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ON •-LICT SIGNED GRAPHS $L_{\bullet_c}(S)$ AND •-LINE SIGNED GRAPHS $L_\bullet(S)$

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Abstract. A *signed graph* (or, in short, *sigraph*) $S = (S^u, \sigma)$ consists of an underlying graph $S^u := G = (V, E)$ and a function $\sigma : E(S^u) \longrightarrow \{+, -\}$, called the signature of S. A *marking* of S is a function $\mu : V(S) \longrightarrow \{+, -\}$. The *canonical marking* of a signed graph S, denoted μ_σ, is given as

$$\mu_\sigma(v) := \prod_{vw \in E(S)} \sigma(vw).$$

The *line graph* of a graph G, denoted $L(G)$, is the graph in which edges of G are represented as vertices, two of these vertices are adjacent if the corresponding edges are adjacent in G. There are three notions of a *line signed graph* of a signed graph $S = (S^u, \sigma)$ in the literature, viz., $L(S)$, $L_\times(S)$ and $L_\bullet(S)$, all of which have $L(S^u)$ as their underlying graph; only the rule to assign signs to the edges of $L(S^u)$ differ. Every edge ee' in $L(S)$ is negative whenever both the adjacent edges e and e' in S are negative, an edge ee' in $L_\times(S)$ has the product $\sigma(e)\sigma(e')$ as its sign and an edge ee' in $L_\bullet(S)$ has $\mu_\sigma(v)$ as its sign, where $v \in V(S)$ is a common vertex of edges e and e'.

The line-cut graph (or, in short, *lict graph*) of a graph $G = (V, E)$, denoted by $L_c(G)$, is the graph with vertex set $E(G) \cup C(G)$, where $C(G)$ is the set of cut-vertices of G, in which two vertices are adjacent if and only if they correspond to adjacent edges of G or one vertex corresponds to an edge e of G and the other vertex corresponds to a cut-vertex c of G such that e is incident with c.

In this paper, we introduce *dot-lict signed graph* (or *•-lict signed graph*) $L_{\bullet_c}(S)$, which has $L_c(S^u)$ as its underlying graph. Every edge uv in $L_{\bullet_c}(S)$ has the sign $\mu_\sigma(p)$, if $u, v \in E(S)$ and $p \in V(S)$ is a common vertex of these edges, and it has the sign $\mu_\sigma(v)$, if $u \in E(S)$ and $v \in C(S)$. We characterize signed graphs on K_p, $p \geq 2$, on cycle C_n and on $K_{m,n}$ which are •-lict signed graphs or •-line signed graphs, characterize signed graphs S so that $L_{\bullet_c}(S)$ and $L_\bullet(S)$ are balanced. We also establish the characterization of signed graphs S for which $S \sim L_{\bullet_c}(S)$, $S \sim L_\bullet(S)$, $\eta(S) \sim L_{\bullet_c}(S)$ and $\eta(S) \sim L_\bullet(S)$, here $\eta(S)$ is negation of S and $\sim$ stands for switching equivalence.

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1. Introduction

For graph theory terminology in general we refer the reader to [6]. All graphs considered here are simple, connected and finite. Let $V(G)$, $E(G)$ and $C(G)$ respectively denote the vertex set, the edge set and the cut-vertex set of a graph G.

A *signed graph* is an ordered pair $S = (S^u, \sigma)$, where $S^u := G = (V, E)$ is a graph called the underlying graph of S and $\sigma : E(S^u) \longrightarrow \{+, -\}$ is a function, called the *signature* of S. In general terms, we say that the edges are *signed* by σ. In a pictorial representation of a signed graph S, its positive edges are shown as bold line segments ('Jorden curves' drawn on the plane) and negative edges as broken line segments as shown in Figure 1. $E^+(S) = \{e \in E(S^u) : \sigma(e) = +\}$ and $E^-(S) = \{e \in E(S^u) : \sigma(e) = -\}$. The elements of $E^+(S)$ ($E^-(S)$) are called *positive* (*negative*) *edges* of S and the set $E(S) = E^+(S) \cup E^-(S)$ is called the edge set of S.

A signed graph in which all the edges are positive, we regard as *all-positive signed graph* (*all-negative signed graph* is defined similarly). A signed graph is said to be *homogeneous* if it is either all-positive or all-negative and *heterogeneous* otherwise. By $d(v)$, we denote degree of $v \in V(S)$, $d(v) = d^+(v) + d^-(v)$, where $d^+(v)$ ($d^-(v)$) denote the positive (negative) degree of v.

A *marking* of S is a function $\mu : V(S) \longrightarrow \{+, -\}$. Sampathkumar introduced the idea of marking the vertices with signs derived from the edges signs in [10], which is $\mu_\sigma : V(S) \longrightarrow \{+, -\}$ given as

$$\mu_\sigma(v) := \prod_{vw \in E(S)} \sigma(vw).$$

This marking is called canonical marking. In this paper, a vertex $v \in V(S)$ of $d^-(v)$ even or $\mu_\sigma(v) = +$ is called *positive vertex* (*negative vertex* is defined similarly).

Lemma 1.1 (**Sampathkumar** [10]). *In any canonically marked signed graph there are an even number of vertices marked negative.*

A cycle in a signed graph is said to be positive (negative) if the product of the signs of its edges is positive (negative); that is, it contains an even (odd) number of negative edges. A signed graph is said to be *balanced*, if every cycle in it is positive [7].

Lemma 1.2 (**Zaslavsky** [14]). *A signed graph in which every chordless cycle is positive, is balanced.*

A cycle in a signed graph is said to be *consistent* with respect to marking μ if it contains an even number of negative vertices and a signed graph is said to be consistent with respect to marking μ if every cycle in it is consistent [11]. Similarly, a cycle of a signed graph is called *canonically consistent (or $\mathcal{C}$-consistent)* if it contains an even number of negative vertices and a signed graph is said to be $\mathcal{C}$-consistent if every cycle in it is $\mathcal{C}$-consistent.

Signed graphs S_1 and S_2 are said to be *isomorphic*, written as $S_1 \cong S_2$, if there is a graph isomorphism $f : S_1^u \to S_2^u$ that preserves edge signs.

One of the important operations on signed graphs involves changing signs of their edges. The *negation* of a signed graph S, denoted by $\eta(S)$, is obtained by negating the sign of every edge of S, i.e., by changing the sign of every edge to its opposite [8].

The idea of *switching* of a signed graph was introduced in connection with structural analysis of marking μ of a signed graph S in [1, 13]. Switching of S with respect to a marking μ is the operation of changing the sign of every edge of S to its opposite whenever its end vertices are of opposite signs. The signed graph obtained in this way is denoted by $\mathbb{S}_\mu(S)$ and is called *μ-switched* signed graph or just switched signed graph.

Further, a signed graph $S_1 = (S_1^u, \sigma)$ *switches* to a signed graph $S_2 = (S_2^u, \sigma')$ (or that S_1 and S_2 are *switching equivalent*) written as $S_1 \sim S_2$ whenever there exists a marking μ of S_1 such that $\mathbb{S}_\mu(S_1) \cong S_2$. Since the definition of switching does not change the underlying graphs of the respective signed graphs, $S_1 \sim S_2$ implies that $S_1^u \cong S_2^u$.

Signed graphs S_1 and S_2 are said to be *weakly isomorphic* (see [12]) or *cycle isomorphic* (see [13]) if there is a graph isomorphism $f : S_1^u \to S_2^u$ that preserves cycle signs. The following result is well known:

Theorem 1.3 (**Zaslavsky** [13]). *Two signed graphs S_1 and S_2 with the same underlying graph are switching equivalent if and only if they are cycle isomorphic.*

2. •-line signed graph and •-lict signed graph

The *line graph* of a graph G, denoted $L(G)$, is the graph in which edges of G are represented as vertices, two of these vertices are adjacent if the corresponding edges are adjacent in G (*c.f.*:[6], *chapter 8*). The following theorem is the well-known characterization of line graphs.

Theorem 2.1. [6] *The following statements are equivalent:*

- **(1)** *G is a line graph.*
- **(2)** *The lines of G can be partitioned into complete subgraphs in such a way that no point lies in more than two of these subgraphs.*

Theorem 2.2 ([6]). *For a connected graph G, $G \cong L(G)$ if and only if G is a cycle.*

There are three notions of a *signed line graph* of a signed graph $S = (S^u, \sigma)$ in the literature, viz., $L(S)$, $L_\times(S)$ and $L_\bullet(S)$, all of which have $L(S^u)$ as their underlying graph; only the rule to assign signs to the edges of $L(S^u)$ differ. Every edge ee' in $L(S)$ is negative whenever both the adjacent edges e and e' in S are negative [5], an edge ee' in $L_\times(S)$ has the product $\sigma(e)\sigma(e')$ as its sign [4] and an edge

ee' in $L_\bullet(S)$ has $\mu_\sigma(v)$ as its sign, where $v \in V(S)$ is a common vertex of edges e and e' [3]. Note that for a graph G, $L(G) \cong L_\times(G) \cong L_\bullet(G)$ as G is an all-positive signed graph. For an all-negative signed graph S in which every vertex is positive, $L_\times(S) \cong L_\bullet(S)$ and are all-positive.

Figure 1 illustrates a signed graph and its line, $\times$-line and $\bullet$-line signed graphs.

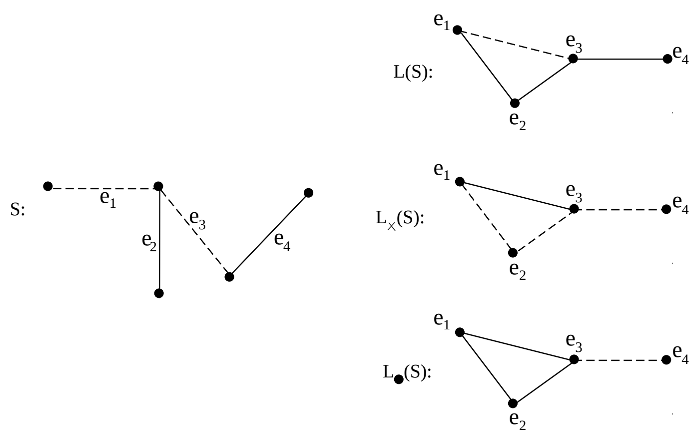

FIGURE 1. A signed graph S and its $L(S)$, $L_\times(S)$ and $L_\bullet(S)$

A signed graph S is said to be a line ($\times$-line or $\bullet$-line) signed graph if there exists a signed graph T whose line ($\times$-line or $\bullet$-line) signed graph $L(T)$ ($L_\times(T)$ or $L_\bullet(T)$) is isomorphic to S; and this signed graph T is called the line ($\times$-line or $\bullet$-line) root of S.

Kulli and Muddebihal [9] introduced the idea of a line-cut graph:

Definition 2.3. *The line-cut graph $L_c(G)$ (also known as the lict graph) of a graph $G = (V, E)$ is a graph having vertex set $E(G) \cup C(G)$, in which two vertices are adjacent if and only if they correspond to adjacent edges of G or one vertex corresponds to an edge e of G and the other vertex corresponds to a cut-vertex c of G such that e is incident with c in G; that is, $L_c(G)$ is the intersection graph $\Omega(E(G) \cup C(G))$. Clearly, $L_c(G) \cong L(G)$ if $C(G) = \phi$.*

A graph G is said to be a lict graph if there exists a graph H whose lict graph $L_c(H)$ is isomorphic to G.

A *clique* of a graph is its maximal complete subgraph. The following theorem gives the characterization of lict graphs.

Theorem 2.4 (**Acharya et al.** [2]). *The following statements are equivalent:*

(1) *$G = (V, E)$ is a lict graph.*

(2) *The edges of G can be partitioned into cliques in such a way that no vertex lies in more than two of these cliques and for each clique G',*

  (i) *if each vertex of G' lies in two cliques of the partition, then $G - E(G')$ is connected; and*

(ii) *if each vertex of G' lies in two cliques of the partition except exactly one vertex (say v) of G' then $G - E(G') - v$ is disconnected. G does not contain a pendant vertex.*

Theorem 2.5 (**Kulli and Muddebihal** [9]). *For a connected graph G, $G \cong L_c(G)$ if and only if G is a cycle.*

In this paper, we introduce $\bullet$-lict signed graph $L_{\bullet_c}(S)$ as follows:

Definition 2.6. *The dot-lict signed graph (or $\bullet$-lict signed graph) of a signed graph $S = (S^u, \sigma)$, is a signed graph $L_{\bullet c}(S) = (L_c(S^u), \sigma')$, where for every edge uv of $L_c(S^u)$*

$$\sigma'(uv) = \begin{cases} \mu_\sigma(p) & \text{if } u, v \in E(S) \text{ and } p \in V(S) \text{ is a common vertex of these edges;} \\ \mu_\sigma(v) & \text{if } u \in E(S) \text{ and } v \in C(S) \end{cases}$$

Figure 2 illustrates a signed graph and its $\bullet$-lict signed graph.

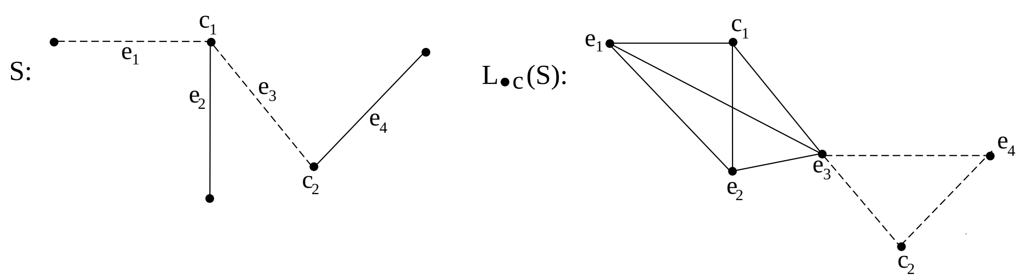

FIGURE 2. A signed graph S and its $\bullet$-lict signed graph $L_{\bullet c}(S)$

Note that for a graph G, $L_c(G) \cong L_{\bullet c}(G)$ as G is an all-positive signed graph. For a signed graph S in which every non-pendant vertex is positive, $L_{\bullet c}(S) \cong L_c(S^u)$.

A signed graph S is said to be a $\bullet$-lict signed graph if there exists a signed graph T whose $\bullet$-lict signed graph $L_{\bullet c}(T)$ is isomorphic to S; and this signed graph T is called the $\bullet$-lict root of S.

3. **Main Results**

Theorem 3.1. *A signed graph $S = (S^u, \sigma)$, on a complete graph $S^u := K_p$, $p \geq 3$, is a $\bullet$-lict signed graph if and only if S is homogeneous or a triangle having two negative edges.*

Proof. Necessity:
Let $S = (S^u, \sigma)$, on a complete graph K_p, $p \geq 3$, be a $\bullet$-lict signed graph. Therefore, $S \cong L_{\bullet c}(T)$ for some signed graph $T = (T^u, \sigma')$. This implies that $S^u \cong L_{\bullet c}(T^u)$, i.e., $K_p \cong L_c(T^u)$. By the definition of lict graph it is clear that

$$T^u = \begin{cases} K_3 \text{ or } K_{1,2} & \text{if p=3;} \\ K_{1,p-1} & \text{if } p \geq 4. \end{cases}$$

- If $T^u := K_3$, then for homogeneous T on T^u, S is all-positive triangle and for heterogeneous T, S is a triangle having two negative edges, as shown in Figure 3.

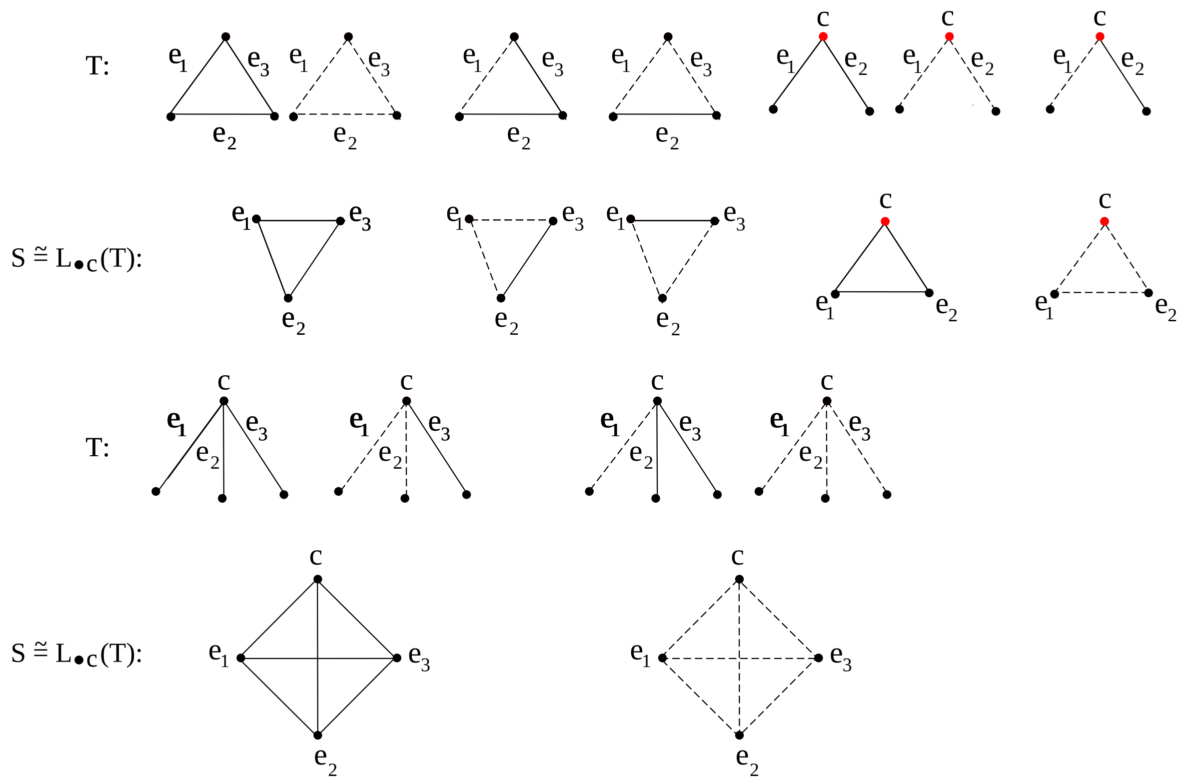

FIGURE 3. Signed graphs S and T such that $S \cong L_{\bullet c}(T)$

- If $T^u := K_{1,p-1}$, $p \geq 3$, then since non-pendant vertex is positive or negative, S is homogeneous. Thus, the necessity follows.

Sufficiency:

Suppose conditions hold. We give the construction of a signed graph T whose $\bullet$-lict signed graph is S. Let $\mathcal{P}(S) = \{S_1, S_2, \ldots, S_n\}$ be the partition of $E(S)$ into homogeneous cliques. The vertices of T^u correspond to the set $\mathcal{P}(S)$ together with the set U of vertices of S belonging only to one of the homogeneous cliques S_i leaving one such vertex for each S_i. Thus $V(T^u) = \mathcal{P}(S) \cup U$, two of these vertices are adjacent whenever they have a nonempty intersection in S. Now, assign '+' ('-') sign to each non-pendant vertex $S_i \in V(T^u)$ if it corresponds to an all-positive (all-negative) S_i in S and take signature of T in such a way that signs assigned to vertices of T^u are preserved under canonical marking of T. For this signed graph T, $S \cong L_{\bullet c}(T)$; that is, S is a $\bullet$-lict signed graph. This completes the proof. □

Figure 4 illustrates construction of a signed graph T such that $S \cong L_{\bullet c}(T)$ for a signed graph S that satisfies sufficiency condition of Theorem 3.1. Note that T need not be unique.

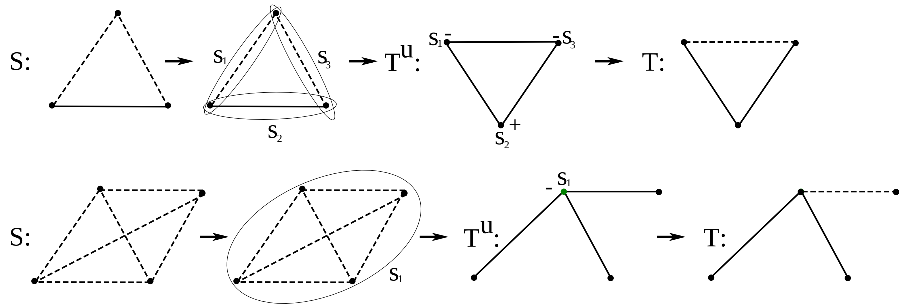

FIGURE 4. The construction of signed graph T from S such that $S \cong L_{\bullet c}(T)$

Theorem 3.2. *A signed graph $S = (S^u, \sigma)$, on a complete graph $S^u := K_p$, $p \geq 2$, is a $\bullet$-line signed graph if and only if S is homogeneous or a triangle having two negative edges.*

Proof. Necessity:
Let $S = (S^u, \sigma)$, on a complete graph K_p, $p \geq 2$, be a $\bullet$-line signed graph. Therefore, $S \cong L_\bullet(T)$ for some signed graph $T = (T^u, \sigma')$. This implies that $S^u \cong L_\bullet(T^u)$, i.e., $K_p \cong L(T^u)$. Clearly,

$$T^u = \begin{cases} K_3 \ \ or \ \ K_{1,3} & \text{if p=3;} \\ K_{1,p} & \text{if } p = 2 \text{ or } p \geq 4. \end{cases}$$

- If $T^u := K_3$, then for homogeneous T on T^u, S is all-positive triangle and for heterogeneous T, S is a triangle having two negative edges, as shown in Figure 5.
- If $T^u := K_{1,p}$, $p \geq 2$, then since non-pendant vertex is positive or negative, S is homogeneous. Thus, the necessity follows.

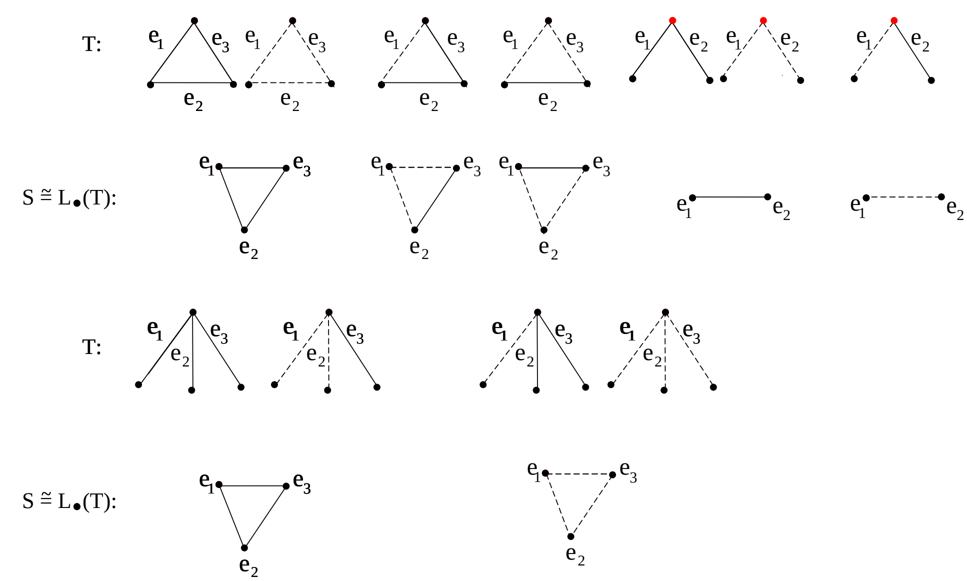

FIGURE 5. Signed graph T such that $S \cong L_\bullet(T)$

Sufficiency:

Suppose conditions hold. We give the construction of a signed graph T whose $\bullet$-line signed graph is S. Let $\mathcal{P}(S) = \{S_1, S_2, \ldots, S_n\}$ be the partition of $E(S)$ into homogeneous complete subgraphs. The vertices of T^u correspond to the set $\mathcal{P}(S)$ together with the set U of vertices of S belonging only to one of the homogeneous complete subgraphs S_i. Thus $V(T^u) = \mathcal{P}(S) \cup U$, two of these vertices are adjacent whenever they have a nonempty intersection. Now, assign '+' ('-') sign to each non-pendant vertex $S_i \in V(T^u)$ if it corresponds to an all-positive (all-negative) S_i in S and take signature of T in such a way that signs assigned to vertices of T^u are preserved under canonical marking of T. For this signed graph T, $S \cong L_\bullet(T)$; that is, S is a $\bullet$-line signed graph. This completes the proof. □

Figure 6 illustrates construction of a signed graph T such that $S \cong L_\bullet(T)$ for a signed graph S that satisfies sufficiency condition of Theorem 3.2.

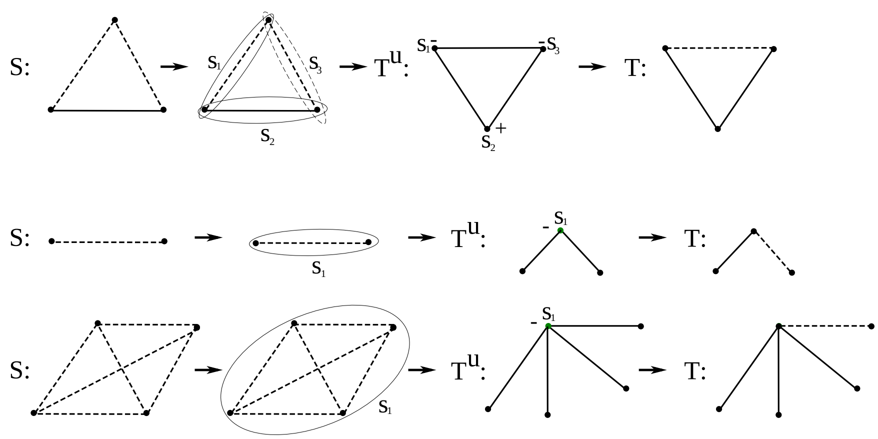

FIGURE 6. The construction of signed graph T from S such that $S \cong L_\bullet(T)$

Theorem 3.3. *A signed graph $S = (S^u, \sigma)$, on a cycle $S^u := C_n$, is $\bullet$-lict signed graph if and only if S is an all-negative triangle or a positive cycle.*

Proof. Necessity:

Let $S = (S^u, \sigma)$, on cycle C_n, be a $\bullet$-lict signed graph. Therefore, $S \cong L_{\bullet c}(T)$ for some signed graph $T = (T^u, \sigma')$. This implies that $S^u \cong L_{\bullet c}(T^u)$, i.e., $C_n \cong L_c(T^u)$. By the definition of lict graph it is clear that

$$T^u = \begin{cases} C_3 \;\; or \;\; K_{1,2} & \text{if n=3;} \\ C_n & \text{if } n \geq 4. \end{cases}$$

- If $T^u := K_{1,2}$, then for T on T^u containing one negative edge, S is an all-negative triangle, as shown in Figure 3 and for other $T's$, S is a positive triangle.
- If $T^u := C_n$, then by the definition of $L_{\bullet c}(T)$, $|E^-(L_{\bullet c}(T))|$ = the number of negative vertices in T. By Lemma 1.1, in any canonically marked signed graph there are an even number of vertices marked negative. Hence $|E^-(L_{\bullet c}(T))|$ = even, i.e., S is a positive cycle.

  Thus the necessity follows.

Sufficiency:
Suppose conditions hold. We give the construction of a signed graph T by the procedure as discussed in the sufficiency of Theorem 3.1. For this signed graph T, $S \cong L_{\bullet c}(T)$; that is, S is a $\bullet$-lict signed graph. This completes the proof. $\square$

Figure 7 illustrates construction of a signed graph T such that $S \cong L_{\bullet c}(T)$ for a signed graph S that satisfies sufficiency condition of Theorem 3.3.

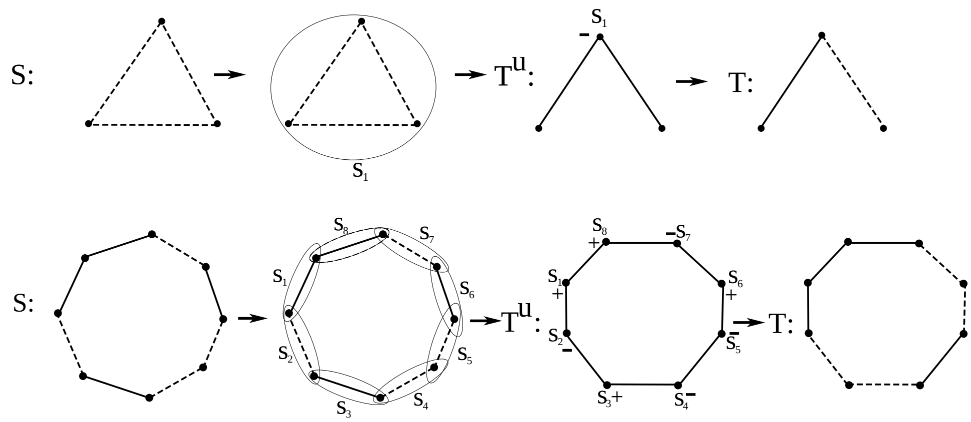

FIGURE 7. The construction of signed graph T from S such that $S \cong L_{\bullet c}(T)$

Corollary 3.4. *A signed graph $S = (S^u, \sigma)$, on a cycle $S^u := C_n$, is $\bullet$-line signed graph if and only if S is an all-negative triangle or a positive cycle.*

Proposition 3.5. *For a signed graph S on cycle C_n, $L_{\bullet c}(S) \cong L_{\bullet}(S)$ is a positive cycle.*

Proof. For a signed graph S on cycle C_n, $L_{\bullet c}(S) \cong L_{\bullet}(S)$, since $C(S) = \phi$. By the definition of $L_{\bullet c}(S)$, $|E^{-}(L_{\bullet c}(S))|$ = the number of negative vertices in S. By Lemma 1.1, in any canonically marked signed graph there are an even number of vertices marked negative. Hence $|E^{-}(L_{\bullet c}(S))|$ = even, i.e., $L_{\bullet c}(S)$ is a positive cycle. $\square$

Theorem 3.6. *A signed graph $S = (S^u, \sigma)$ on a complete bipartite graph $S^u := K_{m,n}$, is a $\bullet$-lict signed graph if and only if S is a positive cycle of order 4.*

Proof. Necessity:
Let $S = (S^u, \sigma)$ on a complete bipartite graph $K_{m,n}$, be a $\bullet$-lict signed graph. Therefore, $S \cong L_{\bullet c}(T)$ for some signed graph T. This implies that $S^u \cong L_{\bullet c}(T^u)$ or $K_{m,n} \cong L_c(T^u)$, i.e., $K_{m,n}$ is a lict graph.

Let v be a cut-vertex of T^u then clearly $d(v) \geq 2$ and by the definition of lict graph, the edges incident with cut-vertex v in T^u induce a complete subgraph $K_{d(v)+1}$, i.e., K_p, $p \geq 3$ in $S^u := K_{m,n}$. Since a complete bipartite graph does not contain any odd cycle, K_p, $p \geq 3$ can not be a subgraph of $K_{m,n}$. Hence, $C(T^u) = \phi$. Therefore $K_{m,n}$ is also a line graph. Since $K_{1,3}$ is a forbidden induced subgraph of a line graph, $m \leq 2$ and $n \leq 2$. Furthermore by Theorem 2.4, a lict graph does not contain a pendant vertex, $K_{m,n} \ncong K_{1,1}$ and $K_{1,2}$. Thus $K_{m,n} \cong C_4$ and by Theorem 3.3, S is a positive cycle of order 4.

Sufficiency:

Suppose S is a positive cycle of order 4 then by Theorem 3.3, S is a $\bullet$-lict signed graph. This completes the proof. □

Corollary 3.7. *A signed graph $S = (S^u, \sigma)$ on a complete bipartite graph $S^u := K_{m,n}$, is a $\bullet$-line signed graph if and only if S is any one of the following:*

- **(i)** *any signed graph on $K_{1,1}$ or $K_{1,2}$*
- **(ii)** *a positive cycle of order 4.*

Theorem 3.8. *For a signed graph S, $L_{\bullet c}(S)$ is balanced if and only if the following conditions hold in S:*

- **(i)** *S is $\mathcal{C}$-consistent and;*
- **(ii)** *each vertex v of $d(v) \geq 3$ and cut-vertex of degree 2 are positive vertices.*

Proof. Necessity:

Suppose for a signed graph S, $L_{\bullet c}(S)$ is balanced, i.e., every cycle in $L_{\bullet c}(S)$ is a positive cycle. By the definition of $L_{\bullet c}(S)$, a cycle Z in S induces a cycle Z' in $L_{\bullet c}(S)$ and $|E^-(Z')|$ = the number of negative vertices in Z. Since Z' is a positive cycle, every cycle Z in S contains an even number of negative vertices; that is, S is $\mathcal{C}$-consistent.

Next, we prove the necessity of condition **(ii)** by contrapositive:

Assume that a vertex $v \in V(S)$ which is of degree≥ 3 or a cut-vertex of degree 2, is a negative vertex, i.e., $d^-(v)$ is odd or $\mu_\sigma(v) = -$. Then by the definition of $L_{\bullet c}(S)$, the edges incident with v will induce an all-negative complete subsignedgraph of order ≥ 3 in $L_{\bullet c}(S)$ that makes $L_{\bullet c}(S)$ unbalanced. Thus, the necessity of **(ii)** follows.

Sufficiency:

A cycle in $L_{\bullet c}(S)$ is induced due to a cycle or a cut-vertex of degree 2 or a vertex of degree≥ 3 or their combinations in S. Suppose conditions **(i)** and **(ii)** hold in S then every chordless cycle in $L_{\bullet c}(S)$ will be positive. By Lemma 1.2, a signed graph in which every chordless cycle is positive, is balanced. Hence $L_{\bullet c}(S)$ is balanced. This completes the proof. □

Corollary 3.9. *$L_\bullet(S)$ is balanced if and only if the following conditions hold in signed graph S:*

- **(i)** *S is $\mathcal{C}$-consistent and;*
- **(ii)** *each vertex v of $d(v) \geq 3$ is a positive vertex.*

Theorem 3.10. *For a signed graph S, $S \sim L_{\bullet c}(S)$ if and only if S is a positive cycle.*

Proof. Suppose for a signed graph S, $S \sim L_{\bullet c}(S)$. This implies that $S^u \cong L_{\bullet c}(S^u)$, i.e., $S^u \cong L_c(S^u)$. By Theorem 2.5, S^u is a cycle and by Proposition 3.5, $L_{\bullet c}(S)$ is a positive cycle. By Theorem 1.3, two signed graphs S_1 and S_2 with the same underlying graph are switching equivalent if and only if they are cycle isomorphic. Hence S is a positive cycle.

Conversely, Suppose S is a positive cycle then by Proposition 3.5, $L_{\bullet c}(S)$ is also a positive cycle. Hence by Theorem 1.3, $S \sim L_{\bullet c}(S)$, as shown in Figure 8. Thus the result follows. □

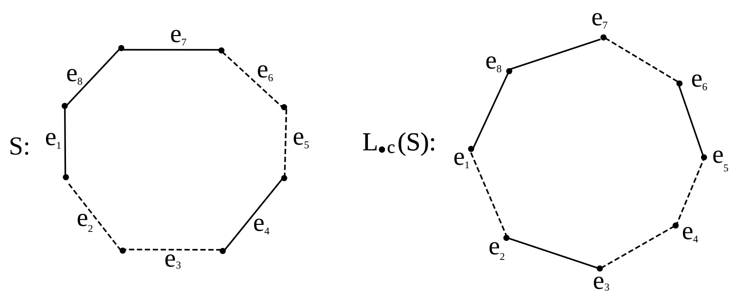

FIGURE 8. A signed graph S such that $S \sim L_{\bullet c}(S)$

Proposition 3.11. *For a signed graph S, $S \sim L_{\bullet}(S)$ if and only if S is a positive cycle.*

Theorem 3.12. *For a signed graph S, $\eta(S) \sim L_{\bullet c}(S)$ if and only if S is a positive even cycle or a negative odd cycle.*

Proof. Necessity:
Suppose for a signed graph S, $\eta(S) \sim L_{\bullet c}(S)$. This implies that $S^u \cong L_{\bullet c}(S^u)$, i.e., $S^u \cong L_c(S^u)$. By Theorem 2.5, S^u is a cycle and by Proposition 3.5, $L_{\bullet c}(S)$ is a positive cycle. By Theorem 1.3, two signed graphs S_1 and S_2 with the same underlying graph are switching equivalent if and only if they are cycle isomorphic. Hence $\eta(S)$ is also a positive cycle; that is, S is a positive even cycle or a negative odd cycle.

Sufficiency:
Suppose S is a positive even cycle or a negative odd cycle. Then clearly $\eta(S)$ is a positive cycle, as shown in Figure 9 and by Proposition 3.5, $L_{\bullet c}(S)$ is a also positive cycle. Hence, by Theorem 1.3, $\eta(S) \sim L_{\bullet c}(S)$. This completes the proof. □

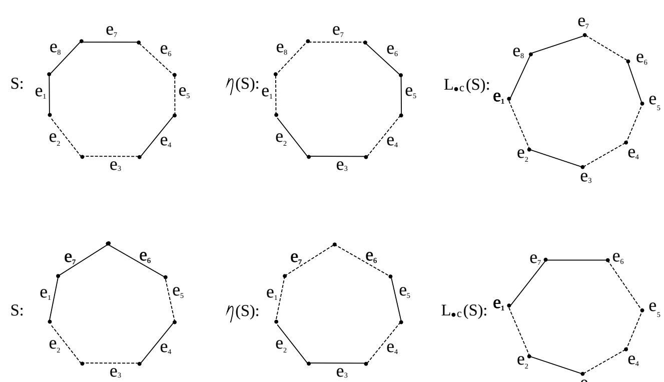

FIGURE 9. A signed graph S such that $\eta(S) \sim L_{\bullet c}(S)$

Proposition 3.13. *For a signed graph S, $\eta(S) \sim L_\bullet(S)$ if and only if S is a positive even cycle or a negative odd cycle.*

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