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WEIGHTED SZEGED INDICES OF SOME GRAPH OPERATIONS

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Abstract. In this paper, the weighted Szeged indices of Cartesian product and Corona product of two connected graphs are obtained. Using the results obtained here, the weighted Szeged indices of the hypercube of dimension n, Hamming graph, C_4 nanotubes, nanotorus, grid, $t-$fold bristled, sunlet, fan, wheel, bottleneck graphs and some classes of bridge graphs are computed.

1. Introduction

All the graphs considered in this paper are simple. A vertex $x \in V(G)$ is said to be *equidistant* from the edge $e = uv$ of G if $d_G(u, x) = d_G(v, x)$, where $d_G(u, x)$ denotes the distance between u and x in G; otherwise, x is a nonequidistant vertex. The degree of a vertex $x \in V(G)$ is denoted by $d_G(x)$.

For an edge $uv = e \in E(G)$, the number of vertices of G whose distance to the vertex u is less than the distance to the vertex v in G is denoted by $n_u(e, G)$; analogously, $n_v(e, G)$ is the number of vertices of G whose distance to the vertex v in G is less than the distance to the vertex u; the vertices equidistant from both the ends of the edge $e = uv$ are not counted.

The two topological indices, namely, the Szeged index and weighted Szeged index of G, denoted by $Sz(G)$ and $Sz_w(G)$, respectively, are defined as follows:

$$Sz(G) = \sum_{e = uv \in E(G)} n_u(e, G)\, n_v(e, G),$$

$$Sz_w(G) = \sum_{e = uv \in E(G)} \Big(d_G(u) + d_G(v)\Big) n_u(e, G)\, n_v(e, G).$$

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The *Cartesian product* $G \,\Box\, H$ of the graphs G and H has the vertex set $V(G \,\Box\, H) = V(G) \times V(H)$ and $(u,x)(v,y)$ is an edge of $G \,\Box\, H$ if $u = v$ and $xy \in E(H)$ or, $uv \in E(G)$ and $x = y$. Notation and definitions which are not given here can be found in [1] or [6]. For graphs G and H the *corona product* $G \circ H$ of two graphs G and H is obtained by taking $|V(G)|$ copies of H and joining each vertex of ith copy with vertex $v_i \in V(G)$.

The Szeged index studied by Gutman [4], Gutman and Dobrynin [5] and Khadikar et. al. [9] is closely related to the Wiener index of a graph. Basic properties of Szeged index and its analogy to the Wiener index are discussed by Klavžar et. al.[8]. It is proved that for a tree T the Wiener index of T is equal to its Szeged index. Ashrafi et. al. [10] have explained the differences between Szeged and Wiener indices of graphs. The mathematical properties and chemical applications of Szeged index are well studied by Dobrynin et. al. [2], Gutman et. al. [3] and Randic et. al. [19]. Recently Pisanski and Randic [18] studied the measuring network bipartivity using Szeged index. Weighted vertex PI index and weighted Szeged index of graph G has been introduced by Ilić and Milosavljevic [11], also they obtained the upper and lower bounds for weighted vertex PI index of graph. Also the exact formula for weighted vertex PI index of Cartesian product of graphs is obtained in [11]. Weighted vertex PI indices of generalized hierarchical product and corona product of two graphs are obtained in [17]. Some topological indices of the corona product of two graphs are studied in [12, 13, 14]. In this paper, the weighted Szeged indices of Cartesian product and corona product of two connected graphs are obtained. Using the results obtained here, the weighted Szeged indices of the hypercube of dimension n, Hamming graph, C_4 nanotubes, nanotorus, grid, $t-$fold bristled, sunlet, fan, wheel, bottleneck graphs and some classes of bridge graphs are computed.

2. Weighted Szeged index of $G\Box H$

In this section, we compute the weighted Szeged index of Cartesian product of two graphs. The following lemma is used in the proof of the main theorem of this section.

Lemma 2.1. *Let G and H be two connected graphs. Then*
(i) $|V(G\Box H)| = |V(G)|\,|V(H)|\,,\ |E(G\Box H)| = |E(G)|\,|V(H)| + |E(H)|\,|V(G)|\,.$
(ii) $d_{G\Box H}((g,h),(g',h')) = d_G(g,g') + d_H(h,h')$.
(iii)$d_{G\Box H}((g,h)) = d_G(g) + d_H(h)$.

For $e = uv \in E(G)$, let $T_G(e;u) = \{x \in V(G)\,|\,d_G(u,x) < d_G(v,x)\}$ and let $T_G(e;v) = \{x \in V(G)\,|\,d_G(v,x) < d_G(u,x)\}$.

Theorem 2.2. *Let G and H be two connected graphs. Then* $Sz_w(G\Box H) = |V(H)|^3\ Sz_w(G) + |V(G)|^3\ Sz_w(H) + 4\,|V(H)|^2\,|E(H)|\ Sz(G) + 4\,|V(G)|^2\,|E(G)|\ Sz(H)$.

Proof. Let $V(G) = \{u_1, u_2, \ldots, u_n\}$ and let $V(H) = \{v_1, v_2, \ldots, v_m\}$. For our convenience, we partition the edge set of $G\Box H$ into two sets, $E_1 = \{(u_r,v_i)(u_r,v_k)\,|\,u_r \in V(G),\, v_iv_k \in E(H)\}$ and $E_2 = \{(u_r,v_i)(u_s,v_i)\,|\,u_ru_s \in E(G),\, v_i \in V(H)\}$, that is, $E_1 = \cup_{u_i \in V(G)} E(\langle X_i\rangle)$ and $E_2 = \cup_{j=1}^{m} E(\langle Y_j\rangle)$, that is, E_1 denotes the edges of the copies of H corresponding to the vertices of G and, E_2 denotes the edges of the copies of G corresponding to the vertices of H.

Let $e = v_iv_k \in E(H)$ and let $v_j \in T_H(e; v_i)$. Then, for any $u_r \in V(G)$ and $e' \in E_1 \subset E(G\Box H)$, the distance of (u_r, v_i) to each vertex of Y_j, is less than its distance to the vertex (u_r, v_k) in $G\Box H$. It can be observed that if some vertex $v_s \notin T_H(e, v_i)$, then all the vertices of the column Y_s are not in $T_{G\Box H}(e'; (u_r, v_i))$ in $G\Box H$. Also if v_r is equidistant to e in H, then every vertex of Y_r is equidistant to e'. Consequently, for the edge $e' \in E_1$ (of $G\Box H$) corresponding to e (in H),

$$n_{(u_r,v_i)}(e', G\Box H) = |V(G)|\; n_{v_i}(e, H) \tag{2.1}$$

$$\text{and similarly, } n_{(u_r,v_k)}(e', G\Box H) = |V(G)|\; n_{v_k}(e, H). \tag{2.2}$$

Hence for E_1 defined as above,

$$\begin{aligned}
&\sum_{(u_r,v_i)(u_r,v_k)\,=\,e'\,\in\,E_1} \Big(d_{G\Box H}((u_r, v_i)) + d_{G\Box H}((u_r, v_k))\Big) n_{(u_r,v_i)}(e', G\Box H)\, n_{(u_r,v_k)}(e', G\Box H) \\
= &\sum_{(u_r,v_i)(u_r,v_k)\,=\,e'\,\in\,E_1} \Big(d_G(u_r) + d_H(v_i) + d_G(u_r) + d_H(v_k)\Big)\, |V(G)|\; n_{v_i}(e, H)\; |V(G)|\; n_{v_k}(e, H), \\
&\text{by (2.1) and (2.2), where } e = v_iv_k \in E(H), \\
= &\;|V(G)| \sum_{v_iv_k\,=\,e\,\in\,E(H)} |V(G)|^2 \,(d_H(v_i) + d_H(v_k))\, n_{v_i}(e, H)\, n_{v_k}(e, H) \\
&+2 \sum_{u_r\,\in\,V(G)} d_G(u_r) \sum_{v_iv_k\,=\,e\,\in\,E(H)} |V(G)|^2\; n_{v_i}(e, H)\, n_{v_k}(e, H), \text{ since } |E_1| = |V(G)|\,|E(H)|, \\
= &\;|V(G)|^3\; Sz_w(H) + 4\,|V(G)|^2\,|E(G)|\; Sz(H).
\end{aligned} \tag{2.3}$$

Since the Cartesian product is commutative, similar to (2.1) and (2.2) we can see that

$$n_{(u_i,v_\ell)}(e', G\Box H) = |V(H)|\; n_{u_i}(e, G) \tag{2.4}$$

$$n_{(u_k,v_\ell)}(e', G\Box H) = |V(H)|\; n_{u_k}(e, G). \tag{2.5}$$

Hence

$$\begin{aligned}
&\sum_{(u_i,v_\ell)(u_k,v_\ell)\,=\,e'\,\in\,E_2} \Big(d_{G\Box H}((u_i, v_\ell)) + d_{G\Box H}((u_k, v_\ell))\Big) n_{(u_i,v_\ell)}(e', G\Box H)\, n_{(u_k,v_\ell)}(e', G\Box H) \\
= &\sum_{(u_i,v_\ell)(u_k,v_\ell)\,=\,e'\,\in\,E_2} \Big(d_G(u_i) + d_H(v_\ell) + d_G(u_k) + d_H(v_\ell)\Big)\, |V(H)|\; n_{u_i}(e, G)\; |V(H)|\; n_{u_k}(e, G), \\
&\text{by (2.4) and (2.5), where } e = u_iu_k \in E(G) \\
= &\;|V(H)| \sum_{u_iu_k\,=\,e\,\in\,E(G)} |V(H)|^2\; (d_G(u_i) + d_G(u_k)) n_{u_i}(e, G)\, n_{u_k}(e, G) + \\
&+2 \sum_{v_\ell\in V(H)} d_H(v_\ell) \sum_{u_iu_k\,=\,e\,\in\,E(G)} |V(H)|^2\; n_{u_i}(e, G)\, n_{u_k}(e, G), \\
&\text{since } |E_2| = |V(H)|\,|E(G(U))|, \\
= &\;|V(H)|^3\; Sz_w(G) + 4\,|V(H)|^2\,|E(H)|\; Sz(G).
\end{aligned} \tag{2.6}$$

Now we shall obtain the $Sz_w(G\Box H)$. By definition,

$$\begin{aligned}
Sz_w(G\Box H) &= \sum_{(x,y)(a,b)\,=\,e'\,\in\, E(G\Box H)} \Big(d_{G\Box H}((x,y))+d_{G\Box H}((a,b))\Big)\left(n_{(x,y)}(e',G\Box H)n_{(a,b)}(e',G\Box H)\right)\\
&= \sum_{(x,y)(x,b)\,=\,e'\,\in\, E_1} \Big(d_{G\Box H}((x,y))+d_{G\Box H}((a,b))\Big)\left(n_{(x,y)}(e',G\Box H)n_{(x,b)}(e',G\Box H)\right)\\
&\quad + \sum_{(x,y)(a,y)\,=\,e'\,\in\, E_2} \Big(d_{G\Box H}((x,y))+d_{G\Box H}((a,b))\Big)\left(n_{(x,y)}(e',G\Box H)n_{(a,y)}(e',G\Box H)\right)\\
&= |V(H)|^3\, Sz_w(G)+|V(G)|^3\, Sz_w(H)+4\,|V(H)|^2\,|E(H)|\, Sz(G)\\
&\quad +4\,|V(G)|^2\,|E(G)|\, Sz(H),\ \text{by (2.3) and (2.6).}
\end{aligned}$$

Let $G_1, G_2, \ldots, G_n$ be graphs with vertex set $V(G_i)$ and edge set $E(G_i)$, $1 \leq i \leq n$. Denote by ${}_{i=1}^{n}G_i$ the Cartesian product of graphs $G_1, G_2, \ldots, G_n$. Clearly, $|V({}_{i=1}^{n}G_i)| = \prod_{i=1}^{n}|V(G_i)|$. By induction on n, one can see that $|E({}_{i=1}^{n}G_i)| = \prod_{i=1}^{n}|V(G_i)|\sum_{i=1}^{n}\frac{|E(G_i)|}{|V(G_i)|}$. In [8], S, Klavžar et al. have proved $Sz({}_{i=1}^{n}G_i) = \sum_{i=1}^{n} Sz(G_i)\prod_{j=1, j\neq i}^{n}|V(G_j)|^3$.

Next we compute a similar result for the weighted Szeged index.

Theorem 2.3. *Let $G_1, G_2, \ldots, G_n$ be connected graphs. Then* $Sz_w({}_{i=1}^{n}G_i) = \sum_{i=1}^{n} Sz_w(G_i)\prod_{j=1, j\neq i}^{n}|V(G_j)|^3 + 4\sum_{i,j=1\, i\neq j}^{n} Sz(G_i)\,|V(G_j)|^2\,|E(G_j)|\prod_{k=1, i\neq k\neq j}^{n}|V(G_k)|^3$.

Proof. The proof follows by mathematical induction. For $n = 2$, the proof follows from Theorem 2.2. Assume that the result hold for n graphs. Then

$$\begin{aligned}
& Sz_w\big({}_{i=1}^{n+1}G_i\big) = Sz_w\big({}_{i=1}^{n}G_i\Box G_{n+1}\big)\\
= {} & \big|V\big({}_{i=1}^{n}G_i\big)\big|^3 Sz_w(G_{n+1}) + |V(G_{n+1})|^3 Sz_w\big({}_{i=1}^{n}G_i\big)\\
& +4\Big(\big|V\big({}_{i=1}^{n}G_i\big)\big|^2\big|E\big({}_{i=1}^{n}G_i\big)\big|\,Sz(G_{n+1}) + |V(G_{n+1})|^2\,|E(G_{n+1})|\,Sz\big({}_{i=1}^{n}G_i\big)\Big)\\
= {} & Sz_w(G_{n+1})\prod_{i=1}^{n}|V(G_i)|^3 + |V(G_{n+1})|^3\sum_{i=1}^{n}Sz_w(G_i)\prod_{j=1, j\neq i}^{n}|V(G_j)|^3\\
& +4\sum_{i,j=1\, i\neq j}^{n} Sz(G_i)\,|V(G_j)|^2\,|E(G_j)|\prod_{k=1, i\neq k\neq j}^{n}|V(G_k)|^3\\
& +4\Big(Sz(G_{n+1})\sum_{i=1}^{n}|V(G_i)|^2\,|E(G_i)|\prod_{j=1, j\neq i}^{n}|V(G_j)|^3 + |V(G_{n+1})|\,|E(G_{n+1})|\sum_{i=1}^{n}Sz(G_i)\prod_{j=1, j\neq i}^{n}|V(G_j)|^3\Big)
\end{aligned}$$

$$= \sum_{i=1}^{n+1} Sz_w(G_i) \prod_{j=1, j\neq i}^{n+1} |V(G_j)|^3 + 4\Big(\sum_{i,j=1,\, i\neq j}^{n} Sz(G_i)\, |V(G_j)|^2\, |E(G_j)| \prod_{k=1,\, i\neq k\neq j}^{n+1} |V(G_k)|^3$$

$$+ \sum_{i\leq j\leq n} Sz(G_i)\, |V(G_j)|^2\, |E(G_j)| \prod_{k=1,\, i\neq k\neq j}^{n+1} |V(G_k)|^3 \Big)$$

$$= \sum_{i=1}^{n} Sz_w(G_i) \prod_{j=1, j\neq i}^{n} |V(G_j)|^3 + 4 \sum_{i,j=1\, i\neq j}^{n} Sz(G_i)\, |V(G_j)|^2\, |E(G_j)| \prod_{k=1, i\neq k\neq j}^{n} |V(G_k)|^3 .$$

The proof of the following corollary directly follows from Theorem 2.3.

Corollary 2.4. *Let G be a connected graph. Then $Sz_w(G^n) = Sz_w(\square_{i=1}^{n} G) = n\,|V(G)|^{3n-4}\,\big\{\,|V(G)|\; Sz_w(G) + 4(n-1)\,|E(G)|\, Sz(G)\big\}$.*

Example 2.5. *Suppose Q_n denotes the hypercube of dimension n. Then by Theorem 2.3, $Sz_w(Q_n) = Sz_w(K_2^n) = n^2\, 2^{(3n-2)}$.*

Let us consider the graph G whose vertices are the $N-$tuples $b_1, b_2, \ldots, b_N$ with $b_i \in \{0, 1, \ldots, n_i - 1\}, n_i \geq 2$, and two vertices be adjacent if the corresponding tuples differ in precisely one place; such a graph is called a *Hamming graph.* It is well-known fact that a graph G is a Hamming graph if and only if it can be written in the form $G = \square_{i=1}^{N} K_{n_i}$ and so the Hamming graph is usually denoted by $H_{n_1,n_2,\ldots,n_N}$. In the following lemma, the weighted Szeged index of a Hamming graph is computed.

It is easy to see that $Sz(K_n) = \frac{n(n-1)}{2}$ and $Sz_w(K_n) = n(n-1)^2$. The proof of the following lemma follows from Theorem 2.3.

Lemma 2.6. *Let G be a Hamming graph with above parameter. Then $Sz_w(H_{n_1 n_2 \ldots n_N}) = \left(\sum_{i=1}^{N} \Big(1 - \frac{1}{n_i}\Big)^2 + \sum_{i,j=1,\ i\neq j}^{N} \frac{(n_i-1)(n_j-1)}{n_i^2} \right) \prod_{i=1}^{N} {n_i}^3.$*

Let C_n and P_n denote the cycle and path on n vertices, respectively. It is known that $Sz(C_n) = \frac{n^3}{4}$ when n is even, and $\frac{n(n-1)^2}{4}$ otherwise and $Sz(P_n) = \binom{n+1}{3}$; see [7]. It can be easily verified that $Sz_w(C_n) = n^3$ when n is even, and $n(n-1)^2$ otherwise and $Sz_w(P_n) = \frac{2(n-1)(n^2+n-3)}{3}$.

Using Theorems 2.2, 2.3 and $Sz_w(P_n)$,$Sz_w(C_n)$,$Sz(P_n)$ and $Sz(C_n)$, we obtain the exact weighted Szeged indices of the following graphs.

Example 2.7. *The graphs $L_n = P_n \square K_2$, $R = P_n \square C_m$, $S = C_m \square C_n$ and $T = P_m \square P_n$ are known as ladder, C_4 nanotubes, C_4 nanotorus and grid, respectively. The exact weigted Szeged indices of these graphs are given below.*

1. $Sz_w(L_n) = 14n^3 - 4n^2 - 24n + 16.$
2. $Sz_w(R) = \begin{cases} \frac{m^3}{3}(10n^3 - 3n^2 - 10n + 6) \text{ if } m \text{ is even} \\ \frac{2m^3}{3}(n^3 + n^2 - 5n + 3) + m(m-1)^2 n^2 (2n-1) \text{ if } m \text{ is odd.} \end{cases}$

3. $Sz_w(S) = \begin{cases} 2n^3m(m-1)^2 + 2m^3n(n-1)^2 \text{ if } m \text{ is odd } n \text{ is odd} \\ 2n^3m(2m^2-2m+1) \text{ if } m \text{ is odd } n \text{ is even} \\ 2m^3n(2n^2-2n+1) \text{ if } m \text{ is even } n \text{ is odd} \\ 4m^3n^3 \text{ if } m \text{ is even } n \text{ is even.} \end{cases}$

4. $Sz_w(T) = \frac{2m^2(n-1)}{3}(2mn^2+2mn-3m-n^2-n) + \frac{2n^2(m-1)}{3}(2nm^2+2mn-3n-m^2-m).$

5. $Sz_w(C_{n_1}\Box C_{n_2}\Box\ldots\Box C_{n_k}) = \begin{cases} k^2\prod_{i=1}^{k} n_i^3 \text{ if each } n_i \text{ is even} \\ k\prod_{i=1}^{k} n_i^3 \sum_{i=1}^{k}\left(1-\frac{1}{n_i}\right)^2 \text{ if each } n_i \text{ is odd.} \end{cases}$

If each $n_i = n$, *then* $Sz_w(C_n^k) = \begin{cases} k^2n^{3k} \text{ if each } n_i \text{ is even} \\ k^2(n-1)^2n^{3k-2} \text{ if each } n_i \text{ is odd.} \end{cases}$

Example 2.8. *Let* $G = C_{n_1}\Box C_{n_2}\Box\ldots\Box C_{n_k}$ *and* $H = C_{m_1}\Box C_{m_2}\Box\ldots\Box C_{m_r}$, *where* $n_i,\ 1\le i\le k$ *are even and* $m_j,\ 1\le j\le r$ *are odd. Using Theorem 2.2 and the above example we obtain the weighted Szeged index of the graph* $G\Box H$.

$Sz_w(G\Box H) = \Big(\prod_{i=1}^{k} n_i^3\Big)\Big(\prod_{j=1}^{r} m_j^3\Big)\Big(k^2+kr+(k+r)\sum_{i=1} r(1-\frac{1}{m_i})^2\Big).$

If each $n_i = n \ge 3$ *is even and* $m_j = m \ge 3$ *is odd, then* $G = C_n^k$ *and* $H = C_m^r$ *and* $Sz_w(G\Box H) = n^{3k}m^{3r}\Big(k^2+kr+(k+r)r(1-\frac{1}{m})^2\Big)$.

Example 2.9. *Using Theorem 2.3, we obtain the exact weighted Szeged index of the grid graph* $P_{n_1}\Box P_{n_2}\Box\ldots\Box P_{n_k}$.

$Sz_w(P_{n_1}\Box P_{n_2}\Box\ldots\Box P_{n_k}) = \frac{2}{3}\Big(\prod_{i=1}^{k} n_i^3\Big)\Big(\sum_{i=1}^{k}\frac{(n_i-1)(n_i^2+n_i-3)}{n_i^3} + \sum_{i,\,j=1,\,i\neq j}^{k}(1-\frac{1}{n_i})(1+\frac{1}{n_i})(1-\frac{1}{n_j})\Big).$

If each $n_i = n$, *then* $Sz_w(P_n^k) = \frac{2k(n-1)n^{3(k-1)}}{3}\big((n^2+n-3)+(k-1)(n+1)\big)$.

3. Weighted Szeged index of the Corona product of graphs

In this section, we compute the weighted Szeged index of the corona product $G\circ H$ of the graphs G and H. The *edge a-Zagreb index* and (a,b)-*Zagreb index* of G are defined, in order, as follows

$$Z_a(G) = \sum_{uv\in E(G)}\Big(d_G(u)^a + d_G(v)^a\Big)$$

$$Z'_{a,b}(G) = \frac{1}{2}\sum_{uv\in E(G)}\Big(d_G(u)^a d_G(v)^b + d_G(u)^b d_G(v)^a\Big).$$

It is not hard to see that $Z_1(G) = M_1(G)$ and $Z'_{1,1} = M_2(G)$, where $M_1(G)$ and $M_2(G)$ are the first and second Zagreb indices of G, respectively. The first and second Zagreb indices were first introduced in [15] also see [16].

For our convenience, we partition the edge set of $G\circ H$ into three sets, $E_1 = \{e\in E(G\circ H)\,|\,e\in E(H_i),\ 1\le i\le n\}$, $E_2 = \{e\in E(G\circ H)\,|\,e\in E(G)\}$ and $E_3 = \{e\in E(G\circ H)\,|\,e = uv,\ u\in V(H_i),\ 1\le i\le n,\ v\in V(G)\}$. It is easy to see that E_1, E_2 and E_3 is a partition of the edge set of $G\circ H$ and also $|E_1| = |V(G)|\,|E(H)|$, $|E_2| = |E(G)|$ and $|E_3| = |V(G)|\,|V(H)|$. Let $t_e(G)$ denote the number of triangles containing the edge e in G. The following lemma is used in the proof of the main theorem of this section.

Lemma 3.1. *Let G and H be graphs. Then*
(1) $|V(G\circ H)| = |V(G)|\,(1+|V(H)|)$ *and* $|E(G\circ H)| = |E(G)| + |V(G)|\,(|V(H)|+|E(H)|)$.
(2) (*i*) *If* $e = uv \in E_1$, *then* $d_{G\circ H}(u) = d_H(u)+1$ *and* $d_{G\circ H}(v) = d_H(v)+1$.
(*ii*) *If* $e = uv \in E_2$, *then* $d_{G\circ H}(u) = d_G(u) + |V(H)|$ *and* $d_{G\circ H}(v) = d_G(v) + |V(H)|$.
(*iii*) *If* $e = uv \in E_3$ *and* $u \in V(H),\ v \in V(G)$, *then* $d_{G\circ H}(u) = d_H(u)+1$ *and* $d_{G\circ H}(v) = d_G(v) + |V(H)|$.

Theorem 3.2. *Let G be connected graph of order n and size p. If H is a graph of order m and size q, then* $Sz_w(G\circ H) = 2n\,Z'_{2,1}(H) - n\sum\limits_{e=uv\in E(H)} t_e(H)(d_H(u)+d_H(v))^2 + n\sum\limits_{e=uv\in E(H)}(d_H(u)+d_H(v))(t_e^2(H)-2t_e(H)) + 2nM_2(H) + 2n\sum\limits_{e\in E(H)} t_e^2(H) + (m+1)^2 Sz_w(G) + 2m(m+1)^2 Sz(G) + (n+nm-1)(2nq+2mp+nm^2+nm) - nM_1(H) - 4pq - 2n(m+1)q$.

Proof. As mentioned in the beginning of this section, we partition the edge of $G\circ H$ into three sets E_1,E_2 and E_3, where $E_1 = \{e \in E(G\circ H)\,|\, e \in E(H_i),\ 1 \le i \le n\}$, $E_2 = \{e \in E(G\circ H)\,|\, e \in E(G)\}$ and $E_3 = \{e \in E(G\circ H)\,|\, e = uv \in,\ u \in V(H_i),\ 1\le i\le n,\ v \in V(G)\}$.

Let $e = uv \in E_1$, if $x \in V(H)$ is non adjacent to both u and v in H, then $d_{G\circ H}(x,u) = d_{G\circ H}(x,v) = 2$. Also if $x \in V(H)$ is adjacent to both u and v in H, then $d_{G\circ H}(x,u) = d_{G\circ H}(x,v) = 1$. For both case x is an equidistant to e in $G\circ H$. Hence $n_u(e, G\circ H) = d_H(u) - t_e(H)$ and $n_v(e, G\circ H) = d_H(v) - t_e(H)$.

$$\begin{aligned}
&\sum_{e=uv\in E_1}\big(d_{G\circ H}(u)+d_{G\circ H}(v)\big)n_u(e,G\circ H)n_v(e,G\circ H)\\
&= n\sum_{e=uv\in E(H)}\big(d_H(u)+d_H(v)+2\big)\big(d_H(u)-t_e(H)\big)\big(d_H(v)-t_e(H)\big),\ \text{by Lemma 3.1}\\
&= n\sum_{e=uv\in E(H)}\big(d_H(u)^2d_H(v)+d_H(u)d_H(v)^2\big) - n\sum_{e=uv\in E(H)} t_e(H)\big(d_H(u)+d_H(v)\big)^2\\
&\quad +n\sum_{e=uv\in E(H)}\big(d_H(u)+d_H(v)\big)\big(t_e^2(H)-2t_e(H)\big) + 2nM_2(H) + 2n\sum_{e\in E(H)} t_e^2(H)\\
&= 2nZ'_{2,1}(H) - n\sum_{e=uv\in E(H)} t_e(H)\big(d_H(u)+d_H(v)\big)^2\\
&\quad +n\sum_{e=uv\in E(H)}\big(d_H(u)+d_H(v)\big)\big(t_e^2(H)-2t_e(H)\big) + 2nM_2(H) + 2n\sum_{e\in E(H)} t_e^2(H) \qquad (3.1)
\end{aligned}$$

Let $e = uv \in E_2$, if $x \in T_G(e;u)$, then all the vertices of the copy of H attached to x are in $T_{G\circ H}(e;u)$. Since $|V(H)| = m$, $n_u(e, G\circ H) = (m+1)n_u(e,G)$ and $n_v(e, G\circ H) = (m+1)n_v(e,G)$.

$$\begin{aligned}
&\sum_{e=uv\in E_2} \big(d_{G\circ H}(u) + d_{G\circ H}(v)\big) n_u(e, G\circ H) n_v(e, G\circ H)\\
&= \sum_{e=uv\in E(G)} \big(d_G(u) + d_G(v) + 2m\big)(m+1)^2 n_u(e,G) n_v(e,G), \text{ by Lemma 3.1}\\
&= (m+1)^2 Sz_w(G) + 2m(m+1)^2 Sz(G). \qquad (3.2)
\end{aligned}$$

Let $e = uv \in E_3$, if $u_1, u_2, \ldots, u_r$ are the vertices adjacent to u in H, then $u_j,\ j = 1, 2, \ldots, r$ is equidistant to e in $G\circ H$. On the other hand every vertex of $G\circ H$ other than $u, u_1, u_2, \ldots, u_r$ are in $T_{G\circ H}(e;v)$. Hence $n_u(e, G\circ H) = 1$ and $n_v(e, G\circ H) = |V(G\circ H)| - (d_H(u) + 1)$.

$$\begin{aligned}
&\sum_{e=uv\in E_3} \big(d_{G\circ H}(u) + d_{G\circ H}(v)\big) n_u(e, G\circ H) n_v(e, G\circ H)\\
&= \sum_{u\in V(H)} \sum_{v\in V(G)} \big(d_H(u) + 1 + d_G(v) + m\big)\big(|V(G\circ H)| - (d_H(u)+1)\big), \text{ by Lemma 3.1}\\
&= (n + nm - 1)(2nq + 2mp + nm^2 + nm) - nM_1(H) - 4pq - 2n(m+1)q. \qquad (3.3)
\end{aligned}$$

Now we shall obtain the $Sz_w(G\circ H)$. By the definition of $Sz_w(G\circ H)$,

$$\begin{aligned}
Sz_w(G\circ H) &= \sum_{e\in E(G\circ H)} \big(d_{G\circ H}(u) + d_{G\circ H}(v)\big) n_u(e, G\circ H) n_v(e, G\circ H)\\
&= 2n\, Z'_{2,1}(H) - n \sum_{e=uv\in E(H)} t_e(H)\big(d_H(u) + d_H(v)\big)^2 + n \sum_{e=uv\in E(H)} \big(d_H(u) + d_H(v)\big)\big(t_e^2(H) - 2t_e(H)\big)\\
&\quad + 2nM_2(H) + 2n \sum_{e\in E(H)} t_e^2(H) + (m+1)^2 Sz_w(G) + 2m(m+1)^2 Sz(G) - nM_1(H) - 4pq\\
&\quad - 2n(m+1)q + (n + nm - 1)(2nq + 2mp + nm^2 + nm), \text{using (3.1), (3.2) and (3.3)}
\end{aligned}$$

by partition $E(G\circ H)$ into E_1, E_2 and E_3 as beginning of the proof.

Using Theorem 3.2, we have the following corollaries.

Corollary 3.3. *Let G be connected graph of order n and size p. If H is a r-regular graph of order m and size q, then* $Sz_w(G\circ H) = 2nr^3q - 4nr(r+1) \sum\limits_{e\in E(H)} t_e(H) + 2n(r+1) \sum\limits_{e\in E(H)} t_e^2(H) + 2nM_2(H) + (m+1)^2 Sz_w(G) + 2m(m+1)^2 Sz(G) + (n+nm-1)(2nq+2mp+nm^2+nm) - nM_1(H) - 4pq - 2n(m+1)q.$

As for a triangle free graph H, $t_e(H) = 0$, for every edge $e \in E(H)$, we have the following corollary.

Corollary 3.4. *Let G be connected graph of order n and size p. If H is a triangle free graph of order m and size q, then* $Sz_w(G\circ H) = 2n\, Z'_{2,1}(H) + 2nM_2(H) + (m+1)^2 Sz_w(G) + 2m(m+1)^2 Sz(G) + (n + nm - 1)(2nq + 2mp + nm^2 + nm) - nM_1(H) - 4pq - 2n(m+1)q.$

Corollary 3.5. *Let G be connected graph of order n and size p. If H is a r-regular triangle free graph of order m and size q, then* $Sz_w(G\circ H) = 2nr^3q + 2nM_2(H) + (m+1)^2 Sz_w(G) + 2m(m+1)^2 Sz(G) + (n + nm - 1)(2nq + 2mp + nm^2 + nm) - nM_1(H) - 4pq - 2n(m+1)q.$

By direct calculations we obtain the first and second Zagreb indices of P_n and C_n. $M_1(C_n) = 4n,\ n \geq 3$, $M_1(P_1) = 0$, $M_1(P_n) = 4n - 6,\ n > 1$ and $M_1(K_n) = n(n-1)^2$. $M_2(P_n) = 4(n-2)$ and $M_2(C_n) = 4n$.

For a given graph G, its $t-$*fold bristled* graph $Brs_t(G)$ is obtained by attaching t vertices of degree 0 to each vertex of G. This graph can be represented as the corona product of G and complement of a complete graph on t vertices. The $t-$fold bristled graph of a given graph is also known as its $t-$thorny graph.

Example 3.6. *Let G be a graph with n vertices and p edges. Then $Sz_w(G \circ \overline{K}_t) = (t+1)^2 Sz_w(G) + 2t(t+1)^2 Sz(G) + (n + nt - 1)(2pt + nt^2 + nt)$.*

From the above formula, the weighted Szeged indices of $t-$fold bristled graph of P_n and C_n can easily be computed.

$$Sz_w(P_n \circ \overline{K}_t) = \frac{(n-1)(t+1)^2}{3}\Big(2(n^2+n-3) + nt(n+1)\Big) + (n+nt-1)\Big(2t(n-1) + nt(t+1)\Big).$$

$$Sz_w(C_n \circ \overline{K}_t) = \begin{cases} \frac{n^3(t+1)^2(t+2)}{2} + nt(n+nt-1)(t+3), & \text{if } n \text{ is even} \\ \frac{n(n-1)^2(t+1)^2(t+2)}{2} + nt(n+nt-1)(t+3), & \text{if } n \text{ is odd.} \end{cases}$$

A special corona graph $C_n \circ K_1$, that is, a cycle with pendant vertex which has $2n$ vertices. This is called *sunlet* graph.

$$Sz_w(C_n \circ K_1) = \begin{cases} 2n(3n^2+4n-2), & \text{if } n \text{ is even} \\ 2n(3n^2-2n-1), & \text{if } n \text{ is odd.} \end{cases}$$

Example 3.7. *Let H be a triangle free graph with m vertices and q edges. Then $Sz_w(K_n \circ H) = 2n\, Z'_{2,1}(H) + 2nM_2(H) - nM_1(H) + n(n-1)(m+1)^2(n+m-1) + (n+nm-1)(2nq + nm(n-1) + nm(m+1)) - 2nq(n+m)$. In particular, $Sz_w(K_1 \circ H) = 2\, Z'_{2,1}(H) + 2M_2(H) - M_1(H) + m^2(m+1) - 2q$.*

Star graph S_{m+1} on $m+1$ vertices is the corona product of K_1 and $\overline{K}_m$. *Fan* graph F_{m+1} and *wheel* graph W_{m+1} on $m+1$ vertices are also corona product of K_1 and P_m and C_m. For the graphs P_m and C_m, $Z'_{2,1}(P_m) = 8m - 18$ and $Z'_{2,1}(C_m) = 8m$. From the above formula the weighted Szeged indices of these graphs are obtained.

$$Sz_w(K_1 \circ \overline{K}_m) = m^2(m+1).$$

$$Sz_w(K_1 \circ P_m) = \begin{cases} m(m-1)^2, & \text{if } m = 2 \\ m^3 + m^2 + 18m - 44, & \text{if } m \geq 3. \end{cases}$$

$$Sz_w(K_1 \circ C_m) = \begin{cases} m^3 + m^2 + 18m - 54, & \text{if } m = 3 \\ m^3 + m^2 + 18m, & \text{if } m > 3. \end{cases}$$

Corona product sometimes appear in chemical literature as plerographs of the usual hydrogen-suppressed molecular graphs knowns as kenographs, see [20] for definitions and more information. For example, for a given graph H the graph $K_2 \circ H$ is called the *bottleneck* graph of H. The Weighted Szeged index of this graph can easily be obtained from Example 3.7.

Example 3.8. *Let H be a triangle free graph with m vertices and q edges. Then* $Sz_w(K_2 \circ H) = 4\,Z'_{2,1}(H) + 4M_2(H) - 2M_1(H) + 2(3m^3 + 8m^2 + 5m + 1) - 2q(m + 2)$. *In particular, the Weighted Szeged index of the bottleneck graph of P_m is equal to* $Sz_w(K_2 \circ P_m) = 6m^3 + 146m^2 - 32m - 86$.

Let $\{G_i\}_{i=1}^n$ be a set of finite pairwise disjoint graphs with $v_i \in V(G_i)$. The *bridge graph*

$$B\{G_1, G_2, \ldots, G_n; v_1, v_2, \ldots, v_n\}$$

of $\{G_i\}_{i=1}^n$ with respect to the vertices $\{v_i\}_{i=1}^n$ is the graph obtained from the graphs $G_1, G_2, \ldots, G_n$ by connecting the vertices v_i and v_{i+1} by an edge for all $i = 1, 2, \ldots, n-1$.

We define $G_n(H, v) = B\{\underbrace{H, H, \ldots, H}_{n\ times}; \underbrace{v, v, \ldots, v}_{n\ times}\}$, ($n$ times) which is the special case of bridge graph. For example, let P_n be the path on n vertices $v_1, v_2, \ldots, v_n$, define $B_n = G_n(P_3, v_2)$, in particular, it is a Polyethene when $n = 4$. As another example, let C_k be the cycle with k vertices and define $T_n = G_n(C_k, v_1)$ (when $k = 3$ and $n = 5$). As a final example , define the bridge graph $J_{n,m+1} = G_n(W_{m+1}, v_1)$, where W_{m+1} is the Wheel graph on $m+1$ vertices $v_1, v_2, \ldots, v_{m+1}$ such that $deg(v_1) = m$ and $deg(v_i) = 3,\ i = 1, 2, \ldots, m+1$. By the definition of corona product, $B_n = P_n \circ \overline{K}_2$, $T_{n,3} = P_n \circ K_2$ and $J_{n,m+1} = P_n \circ C_m$.

Example 3.9. *Using Theorem 3.2, we obtain the weighted Szeged indices of the following graphs.*
1. $Sz_w(B_n) = 2(6n^3 + 15n^2 - 26n + 11)$.
2. $Sz_w(T_{n,3}) = 2(6n^3 + 18n^2 - 31n + 13)$.
3. $Sz_w(J_{n,m+1}) = \frac{(m+1)^2(n-1)(n(n+1)(m+2)-6)}{3} + m(5n^2 + 6n^2m + n^2m^2 - 3mn + 5n + 6)$.

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References

[1] R. Balakrishnan and K. Ranganathan, *A Text Book of Graph Theory*, Springer-Verlag, New York, 2000.

[2] A. A. Dobrynin, I. Gutman and G. Domotor, A Wiener-type graph invariant for some bipartite graphs, *Appl. Math. Lett.*, **8** (1995) 57–62.

[3] I. Gutman, P. V. Khadikar, P. V. Rajput and S. Karmarkar, The Szeged index of polyacenes, *J. Serb. Chem. Soc.*, **60** (1995) 759–764.

[4] I. Gutman, A formula for the Wiener number of trees and its extension to graphs containing cycles, *Graph Theory Notes N. Y.*, **27** (1994) 9–15.

[5] I. Gutman and A. A. Dobrynin, The Szeged index- a success story, *Graph Theory Notes N. Y.*, **34** (1998) 37–44.

[6] W. Imrich and S. Klavžar, *Product graphs: Structure and Recognition*, John Wiley, New York, 2000.

[7] M. H. Khalifeh, H. Yousefi-Azari, A. R. Ashrafi and I. Gutman, The edge Szeged index of product graphs, *Croatica Chemica Acta*, **81** (2008) 277–281.

[8] S. Klavžar, A. Rajapakse and I. Gutman, The Szeged and the Wiener index of graphs, *Appl. Math. Lett.*, **9** (1996) 45–49.

[9] P. V. Khadikar, N. V. Deshpande, P. P. Kale, A. A. Dobrynin and I. Gutman, The Szeged index and an analogy with the Wiener index, *J. Chem. Inf. Comput. Sci.*, **35** (1995) 547–550.

[10] M. J. Nadjafi-Arani, H. Khodashenas and A. R. Ashrafi, On the differences between Szeged and Wiener indices of graphs, *Discrete Math.*, **311** (2011) 2233 –2237.

[11] A. Ilić and N. Milosavljević, The Weighted vertex PI index, *Math. Comput. Modelling*, **57** (2013) 623–631.

[12] Z. Yarahmadi and A. R. Ashrafi, The Szeged, vertex PI, first and second Zagreb indices of corona product of graphs, *Filomat*, **26** (2012) 467–472.

[13] M. Alaeiyan, J. Asadpour and R. Mojarad, Computing of some topological indices of corona product graphs, *Australian J. Basic Appl. Sci.*, **5** (2011) 145–152.

[14] M. Tavakoli and H. Yousefi-Azari, Computing PI and hyper-Wiener indices of corona product of some graphs, *Iranian J. Math. Chem.*, **1** (2010) 131–135.

[15] I. Gutman, B. Ruscić, N. Trinajstić and C. F. Wilcox, Graph theory and molecular orbitals.XII. Acyclic polyenes, *J. Chem. Phys.*, **62** (1975) 3399–3405.

[16] I. Gutman and N. Trinajstić, Graph theory and molecular orbitals. Total ϕ-electron energy of alternant hydrocarbons, *Chem.Phys. Lett.*, **17** (1972) 535–538.

[17] K. Pattabiraman and P. Kandan, *Weighted PI indices of some graph operations*, to appear in *Electronic Notes in Discrete Mathematics*.

[18] T. Pisanski and M. Randić, Use of the Szeged index and the revised Szeged index for measuring network bipartivity, *Discrete Appl. Math.*, **158** (2010) 1936–1944.

[19] M. Randić, M. Nović and D. Plavsić, Common vertex matrix: A novel characterization of molecular graphs by counting, *J. Comput. Chem.*, **34** (2013) 1409–1419.

[20] A. Milicević and N. Trinajstić, *Combinatorial enumeration in chemistry*, in:A. Hincliffe (Ed.), chemical modelling: Application and Theory, **4**, RSC Publishing, Cambridge, 2006 405–469.

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