}}.\frac{1}{(\frac{n}{2re})^{rn}(\sqrt{\pi n/r}(1+\frac{1}{12n}+O(\frac{1}{n^2})))^r}.$

Denote this last expression by DOMT(r,n). The total number of r-regular graphs, as given in [3] is asymptotic to

$$e^{-(r^2-1)/4}\frac{(rn)!}{(rn/2)!2^{rn/2}(r!)^n} = \frac{e^{-(r^2-1)/4}}{2^{rn/2}(r!)^n}.\frac{(\frac{rn}{e})^n\sqrt{2\pi rn}(1+\frac{1}{12n}+O(\frac{1}{n^2}))}{(\frac{rn}{2e})^n\sqrt{\pi rn}(1+\frac{1}{12n}+O(\frac{1}{n^2}))}$$

Denote this last expression by TOTAL(r,n). Then the proportion of r-regular graphs with $d_t = r$, DOMT(r,n)/TOTAL(r,n), is at most

$$(\frac{r!}{r^{r-1}})^n O(n^{\frac{(r-1)(r-2)}{4}}).$$

Since $\frac{r!}{r^{r-1}}$ is less than 1, so the limit DOMT(r,n)/TOTAL(r,n) tends to 0, as desired. This completes the proof. □

4. **Total domatic number and minimum degree**

If G is a graph of order n, then Zelinka [12] gave the following lower bound on the total domatic number

$$d_t(G) \geq \lfloor\frac{n}{n-\delta(G)+1}\rfloor.$$

The proof of the following theorem is essentially similar to the proof of Theorem 3 of [6] and we leave it to the reader.

Theorem 4.1. Let G be a graph of order n with minimum degree δ and maximum degree Δ, and let k be a nonnegative integer. If

$$e(\Delta^2+1)k(1-\frac{1}{k})^{\delta} < 1,$$

then $d_t(G) \geq k$.

For the special case of a regular graph, we obtain a significant improvement of Zelinka's bound.

Corollary 4.2. Let G be an r-regular graph with $r \geq 3$. Then

$$d_t(G) \geq \frac{r}{3 \ln r}.$$

Proof. With $\Delta = \delta = r$ and $k = \frac{r}{3 \ln r}$ we have

$$\begin{aligned} e(\Delta^2 + 1)k(1 - \tfrac{1}{k})^{\delta} &= e(r^2 + 1)k(1 - \tfrac{1}{k})^r \\ &\leq e(r^2 + 1)\tfrac{r}{3 \ln r} e^{-\frac{3 \ln r}{r} r} \\ &= \tfrac{e(r^2+1)}{3r^2 \ln r} \\ &< 1. \end{aligned}$$

Now it follows from Theorem 4.1 that $d_t(G) \geq \frac{r}{3 \ln(r)}$. □

A question that arises naturally is whether the bound in Corollary 4.2 is best possible. For a positive integer r, let $f(r)$ be the minimum total domatic number of all r-regular graphs. By Corollary 4.2 we have $f(r) \geq \frac{r}{3 \ln(r)}$. On the other hand, it follows from [1] that there exist r-regular graphs of order n with total domination number $(1 + o(1))\frac{n \ln(r)}{r}$. According to Theorem B, the total domatic number of those graphs is at most $n/\gamma_t(G) = (1 + o(1))\frac{r}{\ln(r)}$. This proves $f(r) = \Omega(\frac{r}{\ln(r)})$, and the order of magnitude of the bound in Corollary 4.2 is best possible.

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