ON THE UNIMODALITY OF INDEPENDENCE POLYNOMIAL OF CERTAIN CLASSES OF GRAPHS

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Abstract. The independence polynomial of a graph $G$ is the polynomial $\sum i_k x^k$, where $i_k$ denote the number of independent sets of cardinality $k$ in $G$. In this paper we study unimodality problem for the independence polynomial of certain classes of graphs.

1. Introduction

Throughout this paper $G = (V, E)$ is a simple (i.e., finite, undirected, loopless and without multiple edges) graph with vertex set $V = V(G)$ and edge set $E = E(G)$. If $X \subset V$, then $G[X]$ is the subgraph of $G$ spanned by $X$. By $G\setminus W$ we mean the subgraph $G[V \setminus W]$, if $W \subset V(G)$. We also denote by $G - F$ the partial subgraph of $G$ obtained by deleting the edges of $F$, for $F \subset E(G)$, and we write shortly $G - e$, whenever $F = \{e\}$. The neighborhood of a vertex $v \in V$ is the set $N(v) = \{w \in V | vw \in E\}$, and $N[v] = N(v) \cup \{v\}$. A vertex $v$ is pendant if its neighborhood contains only one vertex; an edge $e = uv$ is pendant if one of its endpoints is a pendant vertex. $K_n, P_n$ and $C_n$ denote the complete graph, the path, and the cycle on $n$ vertices, respectively. The disjoint union of the graphs $G_1$ and $G_2$ is the graph $G = G_1 \cup G_2$ having as a vertex set the disjoint union of $V(G_1)$ and $V(G_2)$, and as an edge set the disjoint union of $E(G_1)$ and $E(G_2)$. In particular, $nG$ denotes the disjoint union of $n > 1$ copies of the graph $G$. The corona of the graphs $G$ and $H$ is the graph $G \circ H$ obtained from $G$ and $|V(G)|$ copies of $H$, such that each vertex of $G$ is joined to all vertices of a copy of $H$.

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An independent set of a graph $G$ is a set of vertices where no two vertices are adjacent. The independence number is the size of a maximum independent set in the graph. For a graph $G$ with independence number $\alpha$, let $i_k$ denote the number of independent sets of cardinality $k$ in $G$ ($k = 0, 1, \ldots, \alpha$). The independence polynomial of $G$,

$$I(G; x) = \sum_{k=0}^{\alpha} i_k x^k,$$

is the generating polynomial for the independent sequence $(i_0, i_1, i_2, \ldots, i_\alpha)$. We say that the polynomial $P(x) = a_0 + a_1 x + \cdots + a_n x^n$ is unimodal if there exist $k \in \{0, \ldots, n\}$, called a mode of the sequence such that

$$a_0 \leq a_1 \leq \cdots \leq a_{k-1} \leq a_k \geq a_{k+1} \geq \cdots \geq a_n.$$

A polynomial, $P(x)$, as above is logarithmically concave (or simply log-concave) if for all $k = 1, \ldots, n-1$, we have

$$a_k^2 \geq a_{k-1} a_{k+1}.$$

It is trivial to show that if $P(x)$ is log-concave, then it is unimodal. Unimodality problems arise naturally in many branches of mathematics and have been extensively investigated. See survey article [6] and [18] for details.

Unimodality problems of graph polynomials have always been of great interest to researchers in graph theory. For example, it is conjectured that the chromatic polynomial of a graph is unimodal [17]. Also recently, it is conjectured that the domination polynomial of a graph is unimodal (see [1] or [2]). There has been an extensive literature in recent years on the unimodality problems of independence polynomials (see [10, 11, 12, 13, 14, 15, 16] for instance). Wang and Zhu in [19] established recurrence relations and gave factorizations of independence polynomials for certain classes of graphs, and then studied their unimodalities. Similar to their method we study the unimodality of certain graphs.

A cactus graph is a connected graph in which no edge lies in more than one cycle. So, each block of a cactus graph is either an edge or a cycle. If all blocks of a cactus $G$ are cycles of the same length $m$, the cactus is $m$-uniform. A hexagonal cactus is a 6-uniform cactus, i.e., a cactus in which every block is a hexagon. A vertex shared by two or more hexagons is called a cut-vertex. If each hexagon of a hexagonal cactus $G$ has at most two cut-vertices, and each cut-vertex is shared by exactly two hexagons, we say that $G$ is a chain hexagonal cactus. The number of hexagons in $G$ is called the length of the chain. We call $G$ a polyphenyl hexagonal chain if each hexagon of $G$ has at most two cut-vertices and each cut vertex is shared by exactly one hexagon and one cut-edge. Obviously, a polyphenyl hexagonal chain of length $n$ has $6n$ vertices and $7n-1$ edges. Furthermore, any polyphenyl hexagonal chain of length greater than one has exactly two hexagons with only one cut-vertex. Such hexagons are called terminal hexagons. Any remaining hexagons are called internal hexagons. The polyphenyl ortho-chains, polyphenyl meta-chains, and polyphenyl para-chains of length $n$ is denoted by $\mathcal{O}_n; \mathcal{M}_n$ and $\mathcal{P}_n$, respectively. For more information on chain hexagonal cacti and polyphenyl chains, see [4].
Examples of polyphenyl ortho-chains, polyphenyl meta-chains, and polyphenyl para-chains are shown in Figure 1.

In Section 2 we prove that the independence polynomial of $O_n$ is unimodal. In Section 3 we study the unimodality of the independence polynomials of certain graphs.

**2. Unimodality of independence polynomial of polyphenyl ortho-chains**

In this section we prove that the independence polynomial of polyphenyl ortho-chains $O_n$ is unimodal.

Hoede and Li [9] obtained the following recursive formula for the independence polynomial of a graph.

**Theorem 2.1.** For any vertex $v$ of a graph $G$, $I(G, x) = I(G - v, x) + xI(G - [v], x)$ where $[v]$ is the closed neighborhood of $v$, contains of $v$, together with all vertices incident with $v$.

Using Theorem 2.1 we can obtain the recurrence relations for the independence polynomials of $P_n$, $O_n$ and $M_n$ (3).

**Theorem 2.2.** (3) If $n \geq 2$, then

(i) $I(O_n, x) = (1 + 6x + 9x^2 + 2x^3)I(O_{n-1}, x) - (x^2 + 6x^3 + 11x^4 + 6x^5 + x^6)I(O_{n-2}, x)$

(ii) $I(P_n, x) = (1 + 5x + 6x^2 + 2x^3)I(P_{n-1}, x) + (2x^2 + 9x^3 + 9x^4 + 4x^5 - x^6)I(P_{n-2}, x)$

(iii) $I(M_n, x) = (1 + 6x + 8x^2 + x^3)I(M_{n-1}, x) + (-x - 4x^2 - x^3 - 9x^4 + 6x^5 + x^6)I(M_{n-2}, x)$.

We shall give factorizations of independence polynomial of $O_n$. To explain our approach and obtain our results, we recall some lemmas (see [19]).

**Lemma 2.3.** (7) Let $\{z_n\}_{n \geq 0}$ be a sequence satisfying the linear recurrence relation

$$z_n = az_{n-1} + bz_{n-2}, \quad n = 2, 3, \ldots$$

If $a^2 + 4b > 0$, then the closed form for the sequence is...
where

\[ \lambda_1 = \frac{a + \sqrt{a^2 + 4b}}{2}, \quad \lambda_2 = \frac{a - \sqrt{a^2 + 4b}}{2} \]

are the roots of quadratic equation \( \lambda^2 - a\lambda - b = 0 \).

**Lemma 2.4.** ([5]) Let \( \lambda_1, \lambda_2 \in \mathbb{R} \) and \( n \in \mathbb{N} \).

(i) If \( n \) is odd, then \( \lambda_1^n - \lambda_2^n = (\lambda_1 - \lambda_2) \prod_{s=1}^{\frac{n-1}{2}} \left[ (\lambda_1 + \lambda_n)^2 - 4\lambda_1\lambda_2 \cos^2 \frac{s\pi}{n} \right] \).

(ii) If \( n \) is even, then \( \lambda_1^n - \lambda_2^n = (\lambda_1 - \lambda_2)(\lambda_1 + \lambda_2) \prod_{s=1}^{\frac{n-2}{2}} \left[ (\lambda_1 + \lambda_n)^2 - 4\lambda_1\lambda_2 \cos^2 \frac{(2s-1)\pi}{n} \right] \).

(iii) If \( n \) is odd, then \( \lambda_1^n + \lambda_2^n = \prod_{s=1}^{\frac{n-1}{2}} \left[ (\lambda_1 + \lambda_2)^2 - 4\lambda_1\lambda_2 \cos^2 \frac{(2s-1)\pi}{n} \right] \).

(iv) If \( n \) is even, then \( \lambda_1^n + \lambda_2^n = \prod_{s=1}^{\frac{n}{2}} \left[ (\lambda_1 + \lambda_2)^2 - 4\lambda_1\lambda_2 \cos^2 \frac{s\pi}{2n} \right] \).

**Lemma 2.5.** ([18]) Let \( f(x) \) and \( g(x) \) be polynomials with positive coefficients.

(i) If both \( f(x) \) and \( g(x) \) are log-concave, then so is their product \( f(x)g(x) \).

(ii) If \( f(x) \) is log-concave and \( g(x) \) is unimodal, then their product \( f(x)g(x) \) is unimodal.

(iii) If both \( f(x) \) and \( g(x) \) are symmetric and unimodal, then so is their product \( f(x)g(x) \).

Now we are ready to obtain formula for the independence polynomial of \( \overline{O_n} \):

**Theorem 2.6.** For any even integer \( n \),

\[
I(\overline{O_n}; x) = \prod_{s=1}^{\lfloor \frac{n}{2} \rfloor} \left[ I^2(C_6; x) - 4(x^2 + 6x^3 + 11x^4 + 6x^5 + x^6) \cos^2 \frac{s\pi}{n+1} \right],
\]

and for any odd integer \( n \),

\[
I(\overline{O_n}; x) = I(C_6; x) \prod_{s=1}^{\lfloor \frac{n}{2} \rfloor} \left[ I^2(C_6; x) - 4(x^2 + 6x^3 + 11x^4 + 6x^5 + x^6) \cos^2 \frac{s\pi}{n+1} \right].
\]

**Proof.** By Theorem 2.2

\[
I(\overline{O_n}; x) = (1 + 6x + 9x^2 + 2x^3)I(\overline{O_{n-1}}; x) - (x^2 + 6x^3 + 11x^4 + 6x^5 + x^6)I(\overline{O_{n-2}}; x).
\]

Put \( h_n = I(\overline{O_n}; x) \), then \( h_0 = 1, h_1 = 1 + 6x + 9x^2 + 2x^3 = a, b = -(x^2 + 6x^3 + 11x^4 + 6x^5 + x^6). \) By Lemma 2.3,

\[
h_n = \frac{(a - \lambda_2)\lambda_1^n + (\lambda_1 - a)\lambda_2^n}{(\lambda_1 - \lambda_2)} = \frac{\lambda_1^{n+1} - \lambda_2^{n+1}}{(\lambda_1 - \lambda_2)}.
\]
Thus by Lemma 2.4 for even \( n \) we have

\[
I(O_n; x) = \prod_{s=1}^{\lfloor \frac{n}{2} \rfloor} \left[ I^2(C_6; x) - 4(x^2 + 6x^3 + 11x^4 + 6x^5 + x^6) \cos^2 \frac{s\pi}{n+1} \right],
\]

and for odd \( n \),

\[
I(O_n; x) = I(C_6; x) \prod_{s=1}^{\lfloor \frac{n}{2} \rfloor} \left[ I^2(C_6; x) - 4(x^2 + 6x^3 + 11x^4 + 6x^5 + x^6) \cos^2 \frac{s\pi}{n+1} \right].
\]

We next give a result about the unimodality of the independence polynomials of \( O_n \).

**Theorem 2.7.** \( I(O_n; x) \) is log-concave and therefore unimodal.

**Proof.** First suppose that \( n \) is even. We have

\[
I(O_n; x) = \prod_{s=1}^{\lfloor \frac{n}{2} \rfloor} \left[ I^2(C_6; x) - 4(x^2 + 6x^3 + 11x^4 + 6x^5 + x^6) \cos^2 \frac{s\pi}{n+1} \right].
\]

By substituting \( a = \cos^2 \frac{s\pi}{n+1} \), we have \( 0 \leq a < 1 \) and

\[
I(O_n; x) = \prod_{s=1}^{\lfloor \frac{n}{2} \rfloor} \left( 1 + 12x + (54 - 4a)x^2 + (112 - 24a)x^3 + (105 - 44a)x^4 + (36 - 6a)x^5 + (4 - 4a)x^6 \right).
\]

(2.1)

Since all coefficients of each factor of the above equality is positive, to show the log-concavity of \( I(O_n; x) \), by Lemma 2.5(i) it suffices to show that each factor on the right of (2.1) is log-concave. Now, simple calculations lead us to

\[
(12)^2 = 144 > 1 \cdot (54 - 4a),
\]

\[
1572 - 144a + 8a^2 > 0 \rightarrow (54 - 4a)^2 > 12(112 - 24a),
\]

\[
6874 - 2580a + 400a^2 > 0 \rightarrow (112 - 24a)^2 > (54 - 4a)(105 - 44a),
\]

\[
6993 - 7568a + 1792a^2 > 0 \rightarrow (105 - 44a)^2 > (112 - 24a)(36 - 6a),
\]

\[
876 + 164a - 140a^2 > 0 \rightarrow (36 - 6a)^2 > (105 - 44a)(4 - 4a).
\]

Thus \( I(O_n; x) \) is log-concave and therefore unimodal. The proof for case odd \( n \) is similar to case even \( n \). \( \square \)
3. Unimodality of independence polynomial of certain graphs

In this section we consider graphs of the form $P_n \circ (tK_m)$ and $C_n \circ (tK_m)$, where $n, m, t \in \mathbb{N}$. We study the unimodality of the independence polynomials of these kind of graphs. Firstly we obtain the independence polynomial of $P_n \circ (tK_m)$:

**Theorem 3.1.** Let $n, m, t \in \mathbb{N}$. Then

(i) \[ I(P_n \circ (tK_m); x) = (1 + mx)^t \left( \frac{n+1}{2} \right) \prod_{s=1}^{\left\lfloor \frac{n+1}{2} \right\rfloor} \left( (1 + mx)^t + 4x \cos^2 \frac{s\pi}{n+2} \right). \] (3.1)

(ii) $I(P_n \circ (tK_m); x)$ is log-concave and therefore unimodal.

**Proof.**

(i) Let $h_n = I(P_n \circ (tK_m); x)$. Then by Lemma 2.3
\[ h_n = (1 + mx)^t \cdot h_{n-1} + x(1 + mx)^t \cdot h_{n-2}. \]

Clearly, $h_0 = 1$ and $h_1 = (1 + mx)^t + x$. By Lemma 2.3 we have
\[ h_n = \frac{(1 + mx)^t + x - \lambda^2}{\lambda_1 - \lambda_2} \]
\[ \lambda^n_1 + \left( \lambda_2 - (1 + mx)^t - x \right) \lambda^n_2. \]

Therefore
\[ h_n = \frac{(a + b - \lambda_2) \lambda^n_1 + (\lambda_2 - a - \frac{b}{a}) \lambda^n_2}{\lambda_1 - \lambda_2}, \]

where $a = (1 + mx)^t$ and $b = x(1 + mx)^t$. Note that $a + \frac{b}{a} - \lambda_2 = \frac{\lambda^2}{a}$ and $\lambda_2 - a - \frac{b}{a} = -\frac{\lambda^2}{a}$.

Hence
\[ h_n = \frac{\lambda_1^{n+2} - \lambda_2^{n+2}}{a(\lambda_1 - \lambda_2)}. \]

Thus by Lemma 2.4 for odd $n$ we have
\[ h_n = \frac{(\lambda_1 - \lambda_2) \prod_{s=1}^{\left\lfloor \frac{n+1}{2} \right\rfloor} (\lambda_1 + \lambda_2)^2 - 4\lambda_1 \lambda_2 \cos^2 \frac{s\pi}{n+2}}{a(\lambda_1 - \lambda_2)} = a \prod_{s=1}^{\left\lfloor \frac{n+1}{2} \right\rfloor} \left( a + 4 \frac{b}{a} \cos^2 \frac{s\pi}{n+2} \right). \] (3.2)

and for even $n$,
\[ h_n = \prod_{s=1}^{\left\lceil \frac{n}{2} \right\rceil} (\lambda_1 + \lambda_2)^2 - 4\lambda_1 \lambda_2 \cos^2 \frac{s\pi}{n+2} = a \prod_{s=1}^{\left\lfloor \frac{n}{2} \right\rfloor} \left( a + 4 \frac{b}{a} \cos^2 \frac{s\pi}{n+2} \right). \] (3.3)

Combining (3.2) and (3.3) we obtain:
\[ I(P_n \circ (tK_m); x) = (1 + mx)^t \left( \frac{n+1}{2} \right) \prod_{s=1}^{\left\lfloor \frac{n+1}{2} \right\rfloor} \left( (1 + mx)^t + 4x \cos^2 \frac{s\pi}{n+2} \right). \]
(ii) To show log-concavity of \(I(P_n \circ (tK_m); x)\), it suffices to show that each factor on the right of (3.2) is log-concave. Let 0 \leq a < 1. We claim that the polynomial
\[
(1 + mx)^t + 4ax = 1 + (tm + 4a)x + \frac{t(t-1)}{2}m^2x^2 + \frac{t(t-1)(t-2)}{6}m^3x^3 + ... + mt^tx^t
\]
is log-concave. Actually, since \((1 + mx)^t\) is log-concave, it suffices to prove the inequality
\[
\left[\frac{t(t-1)}{2}m^2\right]^2 \geq (mt + 4a) \left[\frac{t(t-1)(t-2)}{6}m^3\right].
\]
Clearly, it suffices to prove the inequality for \(a = 1\). In this case, the inequality is equivalent to \(mt^2 + (m - 4)t + 16 \geq 0\), which is obviously true for \(m \geq 4\). For \(m = 1, 2\) and 3, we have \(t^2 - 3t + 16 \geq 0\), \(2t^2 - 2t + 16 \geq 0\) and \(3t^2 - t + 16 \geq 0\), respectively which are obviously true. Thus \(I(P_n \circ (tK_m); x)\) is log-concave and therefore is unimodal.

We now consider the independence polynomial of \(C_n \circ (tK_m)\).

**Theorem 3.2.** For \(n \geq 3\)

(i) \(I(C_n \circ (tK_m); x) = (1 + mx)^t \prod_{s=1}^{\left\lfloor \frac{n}{2} \right\rfloor} \left[ (1 + mx)^t + 4x \cos^2 \left(\frac{2s-1}{2n}\right) \pi \right]. \)

(ii) \(I(C_n \circ (tK_m); x)\) is log-concave and therefore unimodal.

**Proof.**

(i) Let \(h_n\) and \(f_n\) be the independence polynomials of \(I(P_n \circ (tK_m); x)\) and \(I(C_n \circ (tK_m); x)\) respectively. By Lemma 2.3 we have
\[
f_n = (1 + mx)^t \cdot h_{n-1} + x(1 + mx)^{2t} \cdot h_{n-3},
\]
Recall that
\[
h_n = \frac{\lambda_1^{n+2} - \lambda_2^{n+2}}{(1 + mx)^t(\lambda_1 - \lambda_2)}.
\]
Therefore using Lemma 2.4 we have
\[
f_n = \frac{\lambda_1^{n+1} - \lambda_2^{n+1}}{(\lambda_1 - \lambda_2)} + x(1 + mx)^t \frac{\lambda_1^{n-1} - \lambda_2^{n-1}}{(\lambda_1 - \lambda_2)} \]
\[
= \frac{\lambda_1^{n+1} - \lambda_2^{n+1}}{(\lambda_1 - \lambda_2)} - \lambda_1 \lambda_2 \frac{\lambda_1^{n-1} - \lambda_2^{n-1}}{(\lambda_1 - \lambda_2)} \]
\[
= \lambda_1^n + \lambda_2^n \]
\[
= (1 + mx)^t \prod_{s=1}^{\left\lfloor \frac{n}{2} \right\rfloor} \left[ (1 + mx)^t + 4x \cos^2 \left(\frac{2s-1}{2n}\right) \pi \right].
\]

(ii) Proof is similar to the proof of Theorem 3.1(ii).
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