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ON THE SPECTRUM OF r-ORTHOGONAL LATIN SQUARES OF DIFFERENT ORDERS

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Abstract. Two Latin squares of order n are orthogonal if in their superposition, each of the n^2 ordered pairs of symbols occurs exactly once. Colbourn, Zhang and Zhu, in a series of papers, determined the integers r for which there exist a pair of Latin squares of order n having exactly r different ordered pairs in their superposition. Dukes and Howell defined the same problem for Latin squares of different orders n and $n+k$. They obtained a non-trivial lower bound for r and solved the problem for $k \geq \frac{2n}{3}$. Here for $k < \frac{2n}{3}$, some constructions are shown to realize many values of r and for small cases ($3 \leq n \leq 6$), the problem has been solved.

1. Introduction

A Latin square of order n is an $n \times n$ array $L = (\ell_{ij})$ on n symbols in which every row and every column of L contains no repeated symbols. A partial Latin square of order n is an $n \times n$ array $L = (\ell_{ij})$ on n symbols with cells that are either empty or contain exactly one symbol and no symbol occurs more than once in any row or column. Two Latin squares L and L' of the same order are orthogonal if $\ell_{ab} = \ell_{cd}$ and $\ell'_{ab} = \ell'_{cd}$, implies $a = c$ and $b = d$. A set of Latin squares $L_1, L_2, \ldots, L_m$ are mutually orthogonal, or a set of MOLS, if for every $1 \leq i < j \leq m$, L_i and L_j are orthogonal. If L and L' are Latin squares of order n, define $L \circ L' = \{(\ell_{ij}, \ell'_{ij}) : 1 \leq i, j \leq n\}$. L and L' are said to be r-orthogonal if $|L \circ L'| = r$. Also define $\mathrm{Ospec}(n) = \{r |$ there exist L and L' such that $|L \circ L'| = r\}$, where L and L' are Latin squares of order n.

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Theorem 1.1. (Ryser's Theorem [7]) *Suppose P is a partial Latin square of order m in which a cell is filled if and only if it lies in the first r rows and s columns. If each symbol appears at least $r+s-m$ times in P, then P can be completed to a Latin square of order m.*

In the rest of the paper, we refer to the condition of Theorem 1.1 as Ryser property.
It was Belyavskaya [1, 2] and [3] (Chapter 6), who first systematically treated the following question: For which integers n and r does a pair of r-orthogonal Latin squares of order n exist?
This has been determined in a series of publications:

Theorem 1.2. [4, 8, 9] *Let n be a positive integer, then*

$$\mathrm{Ospec}(n) = \begin{cases} \{1\}, & n=1 \\ \{2\}, & n=2 \\ \{3,9\}, & n=3 \\ \{4,6,8,9,12,16\}, & n=4 \\ [5,25]-\{6,8,9,20,22,23,24\}, & n=5 \\ [6,36]-\{7,33,35,36\}, & n=6 \\ \left[n,n^2\right]-\{n+1,n^2-1\}, & \text{otherwise.} \end{cases}$$

Dukes and Howell [5], (also see [6]) extended this problem to Latin squares of different orders. If L is a Latin square of order n and L' is a Latin square of order $n+k$, then $L \circ L'$ has the same definition as for Latin squares of the same order. That is, $L \circ L' = \{(\ell_{ij}, {\ell'}_{ij}) | 1 \le i,j \le n\}$. So L is paired against the first n rows and columns of L'. Let $\mathrm{Ospec}(n, n+k) = \{r |$ there exist L and L' such that $|L \circ L'| = r\}$, where L and L' are Latin squares of order n and $n+k$, respectively.

Theorem 1.3. [5] *For positive integers n and k,* $\mathrm{Ospec}(n,n+k) \subseteq [n + k(\lceil \frac{n}{k} \rceil - 1), n^2]$.

We denote the set of expected values of $\mathrm{Ospec}(n,n+k)$ by $J(n,n+k) = [n + k(\lceil \frac{n}{k} \rceil - 1), n^2]$.

Theorem 1.4. [5] *Let n and k be two positive integers, then*

$$\mathrm{Ospec}(n,n+k) = \begin{cases} \left[n,n^2\right], & k \ge n \\ \left[n+k,n^2\right], & \frac{2n}{3} \le k < n. \end{cases}$$

The case $k < \frac{2n}{3}$ is left open. Here we address this question. Our results consist of some constructive methods for finding $\mathrm{Ospec}(n,n+k)$ when $k < \frac{2n}{3}$ which is appeared in Section 2. For small cases, $n = 3,4,5$ and 6, $\mathrm{Ospec}(n,n+k)$ for $k < \frac{2n}{3}$ have been provided in Section 3.

2. Some Constructive methods for $\mathrm{Ospec}(n,n+k)$

In this section, some constructive methods have been provided to realize many values belonging to $\mathrm{Ospec}(n,n+k)$. The following two lemmas are straightforward:

Lemma 2.1. *Suppose that L_1 and L_2 are two r-orthogonal Latin squares of orders n and $n+k$ respectively. Applying any element permutation on L_1 or L_2 does not affect $|L_1 \circ L_2|$.*

Lemma 2.2. *Suppose that L_1 and L_2 are two r-orthogonal Latin squares of orders n and $n+k$ respectively. Let σ be a permutation on the set $\{1, 2, \ldots, n\}$. Applying σ on the rows or columns of L_1 and L_2 simultaneously does not affect $|L_1 \circ L_2|$.*

Definition 2.3. *Let L be a Latin square of order n. The i-*th diagonal *of L is the set of cells which are filled with i.*

Definition 2.4. *Let L_1 and L_2 be two orthogonal Latin squares of order n. For a given i, consider the cells of the i-th diagonal of L_1 and substitute ∞_i in the corresponding cells of L_2. We call this action* complete substitution *of L_2 over the i-th diagonal of L_1.*

Definition 2.5. *Let L_1 and L_2 be two orthogonal Latin squares of order n, and let k be an integer $1 \leq k < n$. For a given i, consider r_i cells of the i-th diagonal of L_1, where $n-k \leq r_i \leq n$ and substitute ∞_i in the corresponding cells of L_2. We call this action r_i-*partial substitution *of L_2 over the i-th diagonal of L_1.*

Part (iii) of the following theorem was proved in [6]. However it seems that the proof does not work for some cases, such as $n = 10$. Here, using similar idea, we prove the same result.

Theorem 2.6. *Let n and k for $k < n$ be two positive integers, then*

(i) *If there exist three MOLS of order n and $k \geq 2$, then for each s, where $1 \leq s \leq k-1$, we have*
$$[n^2 - s(n-1), n^2 - s(n-1) + sk] \subseteq \text{Ospec}(n, n+k).$$

(ii) *Let $n \notin \{2, 6\}$, then*
$$[n^2 - k(n-1), n^2 - k(n-1) + k^2] \subseteq \text{Ospec}(n, n+k).$$

(iii) *If there exist three MOLS of order n, then $n^2 \in \text{Ospec}(n, n+k)$.*

Proof. (i) Let L_1, L_2 and L_3 be three mutually orthogonal Latin squares of order n. For $s \leq i \leq k-1$, apply complete substitution of L_3 over the i-th diagonal of L_1 and name the new partial Latin square L_3'. For $0 \leq i \leq s-1$, apply r_i-partial substitution of L_3' over the i-th diagonal of L_2. Since the resulting partial Latin square has Ryser property, it can be completed to a Latin square, L_3'', of order n+k. Each r_i-partial substitution subtracts $r_i - 1$ from $|L_2 \circ L_3'|$. After applying these substitutions, we have $|L_2 \circ L_3''| = n^2 - \sum_{i=0}^{s-1}(r_i - 1)$. Since $n-k \leq r_i \leq n$, we have $(n^2 - sn + s) \leq |L_2 \circ L_3''| \leq (n^2 - sn + sk + s)$.

(ii) Let L_1 and L_2 be two orthogonal Latin squares of order n. For $0 \leq i \leq k-1$, apply r_i-partial substitution of L_2 over the i-th diagonal of L_1. Since the resulting partial Latin square has Ryser property, it can be completed to a Latin square, L_2', of order $n+k$. The rest of the proof is similar to the proof of part (i), just here instead of s we have k. So $(n^2 - kn + k) \leq |L_1 \circ L_2'| \leq (n^2 - kn + k^2 + k)$ and this completes the proof.

(iii) Let L_1, L_2 and L_3 be three mutually orthogonal Latin squares of order n. For $0 \leq i \leq k-1$ apply complete substitution of L_2 over the i-th diagonal of L_1. Since the resulting partial Latin square has Ryser property, it can be completed to a Latin square, L_2', of order $n+k$. It is obvious that $|L_2' \circ L_3| = n^2$.

□

Theorem 2.7. *Let n and k be two positive integers where $n \notin \{2, 6\}$ and $k < n$, then*

$$[n + k(n-k), n + k(n-k) + k^2] \subseteq \text{Ospec}(n, n+k).$$

Proof. Let L_1 and L_2 be two orthogonal Latin squares of order n. For $0 \leq i \leq k-1$, apply r_i-partial substitution of L_2 over the i-th diagonal of L_1. The resulting partial Latin square has Ryser property, so it can be completed to a Latin square, L_2', of order $n+k$. Each r_i-partial substitution adds r_i to $|L_2 \circ L_2|$. After applying these substitutions, $|L_2 \circ L_2'| = n + \sum_{i=0}^{k-1}(r_i)$. Since $n - k \leq r_i \leq n$, we have $n + k(n-k) \leq |L_2 \circ L_2'| \leq n + kn$. □

In the next theorem we give an improvement of Theorem 3.4 in [5]:

Theorem 2.8. *Let n and k be positive integers with $n/2 < k < n$. If $r \in \text{Ospec}(n)$ and $r > (2n-k+1)(k-1)$, then $r \in \text{Ospec}(n, n+k)$.*

Proof. Since $r > (2n-k+1)(k-1)$ and by using the reasoning mentioned in Theorem 3.4 in [5], $\dfrac{r-(k-1)n}{n-k+1} > k-1$ is achieved. Since we need to have lower bound for the number of u_i's appearance (where $1 \leq i \leq k$) in different ordered pairs and since this number is an integer, we can conclude that this number is at least k. The rest of the proof is similar to the proof of Theorem 3.4 in [5]. □

Corollary 2.9. *Let $m = \min(n + k(n-k), n^2 - k(n-1))$. If $\frac{n}{2} < k < n$, $7 \leq n$ and there exist three MOLS of order n, then $[m, n^2] - \{n^2 - 1\} \subseteq \text{Ospec}(n, n+k)$.*

Proof. By Theorem 2.8 and Theorem 1.2, $[(2n-k+1)(k-1)+1, n^2] - \{n^2-1\} \subseteq \text{Ospec}(n, n+k)$. The intervals in Theorem 2.6 are not disjoint and have intersection for $s \geq 1$, so we can consider union of them as interval $[n^2 - k(n-1), n^2 - (n-1) + k]$. This interval is not disjoint from the set $[(2n-k+1)(k-1)+1, n^2] - \{n^2-1\}$ and union of them makes set $[n^2 - k(n-1), n^2] - \{n^2-1\}$. Now the last set is not disjoint from the interval of Theorem 2.7. Finally we find the set $[\min(n + k(n-k), n^2 - k(n-1)), n^2] - \{n^2-1\}$. □

Theorem 2.10. *Let $n \geq 7$ be a positive integer, then*

$$[n^2 - 3n + 2, n^2 - n + 2] \subseteq \text{Ospec}(n, n+1).$$

Proof. Suppose that r is a positive integer where $n(n-2)+1 \leq r \leq n^2$ and $r \neq n^2 - 1$. By Theorem 1.2, we know that for $r \in [n, n^2] - \{n+1, n^2-1\}$, there exist two r-orthogonal Latin squares of order n. So let L_1 and L_2 be two r-orthogonal Latin squares of order n. Now consider the ordered pairs which have been constructed from superposition of L_1 and L_2. Since $r \geq n(n-2)+1$, by pigeonhole principle, there exists at least one element of L_1 like 1 which appears at least $n-1$ times as the first coordinate of r distinct ordered pairs. Since the ordered pairs are distinct, the second coordinates of the ordered pairs in which their first coordinate is 1 are distinct and moreover make a partial transversal of size $n-1$ in L_2. Replace the elements of this partial transversal with ∞. This constructs a new partial Latin square of order $n+1$ which has Ryser property. So it can be completed to a Latin square, L_2', of order $n+1$. The number of distinct ordered pairs made by superposition of L_2' and L_1 is $r - (n-1) + 1$. It means that $\text{Ospec}(n, n+1)$ contains $[(n^2 - 2n + 1) - (n-1) + 1, n^2 - (n-1) + 1] - \{n^2 - 1 - (n-1) + 1\} =$

$[n^2-3n+3, n^2-n+2]-\{n^2-n+1\}$. To prove $(n^2-n+1) \in \mathrm{Ospec}(n, n+1)$, consider two orthogonal Latin squares L_1 and L_2 of order n. Apply complete substitution of L_2 over 1-st diagonal of L_1, thus a partial Latin square of order $n+1$ is established which has Ryser property. So it can be completed to a Latin square, L'_2, of order $n+1$. The number of distinct ordered pairs made by superposition of L'_2 and L_1 is n^2-n+1. □

Corollary 2.11. *For* $3 \leq n \leq 6$*, we have:*

- $\{7, 8\} \subseteq \mathrm{Ospec}(3, 4)$.
- $\{7, 10, 13, 14\} \subseteq \mathrm{Ospec}(4, 5)$.
- $[13, 18] \cup \{21, 22\} \subseteq \mathrm{Ospec}(5, 6)$.
- $[21, 30] \subseteq \mathrm{Ospec}(6, 7)$.

Proof. It is enough to apply the methods which are used in Theorem 2.10 for the sets which are in Theorem 1.2. It is obvious that for $r \geq n(n-1)+1$, by using the second method, we can conclude that $r-n+1$ is in the spectrum. □

3. Small Cases

In this Section for small cases, $n = 3, 4, 5$ and 6, $\mathrm{Ospec}(n, n+k)$ for $k < \frac{2n}{3}$ have been provided.

Case 1. $n = 3$. Since $k < \frac{2n}{3}$, $k = 1$. By Theorem 1.3, $J(3, 3+1) = J(3, 4) = [5, 9]$.

Lemma 3.1. $9 \notin \mathrm{Ospec}(3, 4)$

Proof. Suppose that M and L are two Latin squares of orders 3 and 4 respectively. By Lemma 2.1 without loss of generality we can assume that $\ell_{4,4} = 0$. If we consider the upper left side 3×3 subsquare of L as L', it should have exactly three entries 0 which together make a diagonal. Also by Lemma 2.2 we can assume that all of these entries 0 are on the main diagonal. Moreover according to Lemma 2.1 we can permute elements of L in such a way that the fourth column of L to become $1, 2, 3, 0$ respectively. Now if the superposition of these two Latin squares has 9 pairs, then obviously all of these pairs should be distinct and therefore each entry 0 of L' should be assigned to $1, 2$ and 0. According to Lemma 2.1, we can arbitrarily permute the elements of M. Therefore suppose that $1, 2$ and 0 appear on the main diagonal in this order. We know such a Latin square will be completed uniquely to:

1	0	2
0	2	1
2	1	0

And L will be as follows:

0	a	b	1
a'	0	c	2
b'	c'	0	3
$*$	$*$	$*$	0

The superposition of these two Latin squares will make these pairs:

$$(1,0), (0,a), (2,b), (0,a'), (2,0), (1,c), (2,b'), (1,c'), (0,0).$$

Now we will consider all of the possible values for a, b and c. Since all of the pairs should be distinct we have:

$$a \neq a', b \neq b', c \neq c'$$

We know that $a = 2$ or 3. Consider the possible two cases:

(1) $a = 2$. Since L is a Latin square, $b = 3$ and $c' = 1$. It is easy to see that L will be completed uniquely and we will have $c = c' = 1$, which is a contradiction.

(2) $a = 3$. We will have $b = 2$. Also we know that $b \neq b'$ therefore $b' = 1$. This completes the Latin square uniquely and we will get $a = a' = 3$ which is a contradiction.

□

Proposition 3.2. $\mathrm{Ospec}(3,4) = [5,8] = J(3,4) - \{9\}$.

Proof. By Theorem 2.7, $\{5,6\} \subseteq \mathrm{Ospec}(3,4)$, by Corollary 2.11, $\{7,8\} \subseteq \mathrm{Ospec}(3,4)$ and by Lemma 3.1, $9 \notin \mathrm{Ospec}(3,4)$. □

Case 2. $n = 4$. Since $k < \frac{2n}{3}$, $k = 1,2$. By Theorem 1.3, $J(4,4+1) = J(4,5) = [7,16]$ and $J(4,4+2) = J(4,6) = [6,16]$.

Lemma 3.3. $15 \notin \mathrm{Ospec}(4,5)$.

Proof. It has been proved by exhaustive search on the Latin squares of order 4 and 5, also we can obtain this result similar to Lemma 3.1. □

Proposition 3.4. $\mathrm{Ospec}(4,5) = [7,16] - \{15\} = J(4,5) - \{15\}$.

Proof. By Theorem 2.6, $\{13,14,16\} \subseteq \mathrm{Ospec}(4,5)$, by Theorem 2.7, $\{7,8\} \subseteq \mathrm{Ospec}(4,5)$, and by Corollary 2.11, $\{7,10,13,14\} \subseteq \mathrm{Ospec}(4,5)$. The remaining numbers are found by computer with an exhaustive search (see Appendix). Non-existence of value 15 is proved in Lemma 3.3. □

Proposition 3.5. $\mathrm{Ospec}(4,6) = [6,16] = J(4,6)$.

Proof. By Theorem 2.6, $[10,16] \subseteq \mathrm{Ospec}(4,6)$ and by Theorem 2.7, $[8,12] \subseteq \mathrm{Ospec}(4,6)$. The remaining numbers are found by computer with exhaustive search. See Appendix. □

Case 3. $n = 5$. Since $k < \frac{2n}{3}$, $k = 1,2,3$. By Theorem 1.3, $J(5,5+1) = J(5,6) = [9,25]$, $J(5,5+2) = J(5,7) = [9,25]$ and $J(5,5+3) = J(5,8) = [8,25]$.

Proposition 3.6. $\mathrm{Ospec}(5,6) = [9,25] = J(5,6)$.

Proof. By Theorem 2.6, $\{21,22,25\} \subseteq \mathrm{Ospec}(5,6)$, by Theorem 2.7, $\{9,10\} \subseteq \mathrm{Ospec}(5,6)$ and by Corollary 2.11, $[13,18] \cup \{21,22\} \subseteq \mathrm{Ospec}(5,6)$. The remaining numbers are found by computer with exhaustive search. See Appendix. □

Proposition 3.7. $\mathrm{Ospec}(5,7) = [9,25] = J(5,7)$.

Proof. By Theorem 2.6, $[17,23] \cup \{25\} \subseteq \mathrm{Ospec}(5,7)$ and by Theorem 2.7, $[11,15] \subseteq \mathrm{Ospec}(5,7)$. The remaining numbers are found by computer with exhaustive search. See Appendix. □

Proposition 3.8. $\text{Ospec}(5,8) = [8,25] = J(5,8)$.

Proof. By Theorem 2.6, $[13,25] \subseteq \text{Ospec}(5,8)$ and by Theorem 2.7, $[11,20] \subseteq \text{Ospec}(5,8)$. The remaining numbers are found by computer with exhaustive search. See Appendix. □

Case 4. $n = 6$. Since $k < \frac{2n}{3}$, $k = 1,2,3$. By Theorem 1.3, $J(6,6+1) = J(6,7) = [11,36]$, $J(6,6+2) = J(6,8) = [10,36]$ and $J(6,6+3) = J(6,9) = [9,36]$.

Theorem 3.9. *For $1 \le k \le 3$, we have*
(i) $[6+6k-k^2, 6+6k] \subseteq \text{Ospec}(6,6+k)$.
(ii) $[34-5k, 34-5k+k^2] \subseteq \text{Ospec}(6,6+k)$.

Proof. Let L_1 and L_2 be the following Latin squares.

$L_1 =$

1	2	3	4	5	0
2	1	0	5	4	3
3	4	1	2	0	5
4	0	5	1	3	2
5	3	2	0	1	4
0	5	4	3	2	1

$L_2 =$

1	4	2	3	5	0
3	0	5	2	1	4
5	2	4	0	3	1
4	1	3	5	0	2
0	3	1	4	2	5
2	5	0	1	4	3

For $0 \le i \le k-1$, we apply r_i-partial substitution of L_1 over the i-th diagonal of L_2. Since the resulting partial Latin square has Ryser property, it can be completed to a Latin square, L_1', of order $6+k$.
(i) Each r_i-partial substitution adds r_i to $|L_1 \circ L_1|$. After applying these substitutions, we have $|L_1 \circ L_1'| = 6 + \sum_{i=0}^{k-1}(r_i)$. Since $6-k \le r_i \le 6$, we have $6 + k(6-k) \le |L_1 \circ L_1'| \le 6+6k$.
(ii) Each r_i-partial substitution subtracts $r_i - 1$ from $|L_2 \circ L_1|$. After applying these substitutions, we have $|L_2 \circ L_1'| = 34 - \sum_{i=0}^{k-1}(r_i - 1)$. Since $6-k \le r_i \le 6$, we have $34 - k(6-1) \le |L_2 \circ L_1'| \le 34 - (6-k-1)$. □

Proposition 3.10. $\text{Ospec}(6,7) = [11,36] = J(6,7)$.

Proof. By Corollary 2.11, $[21,30] \subseteq \text{Ospec}(6,7)$ and by Theorem 3.9, $\{11,12,29,30\} \subseteq \text{Ospec}(6,7)$. The remaining numbers are found by computer with exhaustive search. See Appendix. □

Proposition 3.11. $\text{Ospec}(6,8) = [10,36] = J(6,8)$.

Proof. By Theorem 3.9, $[14,18] \cup [24,28] \subseteq \text{Ospec}(6,8)$. The remaining numbers are found by computer with exhaustive search. See Appendix. □

Proposition 3.12. $\text{Ospec}(6,9) = [9,36] = J(6,9)$.

Proof. By Theorem 3.9, $[15,28] \subseteq \text{Ospec}(6,9)$. The remaining numbers are found by computer with exhaustive search. See Appendix. □

The following table shows the results:

Results	*By Thm.*2.6	*By Thm.*2.7	*By Thm.*3.9	*By Cor.*2.11	*By computer*
$\mathrm{Ospec}(3,4) = [5,8]$	–	$\{5,6\}$	–	$\{7,8\}$	–
$\mathrm{Ospec}(4,5) = [7,16] - \{15\}$	$\{13,14,16\}$	$\{7,8\}$	–	$\{7,10,13,14\}$	$\{9,11,12\}$
$\mathrm{Ospec}(4,6) = [6,16]$	$[10,16]$	$[8,12]$	–	–	$\{6,7\}$
$\mathrm{Ospec}(5,6) = [9,25]$	$\{21,22,25\}$	$\{9,10\}$	–	$[13,18] \cup \{21,22\}$	$\{11,12,19,20,23,24\}$
$\mathrm{Ospec}(5,7) = [9,25]$	$[17,23] \cup \{25\}$	$[11,15]$	–	–	$\{9,10,16,24\}$
$\mathrm{Ospec}(5,8) = [8,25]$	$[13,25]$	$[11,20]$	–	–	$\{8,9,10\}$
$\mathrm{Ospec}(6,7) = [11,36]$	–	–	$\{11,12,29,30\}$	$[21,30]$	$[13,20] \cup [31,36]$
$\mathrm{Ospec}(6,8) = [10,36]$	–	–	$[14,18] \cup [24,28]$	–	$[10,13] \cup [19,23] \cup [29,36]$
$\mathrm{Ospec}(6,9) = [9,36]$	–	–	$[15,28]$	–	$[9,14] \cup [29,36]$

Remark 3.13. *For some values of n and k, we have shown that* $\mathrm{Ospec}(n, n+k) \subsetneq J(n, n+k)$*, such as $n = 3$ and $k = 1$. So at least for small values of n,* $\mathrm{Ospec}(n, n+k) \neq J(n, n+k)$.

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Appendix

The following Latin squares are written in linear form to save space. The i-th set of n digits corresponds to the row i. To show the existence for the remaining numbers, two Latin squares are given in each case.

$n = 4:$ Let the small Latin square be 0123 1032 2301 3210
and the other Latin square with larger order be
01243 10324 23401 34012 42130 ($9 \in \mathrm{Ospec}(4,5)$)
01324 10432 23140 34201 42013 ($11 \in \mathrm{Ospec}(4,5)$)
01423 12034 23140 34201 40312 ($12 \in \mathrm{Ospec}(4,5)$)
012345 103254 234501 325410 450123 541032 ($6 \in \mathrm{Ospec}(4,6)$)
012345 103452 234501 325014 450123 541230 ($7 \in \mathrm{Ospec}(4,6)$)

$n = 5:$ Let the small Latin square be 01234 10342 23410 34021 42103
and the larger one be
541032 450123 103254 012345 234501 325410 ($11 \in \mathrm{Ospec}(5,6)$)
254103 345012 410325 501234 123450 032541 ($12 \in \mathrm{Ospec}(5,6)$)
031524 120435 253140 415302 342051 504213 ($19 \in \mathrm{Ospec}(5,6)$)
021345 130254 243501 352410 405123 514032 ($20 \in \mathrm{Ospec}(5,6)$)
054213 125304 210435 341520 503142 432051 ($23 \in \mathrm{Ospec}(5,6)$)
0123456 1034265 5641023 6405132 2560314 3216540 4352601 ($9 \in \mathrm{Ospec}(5,7)$)
0123456 1034265 5641023 6502134 3250641 2465310 4316502 ($10 \in \mathrm{Ospec}(5,7)$)
5461302 6354120 4506213 1032645 3210564 0123456 2645031 ($16 \in \mathrm{Ospec}(5,7)$)
0123456 2345601 3260145 6051324 1432560 4516032 5604213 ($24 \in \mathrm{Ospec}(5,7)$)
01234567 10342675 23410756 56721034 62175403 37506142 45067321 74653210 ($8 \in \mathrm{Ospec}(5,8)$)
01234567 10342675 23410756 56721034 62075413 47156320 34567102 75603241 ($9 \in \mathrm{Ospec}(5,8)$)
01234567 10342675 23410756 56701234 64175320 35067142 47526013 72653401 ($10 \in \mathrm{Ospec}(5,8)$)
Now let the small Latin square be 01234 10342 23401 34120 42013
and the larger one be
132045 041352 350214 425103 513420 204531 ($24 \in \mathrm{Ospec}(5,6)$)

$n = 6:$ Let the small Latin square be 012345 123450 234501 345012 450123 501234
and the larger one be
5612430 6023541 1245063 0134652 2356104 3460215 4501326 ($15 \in \mathrm{Ospec}(6,7)$)
6103452 0214563 1325604 2436015 4651230 3540126 5062341 ($16 \in \mathrm{Ospec}(6,7)$)
0123546 1234650 2345061 3456102 4560213 5601324 6012435 ($17 \in \mathrm{Ospec}(6,7)$)
0132456 1243560 2354601 3465012 4506123 5610234 6021345 ($19 \in \mathrm{Ospec}(6,7)$)
0163524 1204635 2315046 3426150 4530261 5641302 6052413 ($20 \in \mathrm{Ospec}(6,7)$)
0642153 1053264 2164305 3205416 4316520 5420631 6531042 ($31 \in \mathrm{Ospec}(6,7)$)
01234567 12345076 67450123 74501632 45672301 56723410 23016745 30167254 ($10 \in \mathrm{Ospec}(6,8)$)
01234567 12345670 23456701 34567012 45670123 56701234 67012345 70123456 ($11 \in \mathrm{Ospec}(6,8)$)
01245637 12356740 23467051 34570162 56712304 67023415 45601273 70134526 ($12 \in \mathrm{Ospec}(6,8)$)
01236547 12347650 23450761 34561072 45672103 56703214 67014325 70125436 ($13 \in \mathrm{Ospec}(6,8)$)
01246537 12357640 23460751 34571062 45602173 56713204 67024315 70135426 ($19 \in \mathrm{Ospec}(6,8)$)
01274356 12305467 23416570 34527601 45630712 56741023 67052134 70163245 ($20 \in \mathrm{Ospec}(6,8)$)
01237465 12340576 23451607 34562710 45673021 56704132 67015243 70126354 ($21 \in \mathrm{Ospec}(6,8)$)
01246375 12357406 23460517 34571620 45602731 56713042 67024153 70135264 ($22 \in \mathrm{Ospec}(6,8)$)
01243567 12354670 23465701 34576012 45607123 56710234 67021345 70132456 ($23 \in \mathrm{Ospec}(6,8)$)
01427365 12530476 23641507 34752610 45063721 56174032 67205143 70316254 ($29 \in \mathrm{Ospec}(6,8)$)
01572634 12603745 23714056 34025167 45136270 56247301 67350412 70461523 ($30 \in \mathrm{Ospec}(6,8)$)
01643275 12754306 23065417 34176520 45207631 56310742 67421053 70532164 ($31 \in \mathrm{Ospec}(6,8)$)
02164375 13275406 24306517 35417620 46520731 57631042 60742153 71053264 ($32 \in \mathrm{Ospec}(6,8)$)
05361472 16472503 27503614 30614725 41725036 52036147 63147250 74250361 ($33 \in \mathrm{Ospec}(6,8)$)

02531476 13642507 24753610 46175032 57206143 60317254 35064721 71420365 ($34 \in \mathrm{Ospec}(6,8)$)
01234567 23456701 45671320 67025143 12347056 36702415 50163274 74510632 ($35 \in \mathrm{Ospec}(6,8)$)
07643215 10754326 21065437 32176540 43207651 54310762 65421073 76532104 ($36 \in \mathrm{Ospec}(6,8)$)
013467528 124578630 346701852 457812063 670134285
781245306 235680741 568023174 802356417 ($9 \in \mathrm{Ospec}(6,9)$)
012345678 123450786 234501867 678012345 780126453
841267530 365874021 456738102 507683214 ($10 \in \mathrm{Ospec}(6,9)$)
012345678 123456780 234567801 345678012 456780123
567801234 678012345 780123456 801234567 ($11 \in \mathrm{Ospec}(6,9)$)
017345628 123450786 234501867 678023145 780136452
801267534 345678210 456782301 562814073 ($12 \in \mathrm{Ospec}(6,9)$)
012375846 123486057 234507168 345618270 456720381
567831402 678042513 780153624 801264735 ($13 \in \mathrm{Ospec}(6,9)$)
012675438 123786540 234807651 345018762 456120873
567231084 678342105 780453216 801564327 ($14 \in \mathrm{Ospec}(6,9)$)
064185732 175206843 286317054 307428165 418530276
520641387 631752408 742863510 853074621 ($35 \in \mathrm{Ospec}(6,9)$)
071853642 182064753 203175864 314286075 425307186
536418207 647520318 758631420 860742531 ($36 \in \mathrm{Ospec}(6,9)$)
Now let the small Latin square be 012345 103254 234501 325410 450123 541032
and the larger one be
018435672 120546783 231657804 342768015 453870126
564081237 675102348 786213450 807324561 ($29 \in \mathrm{Ospec}(6,9)$)
016835472 127046583 238157604 340268715 451370826
562481037 673502148 784613250 805724361 ($30 \in \mathrm{Ospec}(6,9)$)
016842357 127053468 238164570 340275681 451386702
562407813 673518024 784620135 805731246 ($31 \in \mathrm{Ospec}(6,9)$)
018437265 120548376 231650487 342761508 453872610
564083721 675104832 786215043 807326154 ($32 \in \mathrm{Ospec}(6,9)$)
074681532 185702643 206813754 317024865 428135076
530246187 641357208 752468310 863570421 ($33 \in \mathrm{Ospec}(6,9)$)
064182753 175203864 286314075 307425186 418536207
520647318 631758420 742860531 853071642 ($34 \in \mathrm{Ospec}(6,9)$)
Now let the small Latin square be 012345 103254 235401 324510 451023 540132
and the larger one be
3204165 2041653 5420316 4156230 0613542 6532401 1365024 ($13 \in \mathrm{Ospec}(6,7)$)
3214065 1042653 5360214 4156302 0623541 6531420 2405136 ($14 \in \mathrm{Ospec}(6,7)$)
3204156 2041365 5320614 4156230 0613542 6532401 1465023 ($18 \in \mathrm{Ospec}(6,7)$)
3402165 2041356 5320641 6135024 1654203 4216530 0563412 ($32 \in \mathrm{Ospec}(6,7)$)
3402165 2041356 5320641 1635024 6154230 4216503 0563412 ($33 \in \mathrm{Ospec}(6,7)$)
2156043 5640312 4263105 0321456 6504231 1432560 3015624 ($34 \in \mathrm{Ospec}(6,7)$)
0425136 2143650 6312504 1560423 3254061 4036215 5601342 ($35 \in \mathrm{Ospec}(6,7)$)
Now let the small Latin square be 123450 234015 301524 452301 510243 045132
and the larger one be
1234506 5016423 4521360 0462135 3640251 2305614 6153042 ($36 \in \mathrm{Ospec}(6,7)$)

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