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NOTE ON EDGE DISTANCE-BALANCED GRAPHS

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Abstract. Edge distance-balanced graphs are graphs in which for every edge $e = uv$ the number of edges closer to vertex u than to vertex v is equal to the number of edges closer to v than to u. In this paper, we study this property under some graph operations.

1. **Introduction**

Let a and b be two adjacent vertices of the graph G, and $e = ab$ the edge connecting them. For an edge $e = ab$ of a graph G, let $n_a^G(e)$ be the number of vertices closer to a than to b. In other words, $n_a^G(e) = |\{u \in V(G)|d(u,a) < d(u,b)\}|$. In addition, let $n_0^G(e)$ be the number of vertices with equal distances to a and b, i. e., $n_0^G(ab) = |\{u \in V(G)|d(u,a) = d(u,b)\}|$.

A graph G is said to be distance-balanced, if $n_a^G(e) = n_b^G(e)$, for each edge $e = ab \in E(G)$, see [1, 5] for details. These graphs were, at least implicitly, first studied by Handa [4] who is considered distance-balanced partial cubes. The term itself, however, is due to Jerebič et al. [7] who is studied distance-balanced graphs in the framework of various kinds of graph products. Let G be a graph, $e = uv \in E(G)$, $m_u^G(e)$ denotes the number of edges lying closer to the vertex u than the vertex v, and $m_v^G(e)$ is defined analogously. Here is our key definition. We call a graph G to be edge distance-balanced, if $m_a^G(e) = m_b^G(e)$ holds for each edge $e = ab \in E(G)$. As examples of edge distance-balanced graphs, we mention the complete graph K_n on $n \geq 2$ vertices and the complete bipartite graph $K_{n,n}$ on $2n$ vertices.

Let G and H be two graphs. The corona product GoH is obtained by taking one copy of G and $|V(G)|$ copies of H; and by joining each vertex of the i^{th} copy of H to the i^{th} vertex of G,

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$i = 1, 2, \cdots, |V(G)|$, see [10, 13]. The Cartesian product $G \times H$ of the graphs G and H has the vertex set $V(G \times H) = V(G) \times V(H)$ and $(a, x)(b, y)$ is an edge of $G \times H$ if $a = b$ and $xy \in E(H)$, or $ab \in E(G)$ and $x = y$.

The cluster $G\{H\}$ is obtained by taking one copy of G and $|V(G)|$ copies of a rooted graph H, and by identifying the root of the i^{th} copy of H with the i^{th} vertex of G, $i = 1, 2, ..., |V(G)|$ [13]. The lexicographic product $G = G[H]$ of graphs G and H with disjoint vertex sets $V(G)$ and $V(H)$ and edge sets $E(G)$ and $E(H)$ is the graph with vertex set $V(G) \times V(H)$ and $u = (u_1, v_1)$ is adjacent with $v = (u_2, v_2)$ whenever (u_1 is adjacent to u_2) or ($u_1 = u_2$ and v_1 is adjacent to v_2), see [6, p. 22]. Suppose G is a simple connected graph. Following Yan et al. [12], we define the graphs $S(G)$ and $R(G)$ as follows:

(a) $S(G)$ is the graph obtained by inserting an additional vertex in each edge of G. Equivalently, each edge of G is replaced by a path of length 2.

(b) $R(G)$ is obtained from G by adding a new vertex corresponding to each edge of G, then joining each new vertex to the end vertices of the corresponding edge.

A regular graph is a graph where each vertex has the same number of neighbors. A regular graph with vertices of degree k is called a $k-$regular graph or regular graph of degree k. A triangle-free graph is a graph containing no graph cycles of length three. Our other notations are standard and taken mainly form [2, 8, 9, 11].

2. **Main Results**

In this section we study the conditions under which the standard graph products produce an edge distance-balanced graph.

Theorem 2.1. Let G and H be edge and vertex distance-balanced graphs. Then $G \times H$ is edge distance-balanced graphs.

Proof. Consider the following partition of $E(G \times H)$:

$$\begin{aligned} A &= \{(a,x)(b,y) \in E(G \times H) | ab \in E(G), x = y\} \\ B &= \{(a,x)(b,y) \in E(G \times H) | a = b, xy \in E(H)\}. \end{aligned}$$

We assume that G and H are edge and vertex distance-balanced graphs, and $e \in A$. Notice that

$$\begin{aligned} m_{(a,x)}^{(G\times H)}(e) &= m_a^G(ab)|V(H)| + n_a^G(ab)|E(H)|, \\ m_{(b,y)}^{(G\times H)}(e) &= m_b^G(ab)|V(H)| + n_b^G(ab)|E(H)|. \end{aligned}$$

Since G is edge and vertex distance-balanced, thus we have $n_a^G(ab) = n_b^G(ab)$ and $m_a^G(ab) = m_b^G(ab)$. Therefore, in this case we have $m_{(a,x)}^{(G\times H)}(e) = m_{(b,y)}^{(G\times H)}(e)$. In a similar way we can see that, for every edge e of B, we have $m_{(a,x)}^{(G\times H)}(e) = m_{(b,y)}^{(G\times H)}(e)$. Then $G \times H$ is edge distance-balanced graphs. $\square$

A graph G is called nontrivial if $|V(G)| > 1$.

Theorem 2.2. The corona product of two arbitrary, nontrivial and connected graphs is not edge distance-balanced.

Proof. Let G and H be nontrivial connected graphs. Assume that $uv = e \in E(GoH)$ such that $u \in V(G)$ and $v \in V(H_i)$, where H_i, $1 \leq i \leq |V(G)|$ is the i^{th} copy of H. Thus,

$$\begin{aligned} m_u^{GoH}(e) &= (|V(G)|-1)|E(H)| + |E(G)| + |V(G)||V(H)| - 1 \\ &+ |\{f \in E(H)|d_H(v,f) \geqslant 2\}| \end{aligned}$$

and $m_v^{GoH}(e) = deg_H(v)$. Therefore, $m_u^{GoH}(e) \neq m_v^{GoH}(e)$ and so GoH is not edge distance-balanced. □

Theorem 2.3. The cluster of two arbitrary, nontrivial and connected graphs is not edge distance-balanced.

Proof. Let G and H be nontrivial connected graphs. Assume that $e = uv \in E(G\{H\})$ such that u is the root of the i^{th} copy of H and $u \neq v \in V(H_i)$. Thus,

$$m_u^{G\{H\}}(e) = |E(H)|(|V(G)|-1) + |E(G)| + m_u^H(e)$$

and $m_v^{G\{H\}}(e) = m_v^H(e)$. Therefore, $m_u^{G\{H\}}(e) \neq m_v^{G\{H\}}(e)$ and so $G\{H\}$ is not edge distance-balanced. □

Theorem 2.4. Let G and H be connected graphs. Then $G[H]$ is edge distance-balanced if G is nontrivial, edge and vertex distance-balanced and H is triangle-free and regular.

Proof. Suppose that G is nontrivial, edge and vertex distance-balanced and H is triangle-free and regular. Consider the following partition of $E(G[H])$.

$$\begin{aligned} A &= \{(a,x)(b,y) \in E(G[H])|ab \in E(G) and\; x,y \in V(H)\}, \\ B &= \{(a,x)(b,y) \in E(G[H])|a = b \in V(G), xy \in E(H)\}. \end{aligned}$$

Let $e = (a,x)(b,y) \in A$. According to the definition of the lexicographic product, it is clear that

$$\begin{aligned} m_{(a,x)}^{G[H]}(e) - m_{(b,y)}^{G[H]}(e) &= (m_a^G(ab) - m_b^G(ab))|V(H)|^2 \\ &+ (n_a^G(ab) - n_b^G(ab))|E(H)| \\ &+ |\{f \in E(H) \mid d_H(y,f) \geqslant 2\}| \\ &- |\{f \in E(H) \mid d_H(x,f) \geqslant 2\}|. \end{aligned}$$

Since G is edge and vertex distance-balanced, then $m_a^G(ab) = m_b^G(ab)$ and $n_a^G(ab) = n_b^G(ab)$ and since H is triangle-free and regular one can see that $|\{f \in E(H) \mid d_H(y,f) \geqslant 2\}| = |\{f \in E(H) \mid d_H(x,f) \geqslant 2\}|$. It follows that $m_{(a,x)}^{G[H]}(e) = m_{(b,y)}^{G[H]}(e)$. We now assume that $e = (a,x)(b,y) \in B$. It follows from the edge structure of $G[H]$ that $m_{(a,x)}^{G[H]}(e) = m_{(b,y)}^{G[H]}(e)$, if H is triangle-free and regular. Therefore, for each $e = (a,x)(b,y) \in E(G[H])$, we have $m_{(a,x)}^{G[H]}(e) = m_{(b,y)}^{G[H]}(e)$ and thus $G[H]$ is edge distance-balanced. □

Theorem 2.5. Let G be a nontrivial connected graph. Then $R(G)$ is edge distance-balanced if and only if G is a path with $|V(G)| = 2$.

Proof. Let G be a path with $|V(G)| = 2$. Then it is clear that $R(G)$ is edge distance-balanced. Conversely, we assume that $R(G)$ is an edge distance-balanced graph, where G be a graph with $|V(G)| > 2$. Then, there is at least an edge $uv = e$ of G such that u is the end vertex of e with $deg_G(u) > 1$ or v is the end vertex of e with $deg_G(v) > 1$. Without loss of generality, we my assume that u is the end vertex of e with $deg_G(u) > 1$. Also, we assume that x is a new vertex corresponding to edge e of G. Then, $m_x^{R(G)}(xu) = 1$ and $m_u^{R(G)}(xu) > 1$. Thus $m_x^{R(G)}(xu) \neq m_u^{R(G)}(xu)$. Therefore $R(G)$, $|V(G)| > 2$, is not an edge distance-balanced graph and hence G is a path with $|V(G)| = 2$. □

Theorem 2.6. Let G be a nontrivial connected graph with a pendant. Then $S(G)$ is not edge distance-balanced.

Proof. Suppose x is a pendent vertex and u is the new vertex such that u and x are adjacent in $S(G)$. Then $m_x^{S(G)}(ux) = 0$ and $m_u^{S(G)}(ux) \geq 1$, proving the result. □

Suppose G and H are graphs with disjoint vertex sets. Following Doslic [3], for given vertices $y \in V(G)$ and $z \in V(H)$ a splice of G and H by vertices y and z, $(G \cdot H)(y; z)$, is defined by identifying the vertices y and z in the union of G and H. Similarly, a link of G and H by vertices y and z is defined as the graph $(G \sim H)(y; z)$ obtained by joining y and z by an edge in the union of these graphs.

Theorem 2.7. Suppose G and H are rooted graphs with respect to the rooted vertices of a and b, respectively. The graph $(G \cdot H)(a; b)$ is edge distance-balanced if and only if for each $e = uv \in E(G)$ and $f = xy \in E(H)$ the following conditions are satisfied:

$$m_u^G(e) - m_v^G(e) = \begin{cases} |E(H)| & if\ d(v,a) < d(u,a) \\ 0 & if\ d(v,a) = d(u,a) \end{cases}, \tag{2.1}$$

$$m_x^H(f) - m_y^H(f) = \begin{cases} |E(G)| & if\ d(y,b) < d(x,b) \\ 0 & if\ d(y,b) = d(x,b) \end{cases}. \tag{2.2}$$

Proof. In the graph $(G \cdot H)(a; b)$, we put $r = a = b$. We partition edges of $(G \cdot H)(a; b)$ into the following two subsets:

$$\begin{aligned} A &= \{e = uv \in E(G.H) | d(v,r) < d(u,r)\}, \\ B &= \{e = uv \in E(G.H) | d(v,r) = d(u,r)\}. \end{aligned}$$

We first assume that $(G \cdot H)(a; b)$ is edge distance-balanced. Suppose $e = uv$ is an arbitrary edge of G. Then $e \in A$ or $e \in B$ and not both. If $e \in A$ then by the hypothesis $m_u^{G \cdot H}(e) = m_v^{G \cdot H}(e)$. On the other hand by the definition of splice, $m_v^{G \cdot H}(e) = m_v^G(e) + |E(H)|$ and $m_u^{G \cdot H}(e) = m_u^G(e)$. Thus, $m_u^G(e) = m_v^G(e) + |E(H)|$ and so $m_u^G(e) - m_v^G(e) = |E(H)|$. Next we assume that $e \in B$. Again by the hypothesis $m_u^{G \cdot H}(e) = m_v^{G \cdot H}(e)$ and by definition of splice we have, $m_v^{G \cdot H}(e) = m_v^G(e)$ and

$m_u^{G.H}(e) = m_u^G(e)$. This implies that $m_u^G(e) = m_v^G(e)$. Therefore, the equation (1) is satisfied. In a similar way we can see that, for every edge e of H the equation (2) is satisfied.

Conversely, suppose that Eqs. (1,2) are satisfied and $e = uv \in A$ is arbitrary. Then $e \in E(G)$ or $e \in E(H)$ and not both. If $e \in E(G)$ then $m_u^{G.H}(e) = m_u^G(e)$ and $m_v^{G.H}(e) = m_v^G(e) + |E(H)|$. This implies that $m_u^{G.H}(e) - m_v^{G.H}(e) = m_u^G(e) - (m_v^G(e) + |E(H)|)$. Since $m_u^G(e) - m_v^G(e) = |E(H)|$, $m_u^{G.H}(e) - m_v^{G.H}(e) = 0$, as desired. Suppose that $e \in E(H)$. Then $m_u^{G.H}(e) = m_u^H(e)$ and $m_v^{G.H}(e) = m_v^H(e) + |E(G)|$, so $m_u^{G.H}(e) - m_v^{G.H}(e) = m_u^H(e) - (m_v^H(e) + |E(G)|)$. But by the hypothesis, $m_u^H(e) - m_v^H(e) = |E(G)|$, so $m_u^{G.H}(e) - m_v^{G.H}(e) = 0$. We now assume that $e \in B$ is arbitrary. If $e \in E(G)$ then by $m_u^{G.H}(e) = m_u^G(e)$ and $m_v^{G.H}(e) = m_v^G(e)$ we have $m_u^{G.H}(e) - m_v^{G.H}(e) = m_u^G(e) - m_v^G(e) = 0$. If $e \in E(H)$ then by $m_u^{G.H}(e) = m_u^H(e)$ and $m_v^{G.H}(e) = m_v^H(e)$ we have $m_u^{G.H}(e) - m_v^{G.H}(e) = m_u^H(e) - m_v^H(e) = 0$. Therefore, for every edge $e = uv \in B$, $m_u^{G.H}(e) = m_v^{G.H}(e)$ and for every edge $e = uv \in E(G \cdot H)$, $m_u^{G.H}(e) = m_v^{G.H}(e)$. This completes the proof. □

Corollary 2.8. Suppose $G_1, G_2, \ldots, G_n$ are connected rooted graphs with root vertices $r_1, \ldots, r_n$, respectively. Then

$$(G_1 \cdot G_2 \cdot \cdots \cdot G_n)(r_1; r_2; \cdots; r_n)$$

is edge distance-balanced if and only if for each i, $1 \le i \le n$, and for each $e = uv \in E(G_i)$ the following system of equations are satisfied:

$$m_u^{G_i}(e) - m_v^{G_i}(e) = \begin{cases} \sum_{j=1, j \neq i}^{n} |E(G_j)| & if\ d(v, r_i) < d(u, r_i) \\ 0 & if\ d(v, r_i) = d(u, r_i) \end{cases}.$$

Proof. Induct on n. □

Theorem 2.9. Suppose G and H are rooted graphs with respect to the rooted vertices of a and b, respectively. The graph $(G \sim H)(a; b)$ is edge distance-balanced if and only if $|E(G)| = |E(H)|$ and for each $e = uv \in E(G)$ and $f = xy \in E(H)$ the following conditions are satisfied:

$$m_u^G(e) - m_v^G(e) = \begin{cases} |E(H)| + 1 & if\ d(v, a) < d(u, a) \\ 0 & if\ d(v, a) = d(u, a) \end{cases},$$

$$m_x^H(f) - m_y^H(f) = \begin{cases} |E(G)| + 1 & if\ d(y, b) < d(x, b) \\ 0 & if\ d(y, b) = d(x, b) \end{cases}.$$

Proof. The proof is similar to Theorem 2.7 and so omitted. □

Corollary 2.10. Suppose $G_1, G_2, \ldots, G_n$ are connected rooted graphs with root vertices $r_1, \ldots, r_n$, respectively. Then $(G_1 \sim G_2 \sim \cdots \sim G_n)(r_1; r_2; \cdots; r_n)$ is edge distance-balanced if and only if for each i, $1 \le i \le n$, $|E(G_i)| = |E(G_1)|$ and for each $e = uv \in E(G_i)$ the following system of equations are satisfied:

$$m_u^{G_i}(e) - m_v^{G_i}(e) = \begin{cases} \sum_{j=1, j \neq i}^{n} |E(G_j)| + \binom{n}{2} & if\ d(v, r_i) < d(u, r_i) \\ 0 & if\ d(v, r_i) = d(u, r_i) \end{cases}.$$

Proof. Induct on n. □

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