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## SOME TOPOLOGICAL INDICES AND GRAPH PROPERTIES

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ABSTRACT. In this paper, by using the degree sequences of graphs, we present sufficient conditions for a graph to be Hamiltonian, traceable, Hamilton-connected or  $k$ -connected in light of numerous topological indices such as the eccentric connectivity index, the eccentric distance sum, the connective eccentricity index.

### 1. Introduction

Let  $G$  be a connected graph with vertex set  $V(G)$  and edge set  $E(G)$  such that  $|V| = n$  and  $|E| = m$ . Let  $d(v)$  be the degree of a vertex  $v$  in  $G$ . Let  $d(u, v)$  be the distance between two vertices  $u$  and  $v$  in  $G$ , that is, the length of the shortest path connecting  $u$  and  $v$  in  $G$ . The eccentricity  $\varepsilon(v)$  of a vertex  $v$  is the maximum distance from  $v$  to any other vertex. Let  $K_n, S_n, P_n$  be a complete graph, a star and a path on  $n$  vertices, respectively.

A cycle  $C$  in a graph  $G$  is called a Hamiltonian cycle of  $G$  if  $C$  contains all the vertices of  $G$ . A graph  $G$  is called *Hamiltonian* if  $G$  has a Hamiltonian cycle. A path  $P$  in a graph  $G$  is called a *Hamiltonian path* of  $G$  if  $P$  contains all the vertices of  $G$ . A graph  $G$  is called *traceable* if  $G$  has a Hamiltonian path. A graph  $G$  is called *Hamilton-connected* if for each pair of vertices in  $G$  there is a Hamiltonian path between them. A graph  $G$  is said to be  *$k$ -connected* (or  *$k$ -vertex connected*) if there does not exist a set of  $k - 1$  vertices whose removal disconnects the graph. If  $G$  and  $H$  are two vertex-disjoint graphs, we use  $G \vee H$  to denote the *join* of  $G$  and  $H$ .

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The topological indices are widely used in organic chemistry and have been found to be useful in chemical documentation, isomer discrimination, structure-property relationships, structure-activity (SAR) relationships and pharmaceutical drug design [14, 23]. In past decades, plenty of mathematical properties of numerous topological indices are reported such as the the matching energy [5, 6], Randić index [24] and the Balaban index [7].

For a connected graph  $G$ , its Wiener index, denoted by  $W(G)$ , is defined as

$$W(G) = \sum_{\{u,v\} \subseteq V(G)} d(u,v) = \frac{1}{2} \sum_{v \in V(G)} D(v).$$

Here  $D(v) = \sum_{u \in V(G)} d_G(u,v)$ . It can be easily verified that  $D(v) \geq d(v) + 2(n-1-d(v))$ . The Wiener index and its modifications are well studied in the past years, see [9, 17, 21, 19, 20].

The *eccentric connectivity index* (ECI) [22] of a connected graph  $G$ , denoted by  $\xi^c(G)$ , is defined as

$$\xi^c(G) = \sum_{v \in V(G)} \varepsilon(v)d(v).$$

The *eccentric distance sum* (EDS) [11] of a connected graph  $G$  is defined as

$$\xi^d(G) = \sum_{v \in V(G)} \varepsilon(v) \cdot D(v).$$

The *connective eccentricity index* (CEI) [10] of a connected graph  $G$  is defined as

$$\xi^{ce}(G) = \sum_{v \in V(G)} \frac{d(v)}{\varepsilon(v)}.$$

The above three topological indices involving eccentricities are widely studied from mathematical view, see [13, 18, 26, 27, 28].

In [25], Yang presented a sufficient condition for a graph to be traceable by using Wiener index. In [12], Hua and Wang presented a sufficient condition for a graph to be traceable by using Harary index. Li [15, 16] presented sufficient conditions in terms of the Harary index and Wiener index for a graph to be Hamiltonian or Hamilton-connected using some proof ideas in [25].

In this paper, as a continuance of the above results, we further study the conditions for a graph to be Hamiltonian, traceable, Hamilton-connected or  $k$ -connected in light of numerous topological indices such as the ECI, EDS and CEI.

## 2. Preliminaries

We first present some lemmas that will be used later.

**Lemma 2.1.** [8] *Let  $G$  be a graph of order  $n \geq 3$  with degree sequence  $d_1 \leq d_2 \leq \dots \leq d_n$ . If  $d_k \leq k < \frac{n}{2} \Rightarrow d_{n-k} \geq n - k$ , then  $G$  is Hamiltonian.*

**Lemma 2.2.** [1] *Let  $G$  be a nontrivial graph of order  $n \geq 4$  with degree sequence  $d_1 \leq d_2 \leq \dots \leq d_n$ . If  $d_k + 1 \leq k < \frac{n+1}{2} \Rightarrow d_{n-k+1} \leq n - k - 1$ , then  $G$  is traceable.*

**Lemma 2.3.** [3] *Let  $G$  be a graph of order  $n \geq 4$  with degree sequence  $d_1 \leq d_2 \leq \dots \leq d_n$ . If  $d_i \leq i + k - 2 \Rightarrow d_{n-k+1} \geq n - i$ , for  $1 \leq i \leq \frac{1}{2}(n - k + 1)$ , then  $G$  is  $k$ -connected.*

**Lemma 2.4.** [2] *Let  $G$  be a graph of order  $n \geq 4$  with degree sequence  $d_1 \leq d_2 \leq \dots \leq d_n$ . If  $2 \leq k \leq \frac{n}{2}, d_{k-1} \leq k \Rightarrow d_{n-k} \geq n - k + 1$ , then  $G$  is Hamilton-connected.*

**Lemma 2.5.** [2, Page 210, Corollary 5] *Let  $G = (X, Y; E)$  be a bipartite graph such that  $X = \{x_1, x_2, \dots, x_n\}, Y = \{y_1, y_2, \dots, y_n\}, n \geq 2$ , and  $d(x_1) \leq d(x_2) \leq \dots \leq d(x_n), d(y_1) \leq d(y_2) \leq \dots \leq d(y_n)$ . If  $d(x_k) \leq k < n \Rightarrow d(y_{n-k}) \geq n - k + 1$ , then  $G$  is Hamiltonian.*

**Lemma 2.6.** [4] *Let  $G$  be a 2-connected graph of order  $n \geq 12$ . If  $m \geq \binom{n-2}{2} + 4$ , then  $G$  is Hamiltonian or  $G = K_2 \vee ((2K_1) \cup K_{n-4})$ .*

**Lemma 2.7.** [4] *Let  $G$  be a 3-connected graph of order  $n \geq 18$ . If  $m \geq \binom{n-3}{2} + 9$ , then  $G$  is Hamiltonian or  $G = K_3 \vee ((3K_1) \cup K_{n-6})$ .*

**Lemma 2.8.** [4] *Let  $G$  be a  $k$ -connected graph of order  $n$ . If  $m \geq \binom{n}{2} - (k + 1)(n - k - 1)/2 + 1$ , then  $G$  is Hamiltonian.*

### 3. Main Results

**Theorem 3.1.** *Let  $G$  be a connected graph of order  $n \geq 6$ .*

- (1) *If  $\xi^c(G) \geq n^3 - 3n^2 + 4n - \frac{4m^2}{n} > 0$ , then  $G$  is Hamiltonian.*
- (2) *If  $\xi^d(G) \leq \frac{1}{n(n-1)}(n^2 + n - 4)^2$ , then  $G$  is Hamiltonian.*
- (3) *If  $\xi^{ce}(G) \geq (n - 1)\frac{n^2 - 3n + 5}{n}$ , then  $G$  is Hamiltonian.*

*Proof.* Suppose that  $G$  is not Hamiltonian, then from Lemma 2.1, there exists an integer  $1 \leq k \leq \frac{n-1}{2}$  such that  $d_k \leq k$  and  $d_{n-k} \leq n - k - 1$ .

- (1) We consider  $\xi^c(G)$ . Since  $\varepsilon(v) \leq n - d(v)$ , from the definition, we have

$$\begin{aligned} \xi^c(G) &= \sum_{v \in V(G)} \varepsilon(v)d(v) \leq \sum_{v \in V(G)} (n - d(v))d(v) \\ &= n \left( \sum_{v \in V(G)} d(v) \right) - \sum_{v \in V(G)} d^2(v) \\ &\leq n \left( \sum_{v \in V(G)} d(v) \right) - \frac{1}{n} \left( \sum_{v \in V(G)} d(v) \right)^2 \\ &= n \left( \sum_{v \in V(G)} d(v) \right) - \frac{4m^2}{n} \\ &\leq n [k^2 + (n - 2k)(n - k - 1) + k(n - 1)] - \frac{4m^2}{n} \\ &= n^2(n - 1) + n [3k^2 - (2n - 1)k] - \frac{4m^2}{n}. \end{aligned}$$

Suppose  $f(x) = 3x^2 - (2n - 1)x$  with  $1 \leq x \leq \frac{1}{2}(n - 1)$ . It is easy to see that  $f_{\max}(x) = \max\{f(1), f(\frac{1}{2}(n-1))\}$ . As  $f(1) = 4 - 2n$ ,  $f(\frac{1}{2}(n-1)) = \frac{1}{4}(1 - n^2)$ ,  $f(\frac{n-1}{2}) - f(1) = -\frac{1}{4}(n-5)(n-3) < 0$ , so we have  $f_{\max}(x) = f(1)$ . Thus,  $\xi^c(G) \leq n^2(n - 1) + n(4 - 2n) - \frac{4m^2}{n} = n^3 - 3n^2 + 4n - \frac{4m^2}{n}$ , so we get the result.

If  $\xi^c(G) = n^3 - 3n^2 + 4n - \frac{4m^2}{n}$ , then all the inequalities in the proof should be equalities, so  $k = 1$ , and hence  $d_1 = 1, d_2 = d_3 = \dots = d_{n-1} = n - 2, d_n = n - 1$ . Thus  $G = K_1 \vee (K_1 \cup K_{n-2})$ , which is not Hamiltonian as stated in [1]. But this graph dose not satisfy  $\sum_{v \in V(G)} d^2(v) = \frac{1}{n} \left( \sum_{v \in V(G)} d(v) \right)^2$ , thus the equality can not hold.

(2) We consider  $\xi^d(G)$ . Since  $\varepsilon(v) \geq \frac{D(v)}{n-1}$ ,  $D(v) \geq d(v) + 2(n - 1 - d(v))$ , from the definition, we have

$$\begin{aligned} \xi^d(G) &= \sum_{v \in V(G)} \varepsilon(v) \cdot D(v) \geq \sum_{v \in V(G)} \frac{D(v)}{n-1} \cdot D(v) \\ &= \frac{1}{n-1} \sum_{v \in V(G)} (D(v))^2 \\ &\geq \frac{1}{n-1} \sum_{v \in V(G)} [4(n-1)^2 - 4(n-1)d(v) + (d(v))^2] \\ &= 4n(n-1) - 4 \sum_{v \in V(G)} d(v) + \frac{1}{n-1} \sum_{v \in V(G)} (d(v))^2 \\ &\geq 4n(n-1) - 4 \sum_{v \in V(G)} d(v) + \frac{1}{n-1} \cdot \frac{1}{n} \left( \sum_{v \in V(G)} d(v) \right)^2 \\ &= \frac{1}{n(n-1)} \left[ \left( \sum_{v \in V(G)} d(v) \right)^2 - 4n(n-1) \sum_{v \in V(G)} d(v) + 4n^2(n-1)^2 \right] \\ &= \frac{1}{n(n-1)} \left[ \sum_{v \in V(G)} d(v) - 2n(n-1) \right]^2. \end{aligned}$$

As  $\sum_{v \in V(G)} d(v) < 2n(n - 1)$ , we have

$$\begin{aligned} \xi^d(G) &\geq \frac{1}{n(n-1)} [k^2 + (n - 2k)(n - k - 1) + k(n - 1) - 2n(n - 1)]^2 \\ &= \frac{1}{n(n-1)} \{2n(n - 1) - [k^2 + (n - 2k)(n - k - 1) + k(n - 1)]\}^2 \\ &= \frac{1}{n(n-1)} [-3k^2 + (2n - 1)k + n^2 - n]^2. \end{aligned}$$

Suppose  $f(x) = -3x^2 + (2n-1)x + n^2 - n$  with  $1 \leq x \leq \frac{1}{2}(n-1)$ . As  $f(1) = n^2 + n - 4$ ,  $f(\frac{n-1}{2}) = \frac{1}{4}(n - 1)(5n + 1)$ ,  $f(\frac{n-1}{2}) - f(1) = \frac{1}{4}(n - 3)(n - 5) > 0$ , so we have  $f_{\min}(x) = \min\{f(1), f(\frac{1}{2}(n - 1))\} = f(1)$ . Thus  $\xi^d(G) \geq \frac{1}{n(n-1)}(n^2 + n - 4)^2$ , and we get the result.

If  $\xi^d(G) = \frac{1}{n(n-1)}(n^2 + n - 4)^2$ , then  $k = 1$ , the remaining is as in the previous proof.

(3) We consider  $\xi^{ce}(G)$ . From the definition, we have

$$\begin{aligned} \xi^{ce}(G) &= \sum_{v \in V(G)} \frac{d(v)}{\varepsilon(v)} \\ &\leq \sum_{v \in V(G)} \frac{n-1}{D(v)} \cdot d(v) \\ &\leq (n-1) \sum_{v \in V(G)} \frac{d(v)}{2(n-1) - d(v)}. \end{aligned}$$

Suppose  $f(x) = \frac{x}{2(n-1)-x}$ , then we have  $f'(x) = \frac{2(n-1)}{[2(n-1)-x]^2} > 0$ , and thus  $f(x)$  is strictly increasing, therefore

$$\begin{aligned} \xi^{ce}(G) &\leq (n-1) \left[ \frac{k^2}{2(n-1) - k} + \frac{(n-2k)(n-k-1)}{2(n-1) - (n-k-1)} + \frac{k(n-1)}{2(n-1) - (n-1)} \right] \\ &= (n-1) \left[ \frac{k^2}{2n-k-2} + \frac{(n-2k)(n-k-1)}{n+k-1} + k \right]. \end{aligned}$$

Since  $1 \leq k \leq \frac{n-1}{2}$ , then  $2n-k-2 - (n+k-1) = n-2k-1 \geq 0$ , so  $\frac{k^2}{2n-k-2} \leq \frac{k^2}{n+k-1}$ . Further,  $\frac{(n-2k)(n-k-1)}{n+k-1} = \frac{(n-2k)(n+k-1-2k)}{n+k-1} = n-2k - \frac{2k(n-2k)}{n+k-1}$ . Therefore,

$$\begin{aligned} \xi^{ce}(G) &\leq (n-1) \left[ \frac{k^2}{n+k-1} - \frac{2k(n-2k)}{n+k-1} - k + n \right] \\ &= (n-1) \left[ \frac{k^2 - 2k(n-2k)}{n+k-1} - k + n \right] \\ &= (n-1) \left\{ \frac{k[4k - (3n-1)]}{n+k-1} + n \right\}. \end{aligned}$$

Suppose  $f(x) = \frac{x[4x-(3n-1)]}{n+x-1}$  with  $1 \leq x \leq \frac{1}{2}(n-1)$ . As  $f(1) = \frac{5-3n}{n}$ ,  $f(\frac{n-1}{2}) = -\frac{1}{3}(n+1)$ ,  $f(\frac{n-1}{2}) - f(1) = -\frac{1}{3n}(n-3)(n-5) < 0$ , so we have  $f_{\max}(x) = \max\{f(1), f(\frac{1}{2}(n-1))\} = f(1)$ . Thus  $\xi^{ce}(G) \leq (n-1) \frac{n^2-3n+5}{n}$ , and we get the result.

If  $\xi^{ce}(G) = (n-1) \frac{n^2-3n+5}{n}$ , then all the inequalities in the proof should be equalities, so  $k = 1$ , and hence  $d_1 = 1, d_2 = d_3 = \dots = d_{n-1} = n-2, d_n = n-1$ . Thus  $G = K_1 \vee (K_1 \cup K_{n-2})$ , which is not Hamiltonian as stated in [1].

On the other hand,  $\varepsilon(v) \geq \frac{D(v)}{n-1}$ , with equality if and only if  $d(v, u)$  (for fixed  $v \in V(G)$ ) is a constant for all  $u \in V(G)$  with  $v \neq u$ . Thus,  $G = K_1 \vee (K_1 \cup K_{n-2})$  can not satisfy it, the equality can not hold. □

**Theorem 3.2.** *Let  $G$  be a connected graph of order  $n \geq 11$ .*

- (1) *If  $\xi^c(G) \geq n^3 - 5n^2 + 10n - \frac{4m^2}{n} > 0$ , then  $G$  is traceable.*
- (2) *If  $\xi^d(G) \leq \frac{1}{n(n-1)}(n^2 + 3n - 10)^2$ , then  $G$  is traceable.*
- (3) *If  $\xi^{ce}(G) \geq (n-1) \frac{n^2-5n+12}{n+1}$ , then  $G$  is traceable.*

*Proof.* Suppose that  $G$  is not traceable, then by Lemma 2.2, there is an integer  $k \leq \frac{n}{2}$  such that  $d_k \leq k-1$  and  $d_{n-k+1} \leq n-k-1$ . Since  $G$  is connected and  $d_k \leq k-1$ , we have  $k \geq 2$ .

(1) We consider  $\xi^c(G)$ . As in Theorem 3.1, we have

$$\begin{aligned}\xi^c(G) &\leq n \left( \sum_{v \in V(G)} d(v) \right) - \frac{4m^2}{n} \\ &\leq n[k(k-1) + (n-2k+1)(n-k-1) + (k-1)(n-1)] - \frac{4m^2}{n} \\ &= n^2(n-1) - \frac{4m^2}{n} + n[3k^2 - (2n+1)k].\end{aligned}$$

Suppose  $f(x) = 3x^2 - (2n+1)x$  with  $2 \leq x \leq \frac{n}{2}$ . As  $f(2) = 10 - 4n$ ,  $f(\frac{n}{2}) = -\frac{1}{4}n(n+2)$ ,  $f(\frac{n}{2}) - f(2) = -\frac{1}{4}(n-10)(n-4) < 0$ , so we have  $f_{\max}(x) = \max\{f(2), f(\frac{n}{2})\} = f(2)$ . Thus  $\xi^c(G) \leq n^2(n-1) - \frac{4m^2}{n} + n(10-4n) = n^3 - 5n^2 + 10n - \frac{4m^2}{n}$ , so we get the result.

If  $\xi^c(G) = n^3 - 5n^2 + 10n - \frac{4m^2}{n}$ , then  $k = 2$ , and hence  $d_1 = d_2 = 1$ ,  $d_3 = \dots = d_{n-1} = n-3$ ,  $d_n = n-1$ . Thus  $G = K_1 \vee (K_{n-3} \cup 2K_1)$ , which is not traceable. But this graph does not satisfy  $\sum_{v \in V(G)} d^2(v) = \frac{1}{n} \left( \sum_{v \in V(G)} d(v) \right)^2$ , thus the equality can not hold.

(2) We consider  $\xi^d(G)$ , as in Theorem 3.1, we have

$$\xi^d(G) \geq \frac{1}{n(n-1)} \left[ \sum_{v \in V(G)} d(v) - 2n(n-1) \right]^2.$$

Since  $2n(n-1) - \sum_{v \in V(G)} d(v) > 0$ , we have

$$\begin{aligned}\xi^d(G) &\geq \frac{1}{n(n-1)} [k(k-1) + (n-2k+1)(n-k-1) + (k-1)(n-1) - 2n(n-1)]^2 \\ &= \frac{1}{n(n-1)} \{2n(n-1) - [k(k-1) + (n-2k+1)(n-k-1) + (k-1)(n-1)]\}^2 \\ &= \frac{1}{n(n-1)} [-3k^2 + (2n+1)k + n(n-1)]^2.\end{aligned}$$

Suppose  $f(x) = -3x^2 + (2n+1)x + n^2 - n$  with  $2 \leq x \leq \frac{n}{2}$ . As  $f(2) = n^2 + 3n - 10$ ,  $f(\frac{n}{2}) = \frac{1}{4}n(5n-2)$ ,  $f(\frac{n}{2}) - f(2) = \frac{1}{4}(n-4)(n-10) \geq 0$ , so we have  $f_{\min}(x) = \min\{f(2), f(\frac{n}{2})\} = f(2)$ . Thus  $\xi^d(G) \geq \frac{1}{n(n-1)}(n^2 + 3n - 10)^2$ , so we get the result.

If  $\xi^d(G) = \frac{1}{n(n-1)}(n^2 + 3n - 10)^2$ , then  $k = 2$ , the remaining is as in the previous proof.

(3) We consider  $\xi^{ce}(G)$ . As in Theorem 3.1,

$$\xi^{ce}(G) \leq (n-1) \sum_{v \in V(G)} \frac{d(v)}{2(n-1) - d(v)}.$$

Suppose  $f(x) = \frac{x}{2(n-1)-x}$ , then  $f'(x) = \frac{2(n-1)}{[2(n-1)-x]^2} > 0$ , so

$$\begin{aligned}\xi^{ce}(G) &\leq (n-1) \left[ \frac{k(k-1)}{2(n-1) - (k-1)} + \frac{(n-2k+1)(n-k-1)}{2(n-1) - (n-k-1)} + \frac{(k-1)(n-1)}{2(n-1) - (n-1)} \right] \\ &= (n-1) \left[ \frac{k(k-1)}{2n-k-1} + \frac{(n-2k+1)(n-k-1)}{n+k-1} + k-1 \right].\end{aligned}$$

Since  $2 \leq k \leq \frac{n}{2}$ , then  $2n - k - 1 - (n + k - 1) = n - 2k \geq 0$ . As  $\frac{(n-2k+1)(n-k-1)}{n+k-1} = \frac{(n-2k+1)(n+k-1-2k)}{n+k-1} = n - 2k + 1 - \frac{2k(n-2k+1)}{n+k-1}$ , therefore,

$$\begin{aligned} \xi^{ce}(G) &\leq (n-1) \left[ \frac{k(k-1)}{n+k-1} - \frac{2k(n-2k+1)}{n+k-1} - k + n \right] \\ &= (n-1) \left[ \frac{k(k-1) - 2k(n-2k+1)}{n+k-1} - k + n \right] \\ &= (n-1) \left\{ \frac{k[4k - (3n+2)]}{n+k-1} + n \right\}. \end{aligned}$$

Suppose  $f(x) = \frac{x[4x-(3n+2)]}{n+x-1}$  with  $2 \leq x \leq \frac{n}{2}$ . It is easy to see that  $f_{\max}(x) = \max\{f(2), f(\frac{n}{2})\}$ . As  $f(2) = \frac{6(2-n)}{n+1}$ ,  $f(\frac{n}{2}) = -\frac{n(n+2)}{3n-2}$ ,  $f(\frac{n}{2}) - f(2) = -\frac{(n-4)(n^2-11n+6)}{(3n-2)(n+1)} < 0$ , so we have  $f_{\max}(x) = f(2)$ . Thus  $\xi^{ce}(G) \leq (n-1)\frac{n^2-5n+12}{n+1}$ , so we get the result.

If  $\xi^{ce}(G) = (n-1)\frac{n^2-5n+12}{n+1}$ , then  $k = 2$ , and hence  $d_1 = d_2 = 1$ ,  $d_3 = \dots = d_{n-1} = n - 3$ ,  $d_n = n - 1$ . Thus  $G = K_1 \vee (K_{n-3} \cup 2K_1)$ , which is not traceable.

On the other hand,  $\varepsilon(v) \geq \frac{D(v)}{n-1}$ , with equality if and only if  $d(v, u)$  (for fixed  $v \in V(G)$ ) is a constant for all  $u \in V(G)$  with  $v \neq u$ . But  $G = K_1 \vee (K_{n-3} \cup 2K_1)$  can not satisfy it, and the equality case can not occur. □

**Theorem 3.3.** *Let  $G$  be a connected graph of order  $n \geq 2$ .*

- (1) *If  $\xi^c(G) \geq n^3 - 3n^2 + 2kn - \frac{4m^2}{n} > 0$ , then  $G$  is  $k$ -connected.*
- (2) *If  $\xi^d(G) \leq \frac{1}{n(n-1)}(n^2 + n - 2k)^2$ , then  $G$  is  $k$ -connected.*
- (3) *If  $\xi^{ce}(G) \geq (n-1)(\frac{3k-3n-1}{n} + n)$ , then  $G$  is  $k$ -connected.*

*Proof.* Suppose that  $G$  is not  $k$ -connected, then from Lemma 2.3, there exists an integer  $1 \leq i \leq \frac{n-k+1}{2}$  such that  $d_i \leq i + k - 2$  and  $d_{n-k+1} \leq n - i - 1$ . Obviously,  $1 \leq k \leq n - 1$ .

(1) We consider  $\xi^c(G)$ , as in Theorem 3.1, we have

$$\begin{aligned} \xi^c(G) &\leq n \left( \sum_{v \in V(G)} d(v) \right) - \frac{4m^2}{n} \\ &\leq n [i(i+k-2) + (n-k-i+1)(n-i-1) + (k-1)(n-1)] - \frac{4m^2}{n} \\ &= n^2(n-1) - \frac{4m^2}{n} + 2n [i^2 - (n-k+1)i]. \end{aligned}$$

Suppose  $f(x) = x^2 - (n-k+1)x$  with  $1 \leq x \leq \frac{n-k+1}{2}$ , then  $f(x) \leq f(1) = k - n$ . Thus  $\xi^c(G) \leq n^2(n-1) - \frac{4m^2}{n} + 2n(k-n) = n^3 - 3n^2 + 2kn - \frac{4m^2}{n}$ , so we get the result.

If  $\xi^c(G) = n^3 - 3n^2 + 2kn - \frac{4m^2}{n}$ , then all the inequalities in the proof should be equalities, so  $i = 1$ ,  $d_1 = k - 1$ ,  $d_2 = \dots = d_{n-k+1} = n - 2$ ,  $d_{n-k+2} = \dots = d_n = n - 1$ . Thus  $G = (K_1 \cup K_{n-k}) \vee K_{k-1}$ , which is not  $k$ -connected. But it can not satisfy  $\sum_{v \in V(G)} d^2(v) = \frac{1}{n} \left( \sum_{v \in V(G)} d(v) \right)^2$ , thus the equality can not hold.

(2) We consider  $\xi^d(G)$ . As in Theorem 3.1,

$$\xi^d(G) \geq \frac{1}{n(n-1)} \left[ \sum_{v \in V(G)} d(v) - 2n(n-1) \right]^2.$$

Since  $2n(n-1) - \sum_{v \in V(G)} d(v) > 0$ , then

$$\begin{aligned} \xi^d(G) &\geq \frac{1}{n(n-1)} [i(i+k-2) + (n-k-i+1)(n-i-1) + (k-1)(n-1) - 2n(n-1)]^2 \\ &= \frac{1}{n(n-1)} \{2n(n-1) - [i(i+k-2) + (n-k-i+1)(n-i-1) + (k-1)(n-1)]\}^2 \\ &= \frac{1}{n(n-1)} [-2i^2 - 2i(-1+k-n) + n(n-1)]^2. \end{aligned}$$

Suppose  $f(x) = -2x^2 - 2x(-1+k-n) + n(n-1)$  with  $1 \leq x \leq \frac{n-k+1}{2}$ ,  $f(1) \leq f(x) \leq f(\frac{n-k+1}{2})$ ,  $f(1) = n(n+1) - 2k$ . Thus  $\xi^d(G) \geq \frac{1}{n(n-1)}(n(n+1) - 2k)^2$ , so we get the result.

If  $\xi^d(G) = \frac{1}{n(n-1)}(n^2 + n - 2k)^2$ , then all the inequalities in the proof should be equalities, so  $i = 1$ , the remaining is as in the previous proof.

(3) We consider  $\xi^{ce}(G)$ , as in Theorem 3.1, we have

$$\xi^{ce}(G) \leq (n-1) \sum_{v \in V(G)} \frac{d(v)}{2(n-1) - d(v)}.$$

Suppose  $f(x) = \frac{x}{2(n-1)-x}$ ,  $f'(x) = \frac{2(n-1)}{[2(n-1)-x]^2} > 0$ , so

$$\begin{aligned} \xi^{ce}(G) &\leq (n-1) \left[ \frac{i(i+k-2)}{2(n-1) - (i+k-2)} + \frac{(n-k-i+1)(n-i-1)}{2(n-1) - (n-i-1)} \right. \\ &\quad \left. + \frac{(k-1)(n-1)}{2(n-1) - (n-1)} \right] \\ &= (n-1) \left[ \frac{i(i+k-2)}{2n-k-i} + \frac{(n-k-i+1)(n-i-1)}{n+i-1} + k-1 \right]. \end{aligned}$$

Since  $1 \leq i \leq \frac{n-k+1}{2}$ , then  $2n-k-i - (n+i-1) = n-k-2i+1 \geq 0$ . Further,  $\frac{(n-k-i+1)(n-i-1)}{n+i-1} = \frac{(n-k-i+1)(n+i-1-2i)}{n+i-1} = n-k-i+1 - \frac{2i(n-k-i+1)}{n+i-1}$ .

Therefore,

$$\begin{aligned} \xi^{ce}(G) &\leq (n-1) \left[ \frac{i(i+k-2)}{n+i-1} - \frac{2i(n-k-i+1)}{n+i-1} - i+n \right] \\ &= (n-1) \left[ \frac{i(i+k-2) - 2i(n-k-i+1)}{n+i-1} - i+n \right] \\ &= (n-1) \left[ \frac{i(2i+3k-3n-3)}{n+i-1} + n \right]. \end{aligned}$$

Suppose  $f(x) = \frac{x(2x+3k-3n-3)}{n+x-1}$  with  $1 \leq x \leq \frac{n-k+1}{2}$ , we can easily compute that  $f_{\max}(x) = f(1) = \frac{3k-3n-1}{n}$ . Thus  $\xi^{ce}(G) \leq (n-1)(\frac{3k-3n-1}{n} + n)$ , so we get the result.

If  $\xi^{ce}(G) = (n-1)(\frac{3k-3n-1}{n} + n)$ , then  $i = 1$ ,  $d_1 = k-1$ ,  $d_2 = \dots = d_{n-k+1} = n-2$ ,  $d_{n-k+2} = \dots = d_n = n-1$ . Thus  $G = (K_1 \cup K_{n-k}) \vee K_{k-1}$ , which is not  $k$ -connected.



On the other hand,  $\varepsilon(v) \geq \frac{D(v)}{n-1}$ , with equality if and only if  $d(v, u)$  (for fixed  $v \in V(G)$ ) is a constant for all  $u \in V(G)$  with  $v \neq u$ . But  $G = (K_1 \cup K_{n-k}) \vee K_{k-1}$  can not satisfy it, the equality can not hold.  $\square$

**Theorem 3.4.** *Let  $G$  be a connected graph of order  $n \geq 7$ .*

- (1) *If  $\xi^c(G) \geq n^3 - 3n^2 + 6n - \frac{4m^2}{n} > 0$ , then  $G$  is Hamilton-connected.*
- (2) *If  $\xi^d(G) \leq \frac{1}{n(n-1)}(n^2 + n - 6)^2$ , then  $G$  is Hamilton-connected.*
- (3) *If  $\xi^{ce}(G) \geq (n-1)(\frac{8}{n} + n - 3)$ , then  $G$  is Hamilton-connected.*

*Proof.* Suppose that  $G$  is not Hamilton-connected, then from Lemma 2.4, there exists an integer  $2 \leq k \leq \frac{n}{2}$  such that  $d_{k-1} \leq k$  and  $d_{n-k} \leq n - k$ .

(1) We consider  $\xi^c(G)$ , as in Theorem 3.1, we have

$$\begin{aligned} \xi^c(G) &\leq n \left( \sum_{v \in V(G)} d(v) \right) - \frac{4m^2}{n} \\ &\leq n [(k-1)k + (n-2k+1)(n-k) + k(n-1)] - \frac{4m^2}{n} \\ &= n^2(n+1) - \frac{4m^2}{n} + n [3k^2 - (2n+3)k]. \end{aligned}$$

Suppose  $f(x) = 3x^2 - (2n+3)x$  with  $2 \leq x \leq \frac{n}{2}$ . As  $f(2) = 6 - 4n$ ,  $f(\frac{n}{2}) = -\frac{1}{4}n(n+6)$ ,  $f(\frac{n}{2}) - f(2) = -\frac{1}{4}(n-6)(n-4) < 0$ , so we have  $f_{\max}(x) = \max\{f(2), f(\frac{n}{2})\} = f(2)$ . Thus,  $\xi^c(G) \leq n^2(n+1) - \frac{4m^2}{n} + n(6-4n) = n^3 - 3n^2 + 6n - \frac{4m^2}{n}$ , so we get the result.

If  $\xi^c(G) = n^3 - 3n^2 + 6n - \frac{4m^2}{n}$ , then  $k = 2$ ,  $d_1 = 2$ ,  $d_2 = d_3 = \dots = d_{n-2} = n - 2$ ,  $d_{n-1} = d_n = n - 1$ . Thus  $G = K_2 \vee (K_1 \cup K_{n-3})$ , which is not Hamilton-connected. But it can not satisfy  $\sum_{v \in V(G)} d^2(v) = \frac{1}{n} \left( \sum_{v \in V(G)} d(v) \right)^2$ , thus the equality can not hold.

(2) We consider  $\xi^d(G)$ , as in Theorem 3.1, we have

$$\xi^d(G) \geq \frac{1}{n(n-1)} \left[ \sum_{v \in V(G)} d(v) - 2n(n-1) \right]^2$$

Since  $2n(n-1) - \sum_{v \in V(G)} d(v) > 0$ , then

$$\begin{aligned} \xi^d(G) &\geq \frac{1}{n(n-1)} [k(k-1) + (n-2k+1)(n-k) + k(n-1) - 2n(n-1)]^2 \\ &= \frac{1}{n(n-1)} \{2n(n-1) - [k(k-1) + (n-2k+1)(n-k) + k(n-1)]\}^2 \\ &= \frac{1}{n(n-1)} [-3k^2 + (2n+3)k + n(n-3)]^2. \end{aligned}$$

Suppose  $f(x) = -3x^2 + (2n+3)x + n(n-3)$  with  $2 \leq x \leq \frac{n}{2}$ . As  $f(2) = n^2 + n - 6$ ,  $f(\frac{n}{2}) = \frac{1}{4}n(5n-6)$ ,  $f(\frac{n}{2}) - f(2) = \frac{1}{4}(n-4)(n-6) > 0$ , so we have  $f_{\min}(x) = \min\{f(2), f(\frac{n}{2})\} = f(2)$ . Thus  $\xi^d(G) \geq \frac{1}{n(n-1)}(n^2 + n - 6)^2$ , so we get the result.

If  $\xi^d(G) = \frac{1}{n(n-1)}(n^2 + n - 6)^2$ , then  $k = 2$ , the remaining is as in the previous proof.

(3) We consider  $\xi^{ce}(G)$ , as in Theorem 3.1, we have

$$\xi^{ce}(G) \leq (n-1) \sum_{v \in V(G)} \frac{d(v)}{2(n-1) - d(v)}.$$

Suppose  $f(x) = \frac{x}{2(n-1)-x}$ ,  $f'(x) = \frac{2(n-1)}{[2(n-1)-x]^2} > 0$ , so

$$\begin{aligned} \xi^{ce}(G) &\leq (n-1) \left[ \frac{k(k-1)}{2(n-1)-k} + \frac{(n-2k+1)(n-k)}{2(n-1)-(n-k)} + \frac{k(n-1)}{2(n-1)-(n-1)} \right] \\ &= (n-1) \left[ \frac{k(k-1)}{2n-k-2} + \frac{(n-2k+1)(n-k)}{n+k-2} + k \right]. \end{aligned}$$

Since  $2 \leq k \leq \frac{n}{2}$ , then  $2n - k - 2 - (n + k - 2) = n - 2k \geq 0$ . Further,  $\frac{(n-2k+1)(n-k)}{n+k-2} = \frac{(n-2k+1)(n+k-2-2k+2)}{n+k-2} = n - 2k + 1 - \frac{(2k-2)(n-2k+1)}{n+k-2}$ .

Therefore,

$$\begin{aligned} \xi^{ce}(G) &\leq (n-1) \left[ \frac{k(k-1)}{n+k-2} - \frac{(2k-2)(n-2k+1)}{n+k-2} - k + n + 1 \right] \\ &= (n-1) \left[ \frac{k(k-1) - (2k-2)(n-2k+1)}{n+k-2} - k + n + 1 \right] \\ &= (n-1) \left\{ \frac{4k^2 - (3n+5)k + 2n + 2}{n+k-2} + n + 1 \right\}. \end{aligned}$$

Suppose  $f(x) = \frac{4x^2 - (3n+5)x + 2n + 2}{n+x-2}$  with  $2 \leq x \leq \frac{n}{2}$ . As  $f(\frac{n}{2}) = -\frac{n^2+n-4}{3n-4}$ ,  $f(2) = \frac{8}{n} - 4$ ,  $f(\frac{n}{2}) - f(2) = -\frac{(n-4)(n^2-7n+8)}{n(3n-4)} < 0$ . So we have  $f_{\max}(x) = \max\{f(2), f(\frac{n}{2})\} = f(2)$ . Thus  $\xi^{ce}(G) \leq (n-1)[f(2) + n + 1] = (n-1)(\frac{8}{n} + n - 3)$ , so we get the result.

If  $\xi^{ce}(G) = (n-1)(\frac{8}{n} + n - 3)$ , then  $k = 2$ ,  $d_1 = 2$ ,  $d_2 = d_3 = \dots = d_{n-2} = n - 2$ ,  $d_{n-1} = d_n = n - 1$ . Thus  $G = K_2 \vee (K_1 \cup K_{n-3})$ , which is not Hamilton-connected.

On the other hand,  $\varepsilon(v) \geq \frac{D(v)}{n-1}$ , with equality if and only if  $d(v, u)$  (for fixed  $v \in V(G)$ ) is a constant for all  $u \in V(G)$  with  $v \neq u$ . Thus,  $G = K_2 \vee (K_1 \cup K_{n-3})$  can not satisfy it, the equality can not hold. □

**Theorem 3.5.** Let  $G = (X, Y; E)$  be a connected bipartite graph with  $X = \{x_1, x_2, \dots, x_n\}$ ,  $Y = \{y_1, y_2, \dots, y_n\}$ , and  $n \geq 2$ . Then we have:

- (1) If  $\xi^d(G) \leq \frac{1}{n(n-1)}(n^2 + 2n - 4)^2 + \frac{n(5n-2)^2}{n-1}$ , then  $G$  is Hamiltonian.
- (2) If  $\xi^{ce}(G) \geq (n-1) \left[ \frac{1}{6}(n^2 - 2n + 2) + \frac{n^2}{3n-2} \right]$ , then  $G$  is Hamiltonian.

*Proof.* Suppose that  $G$  is not Hamiltonian, then from Lemma 2.5, there exists an integer  $k < n$  such that  $d(x_k) \leq k$  and  $d(y_{n-k}) \leq n - k$ . Let  $N(x_1) := \{z_1, z_2, \dots, z_s\}$  be the neighbors of  $x_1$ , where  $s = d(x_1)$ . Then  $d(x_1, z_i) = 1$  for each  $z_i \in N(x_1)$ ,  $d(x_1, x_i) \geq 2$  for each  $x_i$  with  $2 \leq i \leq n$ , and  $d(x_1, y_i) \geq 3$  for each  $y_i \in Y - N(x_1)$ .

(1) We consider  $\xi^d(G)$ . First, we have

$$D(x_1) \geq d(x_1) + 2(n-1) + 3(n-d(x_1)) = 5n - 2 - 2d(x_1).$$

Similarly, for each  $i$  with  $2 \leq i \leq n$  and each  $j$  with  $1 \leq j \leq n$ ,

$$D(x_i) \geq d(x_i) + 2(n - 1) + 3(n - d(x_i)) = 5n - 2 - 2d(x_i).$$

$$D(y_j) \geq d(y_j) + 2(n - 1) + 3(n - d(y_j)) = 5n - 2 - 2d(y_j).$$

Therefore, we have

$$\begin{aligned} \xi^d(G) &= \sum_{v \in V(G)} \varepsilon(v) \cdot D(v) \geq \sum_{v \in V(G)} \frac{D(v)}{n-1} \cdot D(v) = \frac{1}{n-1} \sum_{v \in V(G)} (D(v))^2 \\ &= \frac{1}{n-1} \left[ \sum_{x_i \in X} (D(x_i))^2 + \sum_{y_j \in Y} (D(y_j))^2 \right] \\ &\geq \frac{1}{n-1} \left\{ \sum_{x_i \in X} [(5n-2)^2 - 4(5n-2)d(x_i) + 4(d(x_i))^2] \right. \\ &\quad \left. + \sum_{y_j \in Y} [(5n-2)^2 - 4(5n-2)d(y_j) + 4(d(y_j))^2] \right\} \\ &= \frac{1}{n-1} \left[ 2n(5n-2)^2 - 4(5n-2) \sum_{v \in V(G)} d(v) + 4 \sum_{v \in V(G)} (d(v))^2 \right] \\ &\geq \frac{1}{n-1} \left[ 2n(5n-2)^2 - 4(5n-2) \sum_{v \in V(G)} d(v) + \frac{4}{n} \left( \sum_{v \in V(G)} d(v) \right)^2 \right] \\ &= \frac{1}{n(n-1)} \left[ 2n^2(5n-2)^2 - 4n(5n-2) \sum_{v \in V(G)} d(v) + 4 \left( \sum_{v \in V(G)} d(v) \right)^2 \right] \\ &= \frac{1}{n(n-1)} \left[ n(5n-2) - 2 \sum_{v \in V(G)} d(v) \right]^2 + \frac{n(5n-2)^2}{n-1}. \end{aligned}$$

Since

$$\begin{aligned} 2 \sum_{v \in V(G)} d(v) &\leq 2 [k^2 + (n-k)n + (n-k)^2 + kn] \\ &< 2 [kn + (n-k)n + (n-k)n + kn] \\ &= 4n^2 \leq n(5n-2), \end{aligned}$$

it follows that

$$\begin{aligned} \xi^d(G) &\geq \frac{1}{n(n-1)} [n(5n-2) - 2(k^2 + (n-k)n + (n-k)^2 + nk)]^2 + \frac{n(5n-2)^2}{n-1} \\ &= \frac{1}{n(n-1)} [-4k^2 + 4nk + n(n-2)]^2 + \frac{n(5n-2)^2}{n-1}. \end{aligned}$$

Suppose  $f(x) = -4x^2 + 4nx + n(n-2)$  with  $1 \leq x \leq n-1$ . It is easy to see that  $f_{\min}(x) = \min\{f(1), f(n-1)\}$ . As  $f(1) = f(n-1)$ ,  $f_{\min}(x) = f(1) = n^2 + 2n - 4$ , thus  $\xi^d(G) \geq \frac{1}{n(n-1)}(n^2 + 2n - 4)^2 + \frac{n(5n-2)^2}{n-1}$ .

If  $\xi^d(G) = \frac{1}{n(n-1)}(n^2 + 2n - 4)^2 + \frac{n(5n-2)^2}{n-1}$ , then  $k = 1$ ,  $d(x_1) = 1$ ,  $d(x_2) = \dots = d(x_n) = n$ ,  $d(y_1) = d(y_2) = \dots = d(y_{n-1}) = n-1$ ,  $d(y_n) = n$ . Thus  $G = K_{n,n} - K_{1,n-1}$ , which is not Hamiltonian.

On the other hand,  $\varepsilon(v) \geq \frac{D(v)}{n-1}$ , with equality if and only if  $d(v, u)$  (for fixed  $v \in V(G)$ ) is a constant for all  $u \in V(G)$  with  $v \neq u$ . However,  $G = K_{n,n} - K_{1,n-1}$  can not satisfy it, the equality can not hold.

(2) We consider  $\xi^{ce}(G)$ .

$$D(x_1) \geq d(x_1) + 2(n-1) + 3(n-d(x_1)) = 5n-2-2d(x_1).$$

Similarly, for each  $i$  with  $2 \leq i \leq n$  and each  $j$  with  $1 \leq j \leq n$ ,

$$D(x_i) \geq d(x_i) + 2(n-1) + 3(n-d(x_i)) = 5n-2-2d(x_i).$$

$$D(y_j) \geq d(y_j) + 2(n-1) + 3(n-d(y_j)) = 5n-2-2d(y_j).$$

Therefore,

$$\begin{aligned} \xi^{ce}(G) &= \sum_{v \in V(G)} \frac{d(v)}{\varepsilon(v)} \\ &\leq \sum_{x_i \in X} \frac{n-1}{D(x_i)} \cdot d(x_i) + \sum_{y_j \in X} \frac{n-1}{D(y_j)} \cdot d(y_j) \\ &\leq (n-1) \left[ \sum_{x_i \in X} \frac{d(x_i)}{5n-2-2d(x_i)} + \sum_{y_j \in X} \frac{d(y_j)}{5n-2-2d(y_j)} \right]. \end{aligned}$$

Suppose  $f(x) = \frac{x}{5n-2-2x}$ , then we have  $f'(x) = \frac{5n-2}{(5n-2-2x)^2} > 0$ , so

$$\begin{aligned} \xi^{ce}(G) &\leq (n-1) \left[ \frac{k^2}{5n-2-2k} + \frac{(n-k)n}{5n-2-2n} + \frac{(n-k)^2}{5n-2-2(n-k)} + \frac{kn}{5n-2-2n} \right] \\ &= (n-1) \left[ \frac{k^2}{5n-2-2k} + \frac{(n-k)^2}{3n+2k-2} + \frac{n^2}{3n-2} \right]. \end{aligned}$$

Since  $1 \leq k \leq n-1$ , then,

$$\begin{aligned} \xi^{ce}(G) &\leq (n-1) \left[ \frac{k^2}{5(k+1)-2-2k} + \frac{(n-k)^2}{3(k+1)+2k-2} + \frac{n^2}{3n-2} \right] \\ &= (n-1) \left[ \frac{k^2}{3k+3} + \frac{(n-k)^2}{5k+1} + \frac{n^2}{3n-2} \right] \\ &\leq (n-1) \left[ \frac{k^2}{3k+3} + \frac{(n-k)^2}{3k+3} + \frac{n^2}{3n-2} \right] \\ &= (n-1) \left[ \frac{k^2 + (n-k)^2}{3k+3} + \frac{n^2}{3n-2} \right]. \end{aligned}$$

Suppose  $f(x) = \frac{x^2+(n-x)^2}{3x+3}$  with  $1 \leq x \leq (n-1)$ . It is easy to see that  $f_{\max}(x) = \max\{f(1), f(n-1)\}$ . As  $f(n-1) = \frac{1}{3n}(n^2 - 2n + 2)$ ,  $f(1) = \frac{1}{6}(n^2 - 2n + 2)$ ,  $f(n-1) - f(1) = -\frac{(n-2)(n^2-2n+2)}{6n} < 0$ , so we have  $f_{\max}(x) = f(1)$ .

If  $\xi^{ce}(G) = (n - 1) \left[ \frac{1}{6}(n^2 - 2n + 2) + \frac{n^2}{3n-2} \right]$ , then  $k = 1$ , the remaining is as in the previous proof.  $\square$

**Theorem 3.6.** *Let  $G$  be a 2-connected graph of order  $n \geq 12$ . If  $\xi^d(G) \leq \frac{1}{n(n-1)} [n^2 + 3n - 12]^2$ , then  $G$  is Hamiltonian.*

*Proof.* Suppose that  $G$  is not Hamiltonian and  $G$  is not  $K_2 \vee (2K_1 \cup K_{n-4})$ , then from Lemma 2.6, we have that  $m \leq \binom{n-2}{2} + 3$ . As in Theorem 3.1,

$$\xi^d(G) \geq \frac{1}{n(n-1)} \left[ \sum_{v \in V(G)} d(v) - 2n(n-1) \right]^2.$$

Since  $2n(n-1) - \sum_{v \in V(G)} d(v) > 0$ , then

$$\begin{aligned} \xi^d(G) &\geq \frac{1}{n(n-1)} [2n(n-1) - 2m]^2 \\ &\geq \frac{1}{n(n-1)} [n^2 + 3n - 12]^2. \end{aligned}$$

On the other hand,  $\varepsilon(v) \geq \frac{D(v)}{n-1}$ , with equality if and only if  $d(v, u)$  (for fixed  $v \in V(G)$ ) is a constant for all  $u \in V(G)$  with  $v \neq u$ . Thus,  $K_2 \vee (2K_1 \cup K_{n-4})$  can not satisfy it, the equality can not hold.  $\square$

**Theorem 3.7.** *Let  $G$  be a 3-connected graph of order  $n \geq 18$ . If  $\xi^d(G) \leq \frac{1}{n(n-1)} [n^2 + 5n - 28]^2$ , then  $G$  is Hamiltonian.*

*Proof.* Suppose that  $G$  is not Hamiltonian and  $G$  is not  $K_3 \vee (3K_1 \cup K_{n-6})$ . Then from Lemma 2.7, we have that  $m \leq \binom{n-3}{2} + 8$ . Therefore,

$$\begin{aligned} \xi^d(G) &\geq \frac{1}{n(n-1)} \left[ \sum_{v \in V(G)} d(v) - 2n(n-1) \right]^2 \\ &\geq \frac{1}{n(n-1)} [2n(n-1) - 2m]^2 \\ &\geq \frac{1}{n(n-1)} [n^2 + 5n - 28]^2. \end{aligned}$$

On the other hand,  $\varepsilon(v) \geq \frac{D(v)}{n-1}$ , with equality if and only if  $d(v, u)$  (for fixed  $v \in V(G)$ ) is a constant for all  $u \in V(G)$  with  $v \neq u$ . Thus,  $K_3 \vee (3K_1 \cup K_{n-6})$  can not satisfy it, the equality can not hold.  $\square$

**Theorem 3.8.** *Let  $G$  be a  $k$ -connected graph of order  $n \geq 18$ . If*

$$\xi^d(G) \leq \frac{1}{n(n-1)} [n(n-1) + (k+1)(n-k-1)]^2,$$

*then  $G$  is Hamiltonian.*

*Proof.* Suppose that  $G$  is not Hamiltonian, then from Lemma 2.8, we have that  $m \leq \binom{n}{2} - (k+1)(n-k-1)/2$ . Therefore,

$$\begin{aligned} \xi^d(G) &\geq \frac{1}{n(n-1)} \left[ \sum_{v \in V(G)} d(v) - 2n(n-1) \right]^2 \\ &= \frac{1}{n(n-1)} [2n(n-1) - 2m]^2 \\ &\geq \frac{1}{n(n-1)} [n(n-1) + (k+1)(n-k-1)]^2, \end{aligned}$$

which is a contradiction. This completes the proof.  $\square$

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