Transactions on Combinatorics
ISSN (print): 2251-8657, ISSN (on-line): 2251-8665
Vol. 5 No. 2 (2016), pp. 33-40.

www.combinatorics.ir

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ON THE NUMBER OF CONNECTED COMPONENTS OF DIVISIBILITY GRAPH FOR CERTAIN SIMPLE GROUPS

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Communicated by Mohammad Reza Darafsheh

Abstract. The divisibility graph $\mathscr{D}(G)$ for a finite group G is a graph with vertex set $\mathrm{cs}(G) \setminus \{1\}$ where $\mathrm{cs}(G)$ is the set of conjugacy class sizes of G. Two vertices a and b are adjacent whenever a divides b or b divides a. In this paper we will find the number of connected components of $\mathscr{D}(G)$ where G is a simple Zassenhaus group or an sporadic simple group.

1. **Introduction and Preliminaries**

There are several graphs associated to algebraic structures, specially finite groups, and many interesting results have been obtained recently (see for example [2, 11, 12]). In [5] a new graph namely *divisibility graph* which is related to a set of positive integers have been introduced. The divisibility graph, $\overrightarrow{\mathscr{D}}(X)$, is a graph with vertex set $X^* = X \setminus \{1\}$ and there is an arc between two vertices a and b if and only if a divides b. Since this graph is never strongly connected, we consider its underlying graph $\mathscr{D}(X)$ without changing the name for convenience. Recall that a directed graph $\overrightarrow{\Gamma}$ is said to be *strongly connected* if there exists a directed path between any two vertices of $\overrightarrow{\Gamma}$. Also we use $\mathscr{D}(G)$ instead of $\mathscr{D}(\mathrm{cs}(G))$ where $\mathrm{cs}(G)$ is the set of conjugacy class sizes of elements of G. It is also asked for the structure and especially the number of connected components of this graph (see [5, Question 7]). So our motivation is to answer to this question for some certain simple groups.

All groups considered here are finite. Let H be a subgroup of a group G. We use $[G : H] = \{Hg | g \in G\}$ to denote the right cosets of H in G. The conjugacy class of an element $g \in G$, is denoted by g^G, that is the set of all $x^{-1}gx$, where $x \in G$. The conjugate of $h \in H$ by $g \in G$ and the

MSC(2010): Primary: 05C25; Secondary: 20D05.

Keywords: Divisibility graph, Connected component, Simple groups.

Received: 7 January 2015, Accepted: 9 March 2015.

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conjugate of H by g are $h^g = g^{-1}hg$ and $H^g = g^{-1}Hg$ respectively. By $C_G(g) = \{x \in G | g^x = g\}$, $Z(H) = \{z \in G | g^z = g \text{ for every g} \in G\}$ and $N_G(H) = \{g \in G | H^g = H\}$, we denote the centralizer of g in G, the center of G and the normalizer of H in G respectively. An involution of G is an element of order 2. By H is disjoint from its conjugates we mean that $H \cap H^g = \{e\}$ *or* H, for every $g \in G$. We write $G =< g >$, to show that G is generated by g. For a finite set X we use $|X|$ for cardinality of X and $\gcd(a, b)$ is the greatest common divisor of a and b where a and b are positive integers. For undefined terminology and notation for groups we refer to [13].

Throughout the paper all graphs are finite and simple. Let Γ_1, Γ_2 be two graphs. By $\Gamma = \Gamma_1 + \Gamma_2$ we means Γ is the disjoint union of Γ_1 and Γ_2. The complete graph with n vertices will denote by K_n. For other notation for graphs we refer to [15].

The structure of divisibility graph $\mathscr{D}(G)$, where G is a symmetric group or an alternating group is studied in [1]. It is shown there that $\mathscr{D}(S_n)$ has at most two connected components and if it is not connected then one of its connected components is K_1. Moreover it is proved there that $\mathscr{D}(A_n)$ has at most three connected components and if it is not connected then two of its connected components are K_1.

In this paper we consider $\mathscr{D}(G)$ where G is a simple Zassenhaus group or an sporadic group.

Definition 1.1. [10, Chapter 13] *A permutation group G is a Zassenhaus group if it is doubly transitive in which only the identity fixes three letters.*

Theorem 1.2. [10, Chapter 13] *Let G be a Zassenhaus group of degree $n + 1$ and let N be the subgroup fixing a letter. Then we have*

(i) *N is a Frobenius group with kernel K of order n and complement H.*
(ii) *K is a Hall subgroup of G, K is disjoint from its conjugates, $C_G(k_0) \leq K$ for every $k_0 \in K$ and $N = N_G(K)$.*
(iii) *H is a subgroup of G fixing two letters, H is disjoint from its conjugates in G, and $[N_G(H) : H] = 2$.*
(iv) *$|G| = en(n + 1)$, where $e = |H|$ and e divides $n - 1$.*
(v) *If e is even then K is abelian.*
(vi) *If $e \geq (n - 1)/2$ then K is an elementary abelian p-group for some prime p.*
(vii) *If G is simple and e is odd then H is cyclic. Also G has only one conjugacy class of involutions of size $e(n + 1)$ if n is even and of size en if n is odd.*

Also G has only one conjugacy class of involutions of size $e(n + 1)$ if n is even and of size en if n is odd. In this case we say that G is of type (H, K).

As it is stated in [10, Chapter 16, Page 488], $\mathrm{PSL}(2, q)$, the projective special linear group of dimension two over a finite field $\mathbb{F}_q$ of order $q > 3$, and $\mathrm{Sz}(q)$, Suzuki simple groups of order $q^2(q - 1)(q^2 + 1)$ where $q = 2^m$ and $m > 1$ is an odd number, are the only simple Zassenhaus groups. (For more details about the structure of $\mathrm{PSL}(2, q)$ and $\mathrm{Sz}(q)$ we refer to [7] and [14] respectively.)

According to the notations of Theorem 1.2, the projective special linear groups $\mathrm{PSL}(2,q)$ are Zassenhaus groups of type (H,K) of degree $n+1$ where $|K|=n=q$, $e=(n-1)/2$ if n is odd, and $e=n-1$ otherwise. Also H is a cyclic group that inverted by an involution [10, Chapter 2]. So by Theorem 1.1, $|G|=q(q^2-1)/2$ if q is odd and $|G|=q(q^2-1)$ otherwise. It is proved in [7, Chapter XII.] that the projective special linear group, $\mathrm{PSL}(2,q)$, has a cyclic subgroup namely L of order $(q+1)/\gcd(2,q+1)$ such that $x^{\mathrm{PSL}(2,q)} \cap L = \{x, x^{-1}\}$ for every $x \in L$. L is disjoint from its conjugates in G and $[N_G(L) : L] = 2$.

By Theorem 1.2, we observe that the Suzuki simple groups, $\mathrm{Sz}(q)$, are Zassenhaus groups of type (H,K) with H is cyclic of order $q-1$ and K is a nonabelian 2-group of order q^2 with elementary abelian center of order q and $N=HK$ is the subgroup fixing a letter [10, Page 466]. So by Theorem 1.2, $|G| = q^2(q-1)(q^2+1)$ where $q=2^m$ and $m>1$ is an odd number.

Note that by [8] the Suzuki simple groups, $\mathrm{Sz}(q)$, can be defined as a subgroup of the group $\mathrm{SL}_4(q)$. In this representation, $K = <(\alpha,\beta) \mid \alpha,\ \beta \in \mathbb{F}_q>$ and $Z(K) = \{(0,\beta) \mid \beta \in \mathbb{F}_q\}$ where (α,β) is defined

$$(\alpha,\beta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \alpha & 1 & 0 & 0 \\ \alpha^{1+r}+\beta & \alpha^r & 1 & 0 \\ \alpha^{2+r}+\alpha\beta+\beta^r & \beta & \alpha & 1 \end{bmatrix},$$

with multiplicity $(\alpha,\beta)(\gamma,\delta) = (\alpha+\gamma, \alpha\gamma^r+\beta+\delta)$ and $r^2=2q$. Note that this implies that all $1 \neq x \in Z(K)$ are involutions.

The following three lemmas about Suzuki groups will be useful.

Lemma 1.3. [14, Proposition 16] *$\mathrm{Sz}(q)$ contains abelian subgroups A_0, A_1 and A_2 of order $q-1$ and $q+r+1$ and $q-r+1, (r^2=2q)$, respectively. $C_G(x)=A_i$ for every $x \in A_i$ and $i=0,1,2$. Moreover $[N_G(A_i) : A_i] = 4$ for $i=1,2$.*

Lemma 1.4. [8, Lemma 3.3] *The subgroups A_i's are disjoint from their conjugates for $i=0,1,2$.*

By Theorem 1.2, we see that $H=A_0$.

Lemma 1.5. [14, Lemma 3.2., Part (h)] *Let $x=(\alpha,\beta) \in K$, where $\alpha \neq 0$. Then $C_G(x) = <(\alpha,0), Z(K)>$, where $Z(K) = \{(0,\xi) : \xi \in \mathbb{F}_q\}$.*

Now we prove the following lemma:

Lemma 1.6. *Let G be a group and H a subgroup of G which is disjoint from its conjugates in G. Suppose h is a non identity element of H, $N_G(H)=N$, and $C_G(h)=C$. Then $|N| = |h^G \cap H||C|$.*

Proof. Let $c \in C$. So $h^c = h$ and $H^c \cap H \neq \{e\}$. Since H is disjoint from its conjugates, we can conclude $H^c = H$. Hence $c \in N$ and $C \leq N$. Define a map θ from $[N:C]$ to $h^G \cap H$, by $\theta(Cg) = h^g$. Suppose g_0 is an arbitrary element of G. Since H is disjoint from its conjugates, we conclude $h^{g_0} \in H$ if and only if $g_0 \in N$. Also $Cx = Cy$ for $x,y \in N$ if and only if $xy^{-1} \in C$, which implies that θ is well defined and injective. It is easy to see that θ is also surjective. This completes the proof. $\square$

2. The Structure of Divisibility Graph for Simple Zassenhaus Groups

In this section first we calculate the conjugacy class sizes of elements of the projective special linear groups $\mathrm{PSL}(2,q)$ and the Suzuki groups $\mathrm{Sz}(q)$ where these groups are simple. Then by using [6] we determine the conjugacy class sizes of sporadic simple groups. The structure of divisibility graph for simple Zassenhause groups will investigate and we will list the number of connected component of $\mathscr{D}(G)$ where G is an sporadic simple group in Table 1.

2.1. Divisibility Graphs for $\mathrm{PSL}(2,q)$ and $\mathrm{Sz}(q)$.

First we determine the structure of divisibility graphs for $\mathrm{PSL}(2,q)$ and $\mathrm{Sz}(q)$. Although we may find the conjugacy classes of $\mathrm{PSL}(2,q)$ by using the conjugacy classes of $\mathrm{SL}(2,q)$, (see for example [3, 9]), but we prefer to find the conjugacy classes of these groups directly by using the structure of $\mathrm{PSL}(2,q)$ as the simple Zassenhause groups.

Theorem 2.1. *Let $G = \mathrm{PSL}(2,q)$. Then $\mathscr{D}(G)$ is one of the graphs of the following list:*

(i) $3\mathrm{K}_1$
(ii) $\mathrm{K}_2 + 2\mathrm{K}_1$.

Proof. To calculate the conjugacy class sizes of non identity elements of G, we consider three subgroups H, K and L which their orders are pairwise coprime. So the non identity elements of these groups are not conjugate.

First consider the subgroup H. Let $H =< h_0 >$. By considering the structure of G, we find that H is inverted by an involution. So every elements of H is conjugate to its inverse and $N_G(H) \neq C_G(H)$. Moreover $H \leq C_G(h_0) = C_G(H)$. Since by part (iii) of Theorem 1.2, $[N_G(H) : H] = 2$, then $C_G(h_0) = H$. This implies that $|h_0^G| = q(q+1)$.

Consider an arbitrary element $1 \neq h \in H$. Clearly $C_G(h_0) \leq C_G(h)$. If $C_G(h_0) = C_G(h)$ then $|h^G| = q(q+1)$ and h is different from its inverse as h_0 is. If $C_G(h_0) \neq C_G(h)$ then, since H is disjoint from its conjugate, $C_G(h) = N_G(H)$. So by Lemma 1.6, h has only one conjugate in H. Hence $h = h^{-1}$. This means that h is an involution and $|h^G| = q(q+1)/2$. Therefore if e is even, there exist one conjugacy classes of size $q(q+1)/2$ related to $h_0^{e/2}$ and $(e-2)/2$ conjugacy class of size $q(q+1)$ in G, related to other non identity elements of H. If e is odd then G has $(e-1)/2$ conjugacy classes of size $q(q+1)$ that contains non identity elements of H.

Now consider the subgroup K. First suppose $q = 2^\alpha$. In this case by Theorem 1.2 parts (vi) and (vii), K is an elementary abelian 2-group and G has only one conjugacy class of involutions. So every element of K contains in the unique conjugacy class of involutions of size (q^2-1).

Suppose $q = p^\alpha$ is an odd number. By parts (ii) and (vi) of Theorem 1.2, $K = C_G(y)$ for each $y \in K$. Since $[N : K] = e$, we conclude by Lemma 1.6 that $|y^G \cap K| = e$. So we can partition $K \setminus \{1\}$ into two disjoint subset of length e, each of them contains conjugate elements. Hence in this case, there are two conjugacy classes in G of size $(q^2-1)/2$ that contains elements of K.

Finally consider the subgroup L. Since for every $x \in L$ we have $x^{\mathrm{PSL}(2,q)} \cap L = \{x, x^{-1}\}$ by Lemma 1.6, if $x \in L$ is not an involution then we have $|C_G(x)| = |N_G(L)|/2 = |L|$ and if $x \in L$ is an involution then $|C_G(x)| = |N_G(L)| = 2|L|$. If $|L|$ is even then there exist $(|L|-2)/2$ conjugacy classes in G of size $\gcd(2, q+1)eq$ and one conjugacy class of size $\gcd(2, q+1)eq/2$ that contain elements of L. If $|L|$ is odd, then there exist $(|L|-1)/2$ conjugacy classes in G of size $\gcd(2, q+1)eq$ that contain elements of L.

We must consider two following cases:

(Case 1) $G = \mathrm{PSL}(2, 2^k)$. In this case $e = n-1 = q-1$ is an odd number. By using class equation of G we have;

$$1 \ + \ (|L|-1)/2 \ . \ eq \ + \ (q-1)(q+1) \ + \ ((e-1)/2) \ . \ q(q+1) \ = \\ 1 \ + \ q^2(q-1)/2 \ + \ (q^2-1) \ + \ (q-2)q(q+1)/2 \ = \ |G|.$$

(Case 2) $G = \mathrm{PSL}(2, p^k)$ where p is an odd prime. In this case, $e = (n-1)/2 = (q-1)/2$. First assume that e is odd. In this case we have;

$$1 \ + \ ((|L|-2) \ / \ 2) \ . \ 2eq \ + \ eq \ + \ 2 \ . \ e(q+1) \ + \ ((e-1)/2) \ . \ q(q+1) \ = \\ 1 \ + q(q-1)(q-3)/4 \ + \ q(q-1)/2 \ + \ (q^2-1) \ + \ (q-3)q(q+1)/4 \ = |G|.$$

In case, where e is even, we have;

$$1 \ + \ ((|L|-1)/2) \ . \ 2eq \ + \ 2 \ .e(q+1) \ + \ ((e-2)/2) \ . \ q(q+1) \ + \ q(q+1)/2 = \\ 1 \ + \ q(q-1)^2/4 \ + \ (q^2-1) \ + \ (q-5)q(q+1)/4 \ + \ q(q+1)/2 = |G|.$$

Hence by calculating all conjugacy classes of $\mathrm{PSL}(2,q)$, we see that $\mathscr{D}(\mathrm{PSL}(2,q))$ is $3\mathrm{K}_1$ or $\mathrm{K}_2 + 2\mathrm{K}_1$ (see Figure 1). $\square$

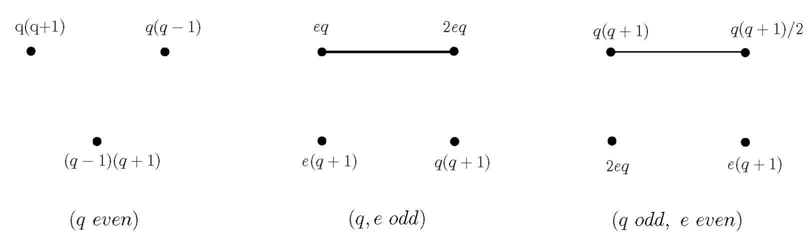

Figure 1. The graph $\mathscr{D}(\mathrm{PSL}(2,q))$

Now we calculate the conjugacy class sizes of Suzuki simple groups.

Theorem 2.2. *Let $G = \mathrm{Sz}(q)$. Then $\mathscr{D}(G) = \mathrm{K}_2 + 3\mathrm{K}_1$.*

Proof. By Lemmas 1.3 and 1.5, G has four subgroups H, K, A_1 and A_2 with pairwise coprime orders.

We start by calculating the conjugacy class sizes of elements of H. By Lemma 1.3, $C_G(h) = H$ for every $1 \neq h \in H$. Since H is disjoint from its conjugates, by Lemma 1.6, h has $[N_G(H) : C_G(h)] = 2$ conjugates in H. So there are $(q-2)/2$ conjugacy classes in G of size $q^2(q^2+1)$ which contains elements of H.

Consider the conjugacy classes of elements of K. Let $x_0 = (\alpha, \beta) \in K$ be an arbitrary element. If $1 \neq x_0 \in Z(K)$, then x_0 is an involution and by Theorem 1.2 part (vii), G has only one class of involutions of size $(q-1)(q^2+1)$. If $x_0 \in K \setminus Z(K)$ then by Lemma 1.5, $C_G(x_0) = \{(x, y) |\ y \in \mathbb{F}_q,\ x = 0 \ or \ \alpha\}$. This means that $|C_G(x_0)| = 2q$ and $|x_0^G| = q(q-1)(q^2+1)/2$. Since K is disjoint from its conjugates, by Lemma 1.6, x_0 has $[K : C_G(x_0)] = q(q-1)/2$ conjugates in $K \setminus Z(K)$. So there exist $(q^2-q)/(q(q-1)/2) = 2$ conjugacy classes of size $q(q-1)(q^2+1)/2$ in G which contains elements of $K \setminus Z(K)$.

Finally we consider two subgroups A_1 and A_2. By Lemma 1.3, A_1 and A_2 are of order $n_1 = q+r+1$ and $n_2 = q-r+1$ respectively. Also $[N_G(A_i) : A_i] = 4$ and for every $x \in A_i$, $C_G(x) = A_i$ where $i = 1, 2$. So for every $x \in A_i, i = 1, 2$, we have $[N_G(A_i) : C_G(x)] = 4$ and by Lemmas 1.4 and 1.6, there are just four elements of A_i which are conjugate with x in G. Hence there are $(q+r)/4$ conjugacy classes of size $q^2(q-1)(q^2+1)/(q+r+1) = q^2(q-1)(q-r+1)$ which contains elements of A_1 and $(q-r)/4$ conjugacy classes of size $q^2(q-1)(q^2+1)/(q-r+1) = q^2(q-1)(q+r+1)$ which contains elements of A_2.

By using the class equation for G we see that

$$\begin{aligned}&1 \ + \ (q-2) \ . \ q^2(q^2+1)/2 \ + \ (q-1)(q^2+1) \ + \ 2 \ . \ q(q-1)(q^2+1)/2 + \\ &(q+r) \ . \ q^2(q-1)(q-r+1)/4 \ + \ (q-r) \ . \ q^2(q-1)(q+r+1)/4 = |G|.\end{aligned}$$

This implies that $\mathscr{D}(\mathrm{Sz}(q)) = \mathrm{K}_2 + 3\mathrm{K}_1$. (See Figure 2.1.) $\square$

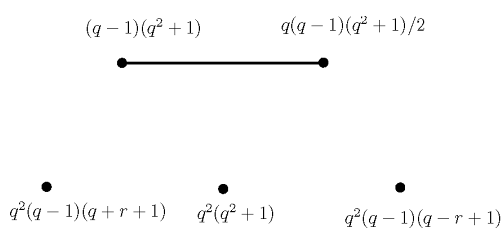

FIGURE 2. The graph $\mathscr{D}(\mathrm{Sz}(q))$

It is easy to see that, in both cases, there is just one connected component of $\mathscr{D}(G)$ not equal to K_1.

2.2. **The Number of Connected Components of Divisibility Graph of Sporadic Simple Groups.**

By using [6], we may find the conjugacy class sizes of all sporadic simple groups. For all sporadic simple groups we will observe that $\mathscr{D}(G)$ is isomorphic to $nK_1 + \mathcal{G}$ where $\mathcal{G} \ncong K_1$. The numbers n for sporadic simple groups are listed in Table 1.

G	n	G	n	G	n	G	n
M_{11}	2	Co_1	1	Fi_{22}	1	J_1	3
M_{12}	1	Co_2	2	Fi_{23}	2	J_2	1
M_{22}	3	Co_3	1	Fi'_{24}	3	J_3	2
M_{23}	2	HN	1	Ru	1	J_4	5
M_{24}	2	M^cL	1	Ly	3	He	1
Th	2	Hs	2	$O'N$	3	M	3
Suz	2	B	1				

Table 1. The number of connected components of $\mathscr{D}(G)$ isomorphic to K_1 for simple sporadic groups.

It seems to us that the divisibility graph for all finite simple groups has at most one connected component not isomorphic to K_1 and all other components are isomorphic to K_1. Therefore we have the following conjecture:

Conjecture 1. *$\mathscr{D}(G)$ has at most one connected component with more than one vertex, where G is a finite simple group.*

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