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EXTREMAL TETRACYCLIC GRAPHS WITH RESPECT TO THE FIRST AND SECOND ZAGREB INDICES

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ABSTRACT. The first Zagreb index, $M_1(G)$, and second Zagreb index, $M_2(G)$, of the graph G is defined as $M_1(G) = \sum_{v\in V(G)} d^2(v)$ and $M_2(G) = \sum_{e=uv\in E(G)} d(u)d(v)$, where $d(u)$ denotes the degree of vertex u. In this paper, the first and second maximum values of the first and second Zagreb indices in the class of all $n-$vertex tetracyclic graphs are presented.

1. Introduction

The aim of this section is to present some introductory materials that will be used throughout this paper. Let G be a connected simple graph with vertex set $V(G)$ and edge set $E(G)$. The cardinalities of $V(G)$ and $E(G)$ are called the order and size of G, respectively.

Let G be a simple graph of order n and size m containing k components. If $m - n + k = 1, 2, 3, 4$ then the graph G is called unicyclic, bicyclic, tricyclic and tetracyclic, respectively. The **first Zagreb index**, $M_1(G)$, and **second Zagreb index**, $M_2(G)$, of the graph G is defined as $M_1(G) = \sum_{v\in V(G)} d^2(v)$ and $M_2(G) = \sum_{e=uv\in E(G)} d(u)d(v)$, where $d(u)$ denotes the degree of vertex u [11]. It is well-known that the inequality $\frac{M_1}{n} \leq \frac{M_2}{m}$ is not true in general. Andova and Petrusevski [1] proved a generalization of this inequality in a special case and Andova et al. [2], presented some classes of graphs that satisfy $\frac{M_1}{n} \leq \frac{M_2}{m}$. In other words, they proved that every graph G whose degrees of vertices are in the interval $[r, r + 2]$, r is integer, satisfies this inequality. In particular, for a given positive integer Δ, $\Delta \geq 5$, they constructed an infinite family of connected graphs of maximum degree Δ such that the inequality is false.

In some research papers the extremal properties of these graph invariants on the set of all bicyclic graphs with a given matching number [16] and bicyclic graphs with a fixed number of pendant vertices [23] are investigated.

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In [17, 18, 22], some extremal graphs of Zagreb indices was obtained in the class of all cacti with a given pendant vertices, qausi-tree and polyominochain graphs, respectively.

Li and Zhou [14] calculated the maximum and minimum Zagreb indices of graphs with connectivity at most k, and Li and Zhang [15], found sharp upper bounds for the Zagreb indices of bipartite graphs with a given diameter. Some bounds for these graph invariants in the class of all graphs can be obtained in [5, 6, 12, 21]. These graph invariants are also investigated for some graph operations in [3, 13, 19, 20]. Moreover, in [8], the authors found the first and second maximum values of the atom−bond connectivity index in the class of all $n-$vertex tetracyclic graphs and in [4] the first, second and third maximum values of the atom−bond connectivity index in the class of all $n-$vertex tricyclic graphs are presented. They also computed [9] the maximum and second maximum of Randić index in the set of all $n-$vertex tetracyclic graphs.

We encourage the interested reader to study [10] and references therein for a complete review on this topic. Our notations are standard and can be taken from the most of books on graph theory. Throughout this paper $TG(n)$ and $TG(n,p)$, $0 \leq p \leq n-5$, denote the set of all tetracyclic n-vertex graph and tetracyclic n-vertex graph containing p pendants, respectively. The tricyclic graphs with greatest Zagreb indices have been determined in Ref. [7]. In the present paper, we continue this research by extending it to the tetracyclic cases. We will compute the first three maximum of M_1 and the first and second maximum of M_2 on the class of $n-$vertex tetracyclic graphs with $n \geq 6$ vertices.

2. Preliminaries

The aim of this section is to prove two elementary results that are crucial throughout this paper.

Lemma 2.1. *Suppose G is a tetracyclic graph containing $p \geq 0$ pendant vertices and at least one non−pendant vertex which is not belonging to a triangle in G. Then there exists a tetracyclic graph G' such that $M_1(G) < M_1(G')$ and $M_2(G) < M_2(G')$.*

Proof. Choose a non-pendant edge $e = uv$ of G that is not belonging to a cycle of length 3 and A is a graph constructed from G by contraction and then deleting the edge $e = uv$. We now construct the graph G' from A by adding a new vertex to A and connecting it to the contracted vertices u and v. Then the graph G' is a simple $n-$vertex tetracyclic graph containing $p+1$ pendant vertices. To prove $M_1(G) < M_1(G')$ and $M_2(G) < M_2(G')$, we assume that $d(u) = s \geq 2$, $d(v) = r \geq 2$, $N_G(u) - \{v\} = \{x_1, \ldots, x_{s-1}\}$ and $N_G(v) - \{u\} = \{y_1, \ldots, y_{r-1}\}$. Therefore,

$$\begin{aligned} M_1(G) - M_1(G') &= r^2 + s^2 - (r+s-1)^2 - 1 \\ &= -2rs - 2 + 2r + 2s < 0, \end{aligned}$$

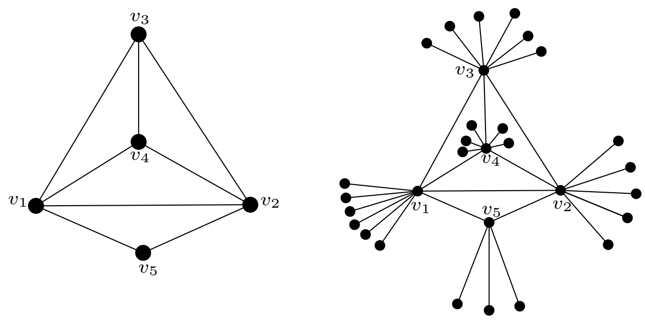
and so $M_1(G) < M_1(G')$. On the other hand,

$$\begin{aligned} M_2(G) - M_2(G') &= \sum_{i=1}^{s-1} sd(x_i) + \sum_{i=1}^{r-1} rd(y_i) + rs \\ &\quad - \Big(\sum_{i=1}^{s-1}(r+s-1)d(x_i) + \sum_{i=1}^{r-1}(r+s-1)d(y_i)\Big) - (r+s-1) \\ &= -\sum_{i=1}^{s-1}(r-1)d(x_i) - \sum_{i=1}^{r-1}(s-1)d(y_i) - (r+s-1) + rs < 0. \end{aligned}$$

This completes our proof. □

The previous lemma shows that for calculation of maximum Zagreb indices among all tetracyclic graphs, we have to consider tetracyclic graphs with a few number of non-pendant vertices.

We now consider the complete graph K_4 and construct a graph F_5 by adding a vertex v_5 and connecting it to two vertices of K_4. Label F_5 in such a way that $d(v_1) = d(v_2) = 4$, $d(v_3) = d(v_4) = 3$ and $d(v_5) = 2$, see Figure 1(a).



(a) (b)

FIGURE 1. a) The Graph F_n.; b) The Tetracyclic Graph $F_n(n_1, n_2, n_3, n_4, n_5)$.

Define $F_n(n_1, n_2, n_3, n_4, n_5)$ to be a graph, Figure 1(b), obtained from F_5 by adding $n_i - 1$ pendant vertices to v_i such that $n_i \geq 1$, $n_1 \geq n_2 \geq n_3 \geq n_4$ and $n_5 = \min\{n_1, n_2, n_3, n_4, n_5\}$, where $1 \leq i \leq 5$. Notice that $\sum_{i=1}^{5} n_i = n$.

Lemma 2.2. *Suppose $G = F_n(n_1, n_2, n_3, n_4, n_5)$ and there are i and j such that $1 \leq i \neq j \leq 5$, $n_i \geq n_j \geq 2$, and b is a pendant vertex adjacent to v_j. Then $M_1(G - v_jb + v_ib) > M_1(G)$.*

Proof. Without loss of generality, we can assume that $i = 1$ and $j = 2$. Define $G_0 = F_n(n_1+1, n_2-1, n_3, n_4, n_5)$. Then

$$\begin{aligned} M_1(G_0) - M_1(G) &= (n_1 - n_1 + 1) + (n_1 + 4)^2 - (n_1 + 3)^2 + (n_2 - n_2 + 1 - 2) \\ &+ (n_2 + 2)^2 - (n_2 + 3)^2 = 2 + 2(n_1 - n_2) > 0\,. \end{aligned}$$

Hence the result. □

3. Tetracyclic Graphs with Five Non−Pendant Vertices

Suppose $TG(n, n-5)$ denotes the set of all tetracyclic graphs with exactly $n - 5$ pendant vertices. In this section, the $n-$vertex tetracyclic graphs with five non−pendant vertices are considered into account and the maximum Zagreb indices of such graphs are computed.

Lemma 3.1. *Suppose $G = F_n(n_1, n_2, n_3, n_4, n_5)$, where $n \geq 5$. Then*

a) *If $n_2 \geq 2$, then $M_2(G) < M_2(F_n(n_1 + 1, n_2 - 1, n_3, n_4, n_5)$,*
b) *If $n_4 \geq 2$, then $M_2(G) < M_2(F_n(n_1, n_2, n_3 + 1, n_4 - 1, n_5)$,*
c) *If $n_5 \geq 2$, then $M_2(G) < M_2(F_n(n_1, n_2, n_3 + 1, n_4, n_5 - 1)$,*

d) *If* $n_5 \geq 2$*, then* $M_2(G) < M_2(F_n(n_1+1, n_2, n_3, n_4, n_5-1)$*,*

e) *If* $n_3 \geq 2$*, then* $M_2(G) < M_2(F_n(n_1+1, n_2, n_3-1, n_4, n_5)$*.*

Proof. **a)** Assume that $n_2 \geq 2$ and $G_0 = F_n(n_1+1, n_2-1, n_3, n_4, n_5)$. Then

$$\begin{aligned} M_2(G_0) - M_2(G) &= [n_1(n_1+4) - (n_1+3)(n_1-1)] \\ &+ [(n_2+2)(n_2-2) - (n_2+3)(n_2-1)] \\ &+ [(n_2+2)(n_1+4) - (n_2+3)(n_1+3)] \\ &+ [(n_3+2)(n_2+2) - (n_2+3)(n_3+2)] \\ &+ [(n_3+2)(n_1+2) - (n_1+3)(n_3+2)] \\ &+ [(n_4+2)(n_2+2) - (n_2+3)(n_4+2)] \\ &+ [(n_4+2)(n_1+4) - (n_1+3)(n_4+2)] \\ &+ [(n_5+1)(n_2+2) - (n_2+3)(n_5+1)] \\ &+ [(n_5+1)(n_1+4) - (n_1+3)(n_5+1)] \\ &= n_1 - n_2 + 1 > 0. \end{aligned}$$

b) Suppose that $n_4 \geq 2$ and $G_1 = F_n(n_1, n_2, n_3+1, n_4-1, n_5)$. Then,

$$\begin{aligned} M_2(G_1) - M_2(G) &= [n_3(n_3+3) - (n_3-1)(n_3+2)] \\ &+ [(n_4-2)(n_4+1) - (n_4+2)(n_4-1)] \\ &+ [(n_3+3)(n_4+1) - (n_3+2)(n_4+2)] \\ &+ [(n_3+3)(n_1+3) - (n_3+2)(n_1+3)] \\ &+ [(n_3+3)(n_2+3) - (n_3+2)(n_2+3)] \\ &+ [(n_4+1)(n_1+3) - (n_4+2)(n_1+3)] \\ &+ [(n_4+1)(n_2+3) - (n_4+2)(n_2+3)] \\ &= n_3 - n_4 + 1 > 0. \end{aligned}$$

c) The proof is similar to part b.

d) Let $n_5 \geq 2$ and $G_3 = F_n(n_1+1, n_2, n_3, n_4, n_5-1)$. Then we have:

$$\begin{aligned} M_2(G_3) - M_2(G) &= [n_1(n_1+4) - (n_1-1)(n_1+3)] \\ &+ [n_5(n_5-2) - (n_5-1)(n_5+1)] \\ &+ [n_5(n_1+4) - (n_1+3)(n_5+1)] \\ &+ [n_5(n_2+3) - (n_5+1)(n_2+3] \\ &+ [(n_1+4) - (n_1+3)][n_2+3+n_3+2+n_4+2] \\ &= n_1 + n_3 + n_4 - n_5 + 5 > 0\,. \end{aligned}$$

Hence the result. □

Corollary 3.2. *Let $G = F_n(n_1, n_2, 1, 1, 1)$ with $n_1, n_2 \geq 2$. Then*

a) $M_1(G) \leq M_1(F_n(n-4, 1, 1, 1, 1))$,

b) $M_2(G) \leq M_2(F_n(n-4, 1, 1, 1, 1))$.

Proof. Suppose $G_1 = F_n(n-4, 1, 1, 1, 1)$. We consider two cases as follows:

a) In this case,

$$\begin{aligned} M_1(G_1) - M_1(G) &= [(n_1+n_2-2) + (n_1+n_2+2)^2 + 16] \\ &- [(n_1-1) + (n_1+3)^2 - (n_2-1) - (n_2+3)^2] \\ &= 2n_1n_2 + 2 - 2n_1 - 2n_2 > 0\,. \end{aligned}$$

b) By Theorem 3.1 and putting $n_3 = n_4 = n_5 = 1$, one can see that removing each pendant edge from the vertex v_2 and adding it to vertex v_1, the second Zagreb index will be increased. Therefore,

$$\begin{aligned} M_2(F_n(n_1, n_2, 1, 1, 1)) &< M_2(F_n(n_1+1, n_2-1, 1, 1, 1)) \\ &< \cdots \\ &< M_2(F_n(n_1+n_2-2, 2, 1, 1, 1)) \\ &< M_2(F_n(n_1+n_2-1, 1, 1, 1, 1)), \end{aligned}$$

which completes the proof. □

Theorem 3.3. *Suppose $G = F_n(n_1, n_2, n_3, n_4, n_5)$, where $n \geq 5$. Then, $M_1(G) \leq n^2 - n + 34$ and $M_2(G) \leq n^2 + 6n + 34$. Moreover, equalities hold if and only if $G = F_n(n-4, 1, 1, 1, 1)$.*

Proof. Assume that $H = F_n(n_1, n_2, n_3, n_4, n_5)$ has maximum value of M_1. If $n_2 \geq 2$, $n_3 \geq 2$, $n_4 \geq 2$ or $n_5 \geq 2$ then one can construct a graph K such that $M_1(H) < M_1(K)$, contradict by maximality of H. Hence $H = F_n(n-4, 1, 1, 1, 1)$ and $M_1(G) \leq n^2 - n + 34$. The equality holds if and only if $G = F_n(n-4, 1, 1, 1, 1)$. The proof of M_2 is similar to the part (a) and so it is omitted. □

Let us consider the wheel graph W_5 with center v_1 of degree four. Construct the graph $W_n(n_1, n_2, n_3, n_4, n_5)$ from W_5 by adding $n_i - 1$ pendant vertices to each v_i in such a way that $\sum_{i=1}^{5} n_i = n$, $n_1 = \max\{n_1, n_2, n_3, n_4, n_5\}$ and $n_i \geq 1$, where $i \leq 5$, Figure 2.

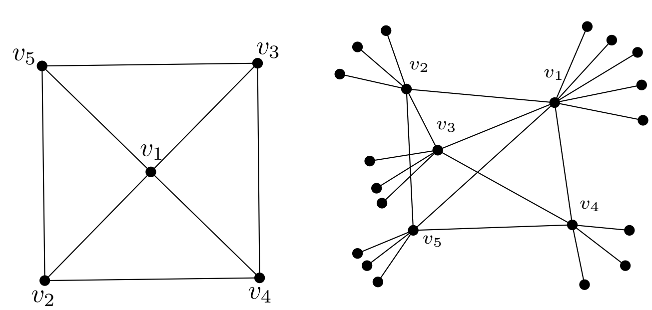


FIGURE 2. The Graph $W_n(n_1, n_2, n_3, n_4, n_5)$.

Lemma 3.4. *Suppose $G = W_n(n_1, n_2, n_3, n_4, n_5)$ such that for some i and j, $1 \leq i \neq j \leq 5$, $n_i \geq n_j \geq 2$. We also assume that b is a pendant vertex adjacent to v_j. Then*

$$M_1(G - v_jb + v_ib) > M_1(G)\,,\quad M_2(G - v_jb + v_ib) > M_2(G).$$

Proof. Without losing generality, we assume that $i = 1$ and $j = 2$. Define:

$$G_0 = W_n(n_1 + 1, n_2 - 1, n_3, n_4, n_5).$$

Therefore,

$$\begin{aligned} M_1(G_0) - M_1(G) &= (n_1 - (n_1 - 1)) + (n_1 + 4)^2 - (n_1 + 3)^2 \\ &+ (n_2 - 2 - (n_2 - 1)) + (n_2 + 1)^2 - (n_2 + 2)^2 \\ &= 2n_1 - 2n_2 + 4 > 0\,. \end{aligned}$$

The second part can be proved in a similar way. □

Lemma 3.5. *Suppose $G = W_n(n_1, n_2, 1, 1, 1)$, where $n_1, n_2 \geq 2$. Then,*

$$M_1(G) \leq M_1(W_n(n-4, 1, 1, 1, 1)) \quad and \quad M_2(G) \leq M_2(W_n(n-4, 1, 1, 1, 1)).$$

Proof. Define $G_1 = W_n(n - 4, 1, 1, 1, 1)$. Then,

$$\begin{aligned} M_1(G_1) - M_1(G) &= (n_1 + n_2 - 2) + (n_1 + n_2 + 2)^2 + 9 - (n_1 - 1) - (n_1 + 3)^2 \\ &- (n_2 - 1) - (n_2 + 2) = 2n_1n_2 - 2n_1 = 2n_1(n_2 - 1) > 0. \end{aligned}$$

The second part can be proved similarly. □

Theorem 3.6. *Suppose $G = W_n(n_1, n_2, n_3, n_4, n_5)$, where $n \geq 5$. Therefore,*

$$M_1(G) \leq n^2 - n + 32 \quad and \quad M_2(G) \leq n^2 + 6n + 29.$$

In addition, the equalities hold if and only if $G = W_n(n - 4, 1, 1, 1, 1)$.

Proof. The proof is similar to Theorem 3.3 and so it is omitted. □

Corollary 3.7. *Among all graphs in $TG(n, n-5)$, the graph $F_n(n-4, 1, 1, 1, 1)$ has maximum value of the first and second Zagreb indices.*

4. Tetracyclic Graphs with Six Non−Pendant Vertices

In this section, the maximum value of $n-$vertex tetracyclic graphs with exactly six non− pendant vertices is computed, Figure 3. To do this, we have to introduce some tetracyclic graphs. Suppose $Q(6; 3, 3, 3, 3)$ is a tetracyclic graph such that all of its cycles of length 3 have a common edge. The graph $Q_n(n_1, n_2, \ldots, n_6)$, Figure 4, is obtained from $Q(6; 3, 3, 3, 3)$ by adding $n_i - 1$ pendant vertices to v_i, $1 \leq i \leq 6$, $n_i \geq 1$, $n_1 \geq n_2 \geq n_3, n_3 = \max\{n_3, n_4, n_5, n_6\}$ and $\sum_{i=1}^{6} n_i = n$.

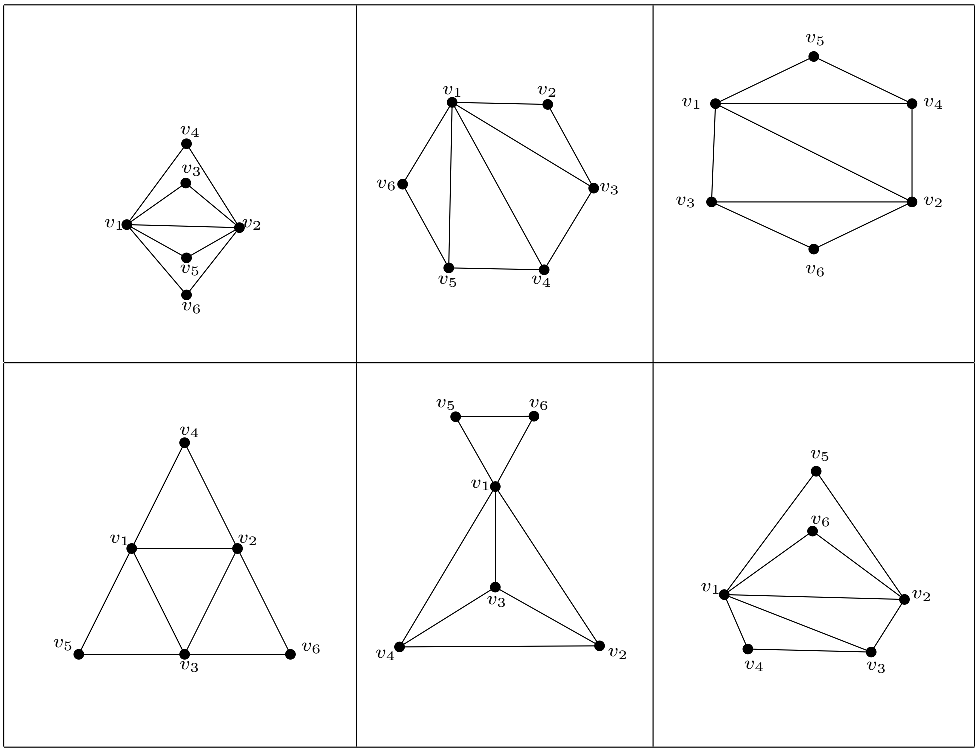
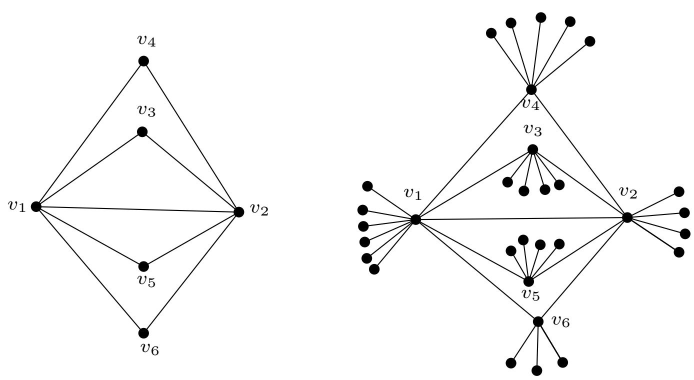


Figure 3. The Tetracyclic Graphs with Six Non−Pendant Vertices.



(**a**) (**b**)

Figure 4. **a**) The Graph $Q(6; 3, 3, 3, 3)$. **b**) The Graph $Q_n(n_1, n_2, n_3, n_4, n_5, n_6)$.

Lemma 4.1. *Suppose $G = Q(n_1, n_2, n_3, n_4, n_5, n_6)$, v_i and v_j are two fixed non−pendant vertices of G and b is a pendant vertex adjacent to v_j. We also assume that $n_i \geq n_j \geq 2$, where $1 \leq i \neq j \leq 6$. Then $M_1(G - v_jb + v_ib) > M_1(G)$.*

Proof. Without loss of generality, we can assume that $i = 1$ and $j = 2$. Suppose $G_0 = Q(n_1 + 1, n_2 - 1, n_3, n_4, n_5, n_6)$. Therefore,

$$M_1(G_0) - M_1(G) = (n_1 - n_1 + 1) + (n_1 + 5)^2 - (n_1 + 4)^2 + (n_2 - n_2 + 1 - 2)$$

$$+\ (n_2+3)^2-(n_2+4)^2 = 2+2(n_1-n_2) > 0,$$

as desired. □

Lemma 4.2. *Suppose $G = Q_n(n_1, n_2, n_3, n_4, n_5, n_6)$, $n \geq 6$. Then*

a) *If $n_2 \geq 2$, then $M_2(G) < M_2(Q(n_1+1, n_2-1, n_3, n_4, n_5, n_6))$,*
b) *If $n_4 \geq 2$, then $M_2(G) < M_2(Q(n_1, n_2, n_3+1, n_4-1, n_5, n_6))$,*
c) *If $n_5 \geq 2$, then $M_2(G) < M_2(Q(n_1, n_2, n_3+1, n_4, n_5-1, n_6))$,*
d) *If $n_6 \geq 2$, then $M_2(G) < M_2(Q(n_1+1, n_2, n_3, n_4, n_5, n_6-1))$.*

Proof. **a)** Assume that $n_2 \geq 2$ and $G_0 = Q(n_1+1, n_2-1, n_3, n_4, n_5, n_6)$. So,

$$\begin{aligned} M_2(G_0) - M_2(G) &= [n_1(n_1+5)-(n_1-1)(n_1+4)] \\ &+ [(n_2-2)(n_2+3)-(n_2-1)(n_2+4)] \\ &+ [(n_3+1)(n_1+5)-(n_3+1)(n_1+4)] \\ &+ [(n_3+1)(n_2+3)-(n_3+1)(n_2+4)] \\ &+ [(n_4+1)(n_1+5)-(n_4+1)(n_1+4)] \\ &+ [(n_4+1)(n_2+3)-(n_4+1)(n_2+4)] \\ &+ [(n_2+3)(n_1+5)-(n_2+4)(n_1+4)] \\ &+ [(n_5+1)(n_2+3)-(n_5+1)(n_2+4)] \\ &+ [(n_5+1)(n_1+5)-(n_5+1)(n_1+4)] \\ &+ [(n_6+1)(n_2+3)-(n_6+1)(n_2+4)] \\ &+ [(n_6+1)(n_1+5)-(n_6+1)(n_1+4)] \\ &= n_1-n_2+1 > 0. \end{aligned}$$

b) If $n_4 \geq 2$ and $G_1 = Q(n_1, n_2, n_3+1, n_4-1, n_5, n_6)$ then

$$\begin{aligned} M_2(G_1) - M_2(G) &= [n_3(n_3+2)-(n_3-1)(n_3+1)] \\ &+ [(n_4-2)(n_4)-(n_4-1)(n_4+1)] \\ &+ [(n_3+2)(n_1+4)-(n_3+1)(n_1+4)] \\ &+ [(n_3+2)(n_2+4)-(n_3+1)(n_2+4)] \\ &+ [(n_4)(n_1+4)-(n_4+1)(n_1+4)] \\ &+ [(n_4)(n_2+4)-(n_4+1)(n_2+4)] \\ &= 2n_3+2-2n_4 > 0. \end{aligned}$$

Other cases can be proved in a similar way. □

Corollary 4.3. *Let $G = Q(n_1, n_2, 1, 1, 1, 1)$, where $n_1, n_2 \geq 2$. Then*

a) $M_1(G) \leq M_1(Q(n-5, 1, 1, 1, 1, 1)) = n^2 - n + 36$,
b) $M_2(G) \leq M_2(Q(n-5, 1, 1, 1, 1, 1)) = n^2 + 6n + 33$.

Proof. To prove (a), we note that

$$\begin{aligned} M_1(Q(n_1,n_2,1,1,1,1)) \;-\; M_1(Q(n_1+n_2-1,1,1,1,1)) &= [(n_1+4)^2+(n_2+4)^2+n_1+n_2-2] \\ &- [(n_1+n_2-2)+(n_1+n_2+3)^2+25] \\ &= 2n_1+2n_2-1-2n_1n_2 < 0. \end{aligned}$$

The part (b) can be proven by a similar argument. □

Theorem 4.4. *Suppose $G \in Q(n_1,n_2,n_3,n_4,n_5,n_6)$. Then*

$$M_1(G) \le n^2 - n + 36 \qquad ; \qquad M_2(G) \le n^2 + 6n + 33.$$

Moreover, the equalities hold if and only if $G \cong Q(n-5,1,1,1,1,1)$.

Proof. Apply Lemma 4.1, Lemma 4.2 and Corollary 4.3. □

We now introduce another class of tetracyclic graphs with six non−pendant vertices. Consider the graph C_6. Choose an arbitrary vertex and connect it to other nonadjacent vertices in such a way that $d(v_1) = 5, d(v_3) = d(v_4) = d(v_5) = 3$ and $d(v_2) = d(v_6) = 2$. This graph is denoted by E_6, see Figure 5. We now construct the graph $G = E_n(n_1,n_2,n_3,n_4,n_5,n_6)$ in a similar method as $Q(n_1,n_2,n_3,n_4,n_5,n_6)$, where $n_1 = \max\{n_i\}_{i=1}^{i=6}$.

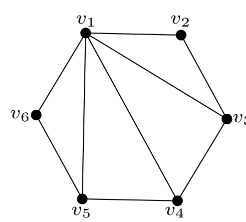


FIGURE 5. The E_6 Graph.

Lemma 4.5. *Suppose $G = E_n(n_1,n_2,n_3,n_4,n_5,n_6)$ in which $n_i \ge 2$, $1 \le i \le 6$. Then,*

1) $M_1(G) < M_1(E_n(n_1+1,n_2-1,n_3,n_4,n_5,n_6))$,
2) $M_2(G) < M_2(E_n(n_1+1,n_2-1,n_3,n_4,n_5,n_6))$,
3) $M_1(G) < M_1(E_n(n_1+1,n_2,n_3-1,n_4,n_5,n_6))$,
4) $M_2(G) < M_2(E_n(n_1+1,n_2,n_3-1,n_4,n_5,n_6))$,
5) $M_1(G) < M_1(E_n(n_1+1,n_2,n_3,n_4-1,n_5,n_6))$,
6) $M_2(G) < M_2(E_n(n_1+1,n_2,n_3,n_4-1,n_5,n_6))$,
7) $M_1(G) < M_1(E_n(n_1+1,n_2,n_3,n_4,n_5-1,n_6))$,
8) $M_2(G) < M_2(E_n(n_1+1,n_2,n_3,n_4,n_5-1,n_6))$,
9) $M_1(G) < M_1(E_n(n_1+1,n_2,n_3,n_4,n_5,n_6-1))$,
10) $M_2(G) < M_2(E_n(n_1+1,n_2,n_3,n_4,n_5,n_6-1))$.

Proof. Suppose that $G_1 = E_n(n_1+1, n_2-1, n_3, n_4, n_5, n_6)$. Then

$$\begin{aligned} M_1(G_1) - M_1(G) &= n_1 + (n_1+5)^2 - (n_1-1) - (n_1+4)^2 \\ &\quad + n_2 - 2 + (n_2+1)^2 - (n_2-1) - (n_2+2)^2 \\ &= 2(n_1 - n_2 + 3) > 0. \end{aligned}$$

The other cases can be proved in similar ways. □

The following corollary is an immediate consequence of Lemma 4.5:

Corollary 4.6. *Suppose $G = E_n(n_1, n_2, 1, 1, 1, 1)$, where $n_1 \geq n_2 \geq 2$. Then*

$$M_1(G) < M_1(E_n(n_1+n_2-1, 1, 1, 1, 1, 1)), \;\; M_2(G) < M_2(E_n(n_1+n_2-1, 1, 1, 1, 1, 1)).$$

Theorem 4.7. *Let $G = E_n(n_1, n_2, n_3, n_4, n_5, n_6)$ with $n \geq 6$. Then*

$$M_1(G) \leq n^2 - n + 30\,, \quad M_2(G) \leq n^2 + 6n + 23\,.$$

Moreover, equalities hold if and only if $G \cong E_n(n-5, 1, 1, 1, 1, 1)$.

Next we introduce a new class of tetracyclic graphs with six non−pendant vertices. Consider the graph H_1 depicted in Figure 6. Then we add $n_i - 1$ pendant vertices to each vertex v_i, $1 \leq i \leq 6$, such that $n_i \geq 1$, $n_1 = \max\{n_1, \ldots, n_6\}$ and $\sum_{i=1}^{6} n_i = n$. The resulting tetracyclic graph and the set of all such graphs are denoted by $H_1(n_1, n_2, n_3, n_4, n_5, n_6)$ and L_1, respectively.

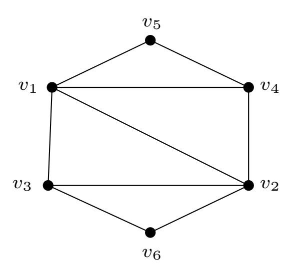


FIGURE 6. The Graph H_1.

Lemma 4.8. *Suppose $G = H_1(n_1, n_2, n_3, n_4, n_5, n_6)$, v_i and v_j are two non−pendant vertices, $1 \leq i \neq j \leq 6$, b is a pendant vertex adjacent to v_j and $n_i \geq n_j \geq 2$. Then,*

$$M_1(G - v_jb + v_ib) > M_1(G),$$

$$M_2(G - v_jb + v_ib) > M_2(G).$$

Proof. Without loss of generality, we can assume that $i = 1$, $j = 2$. Define

$$G_0 = H_1(n_1+1, n_2-1, n_3, n_4, n_5, n_6).$$

Then,

$$\begin{aligned} M_1(G) - M_1(G_0) &= (n_1+3)^2 - (n_2+3)^2 - (n_1+4)^2 - (n_2+2)^2 \\ &= -2 - 2n_1 + 2n_2 < 0, \\ M_2(G) - M_2(G_0) &= (n_1-1)(n_1+3) + (n_2-1)(n_2+3) + (n_1+3)(n_2+n_3+n_4+8) \\ &+ (n_2+3)(n_3+n_4+n_5+5) - n_1(n_1+4) - (n_2-2)(n_2+2) \\ &- (n_1+4)(n_2+n_3+n_4+n_6+7) - (n_2+2)(n_3+n_4+n_5+5) \\ &= -2n_1 + n_2 + n_5 - n_6 - 1 < 0, \end{aligned}$$

which completes the proof. □

Theorem 4.9. *Among all graphs in L_1, the graph $G = H_1(n-5,1,1,1,1,1)$ has maximum first and second Zagreb indices with the following Zagreb values:*

$$M_1(G) = n^2 - 3n + 40, \qquad M_2(G) = n^2 + 4n + 32.$$

Proof. If we eliminate a pendant edge connect to the vertex v_i, $i \geq 1$, and adding it to the vertex v_1 then By Lemma 4.8, the Zagreb values will be increased. □

Consider the graph H_2 which is depicted in Figure 7 and add $n_i - 1$ pendant vertices to each vertex v_i, $1 \leq i \leq 6$, in such a way that $n_i \geq 1$, $n_1 = \max\{n_1, \ldots, n_6\}$, $n_5 \geq n_6$ and $\sum_{i=1}^{6} n_i = n$. The resulting tetracyclic graph and the set of all such graphs are denoted by $H_2(n_1, n_2, n_3, n_4, n_5, n_6)$ and L_2, respectively.

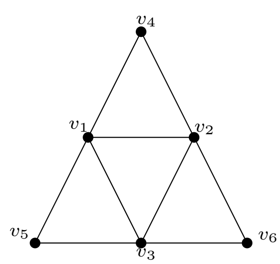


FIGURE 7. The Graph H_2.

Lemma 4.10. *Suppose $G = H_2(n_1, n_2, n_3, n_4, n_5, n_6)$ contains two non−pendant vertices v_i and v_j, b is a pendant vertex adjacent to v_j, $1 \leq i \neq j \leq 6$ and $n_i \geq n_j \geq 2$. Then,*

$$M_1(G - v_jb + v_ib) > M_1(G),$$
$$M_2(G - v_jb + v_ib) > M_2(G).$$

Proof. Without loss of generality, we can assume that $i = 1$ and $j = 2$. Define $G_0 = H_2(n_1 + 1, n_2 - 1, n_3, n_4, n_5, n_6)$. Then,

$$\begin{aligned}
M_1(G) - M_1(G_0) &= (n_1+3)^2 - (n_2+3)^2 - (n_1+4)^2 - (n_2+2)^2 \\
&= -2 - 2n_1 + 2n_2 < 0, \\
M_2(G) - M_2(G_0) &= (n_1-1)(n_1+3) + (n_2-1)(n_2+3) \\
&+ (n_1+3)(n_2+n_3+n_4+n_5+8) \\
&+ (n_2+3)(n_3+n_4+n_6+7) \\
&- n_1(n_1+4) - (n_2-2)(n_2+2) \\
&- (n_1+4)(n_2+n_3+n_4+n_5+7) \\
&- (n_2+2)(n_3+n_4+n_6+7) \\
&= -2n_1 + n_2 - n_5 + n_6 + 1 < 0.
\end{aligned}$$

This completes the proof. □

Theorem 4.11. *The graph $G = H_2(n-5, 1, 1, 1, 1, 1)$ has maximum first and second Zagreb indices in the class L_2 and the Zagreb values are as follows:*

$$M_1(G) = n^2 - 3n + 42, \qquad M_2(G) = n^2 + 4n + 36.$$

Proof. The proof is similar to Theorem 4.9 and so omitted. □

We now consider the graph H_3 depicted in Figure 8. Add $n_i - 1$ pendant vertices to each vertex v_i, $1 \leq i \leq 6$, such that $n_i \geq 1$, $n_1 = \max\{n_1, \ldots, n_6\}$, $n_5 \geq n_6$ and $\sum_{i=1}^{6} n_i = n$. The resulting tetracyclic graph will be denoted by $H_3(n_1, n_2, n_3, n_4, n_5, n_6)$. Define L_3 to be the set of all such graphs.

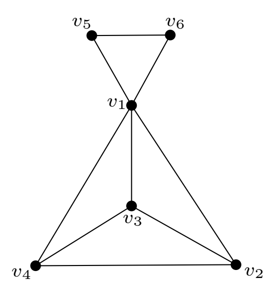


Figure 8. The Graph H_3.

Lemma 4.12. *Suppose $G = H_3(n_1, n_2, n_3, n_4, n_5, n_6)$ is containing two non−pendant vertices v_i and v_j, $1 \leq i \neq j \leq 6$, b is a pendant vertex adjacent to v_j and $n_i \geq n_j \geq 2$. Then,*

$$M_1(G - v_j b + v_i b) > M_1(G),$$
$$M_2(G - v_j b + v_i b) > M_2(G).$$

Proof. Without loss of generality, we can assume that $i = 1$ and $j = 2$. Define $G_0 = H_3(n_1+1, n_2-1, n_3, n_4, n_5, n_6)$. Then,

$$M_1(G_0) - M_1(G) = 2(n_1 - n_2 + 3) > 0,$$

$$M_2(G)) - M_2(G) = n_1 + n_2 + n - 5 + n_6 + 4 > 0,$$

proving the lemma. □

Theorem 4.13. *The graph* $G = H_3(n-5,1,1,1,1,1)$ *has maximum first and second Zagreb indices in* L_3 *with the following Zagreb values:*

$$M_1(G) = n^2 - n + 30, \qquad M_2(G) = n^2 + 6n + 24.$$

Proof. The proof is similar to the Theorem 4.9 and so omitted. □

Suppose H_4 is the graph depicted in Figure 9. Add $n_i - 1$ pendant vertices to each vertex v_i, $1 \le i \le 6$, such that $n_i \ge 1$, $n_1 = \max\{n_1,\ldots,n_6\}$, $n_5 \ge n_6$ and $\sum_{i=1}^{6} n_i = n$. The resulting tetracyclic graph is denoted by $H_4(n_1,n_2,n_3,n_4,n_5,n_6)$. Define L_4 to be the set of all such graphs.

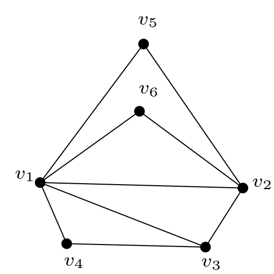


FIGURE 9. The Graph H_4.

Lemma 4.14. *Suppose* $G = H_4(n_1,n_2,n_3,n_4,n_5,n_6)$ *containing non−pendant vertices* v_i *and* v_j, b *is a pendant vertex adjacent to* v_j, $1 \le i \ne j \le 6$ *and* $n_i \ge n_j \ge 2$. *Then,*

$$M_1(G - v_jb + v_ib) > M_1(G),$$

$$M_2(G - v_jb + v_ib) > M_2(G).$$

Theorem 4.15. *The graph* $G = H_4(n-5,1,1,1,1,1)$ *attain the maximum first and second Zagreb indices in* L_4*, and the Zagreb values are as follows:*

$$M_1(G) = n^2 - n + 37, \qquad M_2(G) = n^2 + 6n + 27.$$

Proof. The proof is similar to the Theorem 4.9 and so omitted. □

Theorem 4.16. *Suppose* $G \in TG(n, n-6)$*, where* $n \ge 6$. *Then,*

$$M_1(G) \le n^2 - n + 36\,, \qquad M_2(G) \le n^2 + 6n + 33.$$

Moreover, the equalities hold if and only if $G = Q(n-5,1,1,1,1,1)$.

Proof. Apply Theorems 4.4, 4.7, 4.9, 4.11, 4.13 and 4.15. □

5. **Tetracyclic Graphs with Seven Non−Pendant Vertices**

In this section, we investigate graphs with seven non-pendant vertices, see Figure 10. Consider the graph B_j with vertex set $V_j = \{v_{j1}, \ldots, v_{j7}\}$, $1 \leq j \leq 8$. When there is no confusion, we use the name $v_1, v_2, \ldots, v_7$ for vertices of V_j. Construct the graph $B_j(n_1, n_2, n_3, n_4, n_5, n_6, n_7)$, $1 \leq j \leq 8$, by adding $n_i - 1$ pendant vertices to the vertex v_{ji} of the graph B_j, $1 \leq i \leq 7$, in such a way that $n_1 \geq n_2 \geq \cdots \geq n_7 \geq 1$ and $\sum_{i=1}^{7} n_i = n$. This class of graphs is denoted by **B**.

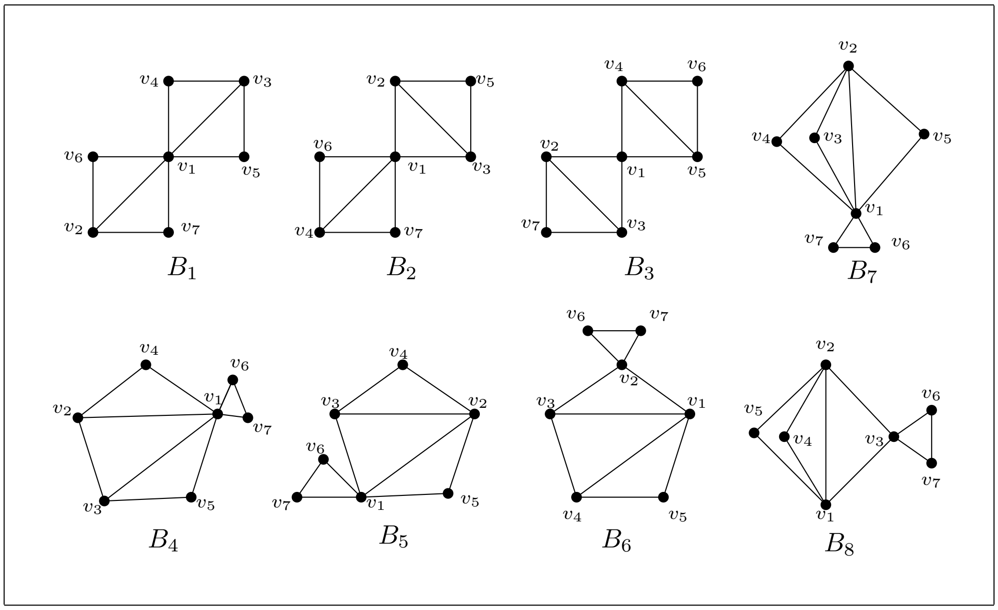


FIGURE 10. The Tetracyclic Graphs with Seven Non−Pendant Vertices.

Lemma 5.1. *Suppose $G = B_1(n_1, n_2, n_3, n_4, n_5, n_6, n_7)$ and there are i and j such that $1 \leq i \neq j \leq 7$ and $n_i \geq n_j \geq 2$. We also assume that b is a pendant vertex adjacent to the vertex v_j. Then,*

$$M_1(G - v_jb + v_ib) > M_1(G),$$
$$M_2(G - v_jb + v_ib) > M_2(G).$$

Proof. Without loss of generality, we can assume that $i = 1$ and $j = 2$. Define $G_0 = B_1(n_1 + 1, n_2 - 1, n_3, n_4, n_5, n_6, n_7)$. Then,

$$\begin{aligned} M_1(G) - M_1(G_0) &= (n_1 + 5)^2 + (n_2 + 2)^2 + n_1 + n_2 - 2 - (n_1 + 6)^2 \\ &- (n_2 + 1)^2 - n_1 - n_2 + 2 \\ &= -8 - 2n_1 + 2n_2 < 0. \end{aligned}$$

On the other hand,

$$M_2(G) - M_2(G_0) = -n_1 + n_2 - n_3 - n_4 - n_5 - 1 < 0,$$

proving the lemma. □

Lemma 5.2. *If* $G = B_1(n_1, n_2, n_3, n_4, n_5, n_6, n_7)$ *then*

$$M_1(G) \leq M_1(B_1(n-6,1,1,1,1,1,1)) = n^2 - n + 28,$$
$$M_2(G) \leq M_2(B_1(n-6,1,1,1,1,1,1)) = n^2 + 6n + 17.$$

Proof. Choose $i, i \neq 1$. Define a graph C constructed from G by removing each edge adjacent to v_i and connect them to v_1. Then by Lemma 5.1, $M_1(C) > M_1(G)$ and $M_2(C) > M_2(G)$. By continuing this process, we find the graph $B_1(n-6,1,1,1,1,1,1)$ with maximum Zagreb indices. □

Lemma 5.3. *Suppose* $G = B_r(n_1, n_2, n_3, n_4, n_5, n_6, n_7)$, $2 \leq r \leq 8$. *We also assume that there are* i *and* j *such that* $1 \leq i \neq j \leq 7$, $n_i \geq n_j \geq 2$, *and* v_j *has a neighbor* b *which is a pendant vertex. Then,*

$$M_1(G - v_jb + v_ib) > M_1(G),$$
$$M_2(G - v_jb + v_ib) > M_2(G).$$

Proof. The proof is similar to Lemma 5.1. □

Theorem 5.4. *Suppose* $G \cong B_r(n_1, n_2, n_3, n_4, n_5, n_6, n_7)$, $2 \leq r \leq 8$. *Then,*

(1) *If* $r = 2$ *then* $M_1(G) \leq M_1(B_2(n-6,1,1,1,1,1,1)) = n^2 - 3n + 36$ *and* $M_2(G) \leq M_2(B_2(n-6,1,1,1,1,1,1)) = n^2 + 4n + 21$,
(2) *If* $r = 3$ *then* $M_1(G) \leq M_1(B_3(n-6,1,1,1,1,1,1)) = n^2 - 5n + 46$ *and* $M_2(G) \leq M_2(B_3(n-6,1,1,1,1,1,1)) = n^2 + 2n + 27$,
(3) *If* $r = 4$ *then* $M_1(G) \leq M_1(B_4(n-6,1,1,1,1,1,1)) = n^2 - n + 28$ *and* $M_2(G) \leq M_2(B_4(n-6,1,1,1,1,1,1)) = n^2 + 6n + 18$,
(4) *If* $r = 5$ *then* $M_1(G) \leq M_1(B_5(n-6,1,1,1,1,1,1)) = n^2 - 3n + 38$ *and* $M_2(G) \leq M_2(B_5(n-6,1,1,1,1,1,1)) = n^2 + 4n + 26$,
(5) *If* $r = 6$ *then* $M_1(G) \leq M_1(B_6(n-6,1,1,1,1,1,1)) = n^2 - 5n + 48$ *and* $M_2(G) \leq M_2(B_6(n-6,1,1,1,1,1,1)) = n^2 + 2n + 32$,
(6) *If* $r = 7$ *then* $M_1(G) \leq M_1(B_7(n-6,1,1,1,1,1,1)) = n^2 - n + 30$ *and* $M_2(G) \leq M_2(B_7(n-6,1,1,1,1,1,1)) = n^2 + 6n + 21$,
(7) *If* $r = 8$ *then* $M_1(G) \leq M_1(B_8(n-6,1,1,1,1,1,1)) = n^2 - 5n + 50$ *and* $M_2(G) \leq M_2(B_8(n-6,1,1,1,1,1,1)) = n^2 + 2n + 37$.

Theorem 5.5. *Among all graphs in* $TG(n, n-7)$, $B_7(n-6,1,1,1,1,1,1)$ *has maximum Zagreb indices.*

Proof. Apply Table 1. □

G	$B_1(n-6,1,\dots,1)$	$B_2(n-6,1,\dots,1)$	$B_3(n-6,1,\dots,1)$	$B_7(n-6,1,\dots,1)$
$M_1(G)$	n^2-n+28	$n^2-3n+36$	$n^2-5n+46$	n^2-n+30
$M_2(G)$	$n^2+6n+17$	$n^2+4n+21$	$n^2+2n+27$	$n^2+6n+21$
G	$B_4(n-6,1,\dots,1)$	$B_5(n-6,1,\dots,1)$	$B_6(n-6,1,\dots,1)$	$B_8(n-6,1,\dots,1)$
$M_1(G)$	n^2-n+28	$n^2-3n+38$	$n^2-5n+48$	$n^2-5n+50$
$M_2(G)$	$n^2+6n+18$	$n^2+4n+26$	$n^2+2n+32$	$n^2-2n+49$

Table 1. The Zagreb Indices of $B_r(n-6,1,\dots,1)$, $1 \le r \le 8$.

6. Tetracyclic Graphs with Eight Non−Pendant Vertices

In this section, the tetracyclic graphs with eight non−pendant vertices are investigated. Consider the graph D_j with vertex set $V_j = \{v_{j1},\dots,v_{j8}\}$, $1 \le j \le 7$. When there is no confusion, we use the name $v_1, v_2, \dots, v_8$ for vertices of V_j. Construct the graph $D_j(n_1,n_2,n_3,n_4,n_5,n_6,n_7,n_8)$, $1 \le j \le 7$, by adding $n_i - 1$ pendant vertices to the vertex v_{ji} of the graph D_j, $1 \le i \le 8$, in such a way that $n_1 \ge n_2 \ge \cdots \ge n_7 \ge n_8 \ge 1$ and $\sum_{i=1}^{8} n_i = n$. This class of graphs is denoted by $\mathbf{D}$.

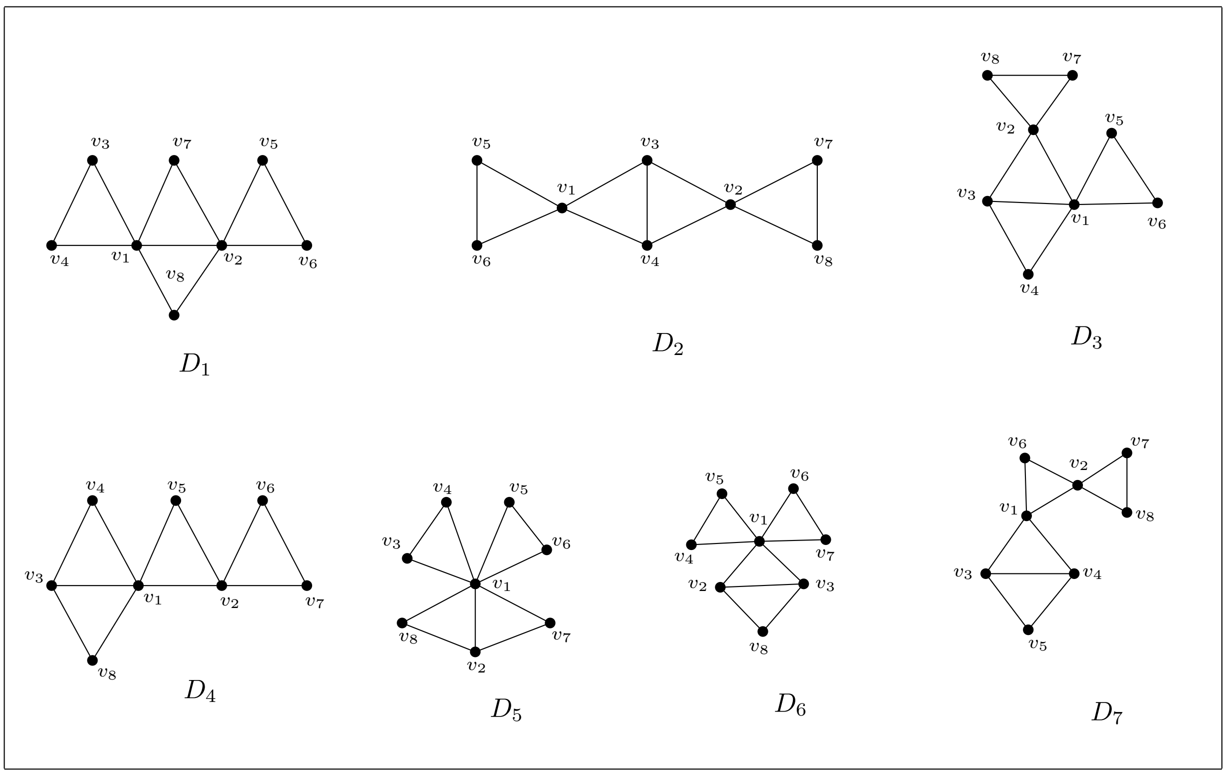


Figure 11. The Tetracyclic Graphs on Eight Non−Pendant Vertices.

Lemma 6.1. *Suppose $G = D_1(n_1,n_2,n_3,n_4,n_5,n_6,n_7,n_8)$ and there are i and j such that $1 \le i \ne j \le 8$, $n_i \ge n_j \ge 2$ and b is a pendant vertex adjacent to v_j. Then,*

$$M_1(G - v_jb + v_ib) > M_1(G),$$

$$M_2(G - v_jb + v_ib) > M_2(G).$$

Proof. Without loss of generality, we can assume that $i = 1$ and $j = 2$. Define $G_0 = D_1(n_1 + 1, n_2 - 1, n_3, n_4, n_5, n_6, n_7, n_8)$. So

$$\begin{aligned} M_1(G) - M_1(G_0) &= (n_1 + 4)^2 + (n_2 + 4)^2 + n_1 + n_2 - 2 \\ &- (n_1 + 5)^2 - (n_2 + 3)^2 - n_1 - n_2 + 2 \\ &= -2n_1 + 2n_2 - 2 < 0. \end{aligned}$$

On the other hand,

$$\begin{aligned} M_2(G) - M_2(G_0) = &[(n_1 - 1)(n_1 + 4) - n_1(n_1 + 5)] \\ &+ [(n_2 - 1)(n_2 + 4) - (n_4 - 2)(n_2 + 3)] \\ &+ [(n_1 + 4)(n_2 + 4) - (n_1 + 5)(n_2 + 3)] \\ &- (n_3 + n_4) + n_5 + n_6 \\ &= -n_1 + n_2 + n_5 + n_6 - n_3 - n_4 - 1 < 0, \end{aligned}$$

which completes the proof. □

Lemma 6.2. *If $G \cong D_1(n_1, n_2, n_3, n_4, n_5, n_6, n_7, n_7, n_8)$, then*

$$M_1(G) \leq M_1(D_1(n - 7, 1, 1, 1, 1, 1, 1, 1)) = n^2 - 5n + 50,$$
$$M_2(G) \leq M_2(D_1(n - 7, 1, 1, 1, 1, 1, 1, 1)) = n^2 + 2n + 33.$$

Proof. By removing each edge from v_i, $i \neq 1$, and connecting them to v_1, we find another graph with greater Zagreb indices. By continuing this process, it will be proved that the graph $D_1(n - 7, 1, 1, 1, 1, 1, 1, 1)$ has maximum Zagreb indices. □

Lemma 6.3. *Suppose $G = D_r(n_1, n_2, n_3, n_4, n_5, n_6, n_7, n_8)$, $2 \leq r \leq 8$, and there are i and j such that $1 \leq i \neq j \leq 8$, $n_i \geq n_j \geq 2$ and b is a pendant vertex adjacent to v_j. Then,*

$$M_1(G - v_jb + v_ib) > M_1(G),$$
$$M_2(G - v_jb + v_ib) > M_2(G).$$

Proof. The proof is similar to Lemma 6.1. □

Theorem 6.4. *If $G \cong D_r(n_1, n_2, n_3, n_4, n_5, n_6, n_7, n_8)$, $2 \leq r \leq 7$, then*

(1) *If $r = 2$ then $M_1(G) \leq M_1(D_2(n - 7, 1, 1, 1, 1, 1, 1, 1)) = n^2 - 7n + 58$ and $M_2(G) \leq M_2(D_2(n - 7, 1, 1, 1, 1, 1, 1, 1)) = n^2 - 2n + 49$,*
(2) *If $r = 3$ then $M_1(G) \leq M_1(D_3(n - 7, 1, 1, 1, 1, 1, 1, 1)) = n^2 - 5n + 46$ and $M_2(G) \leq M_2(D_3(n - 7, 1, 1, 1, 1, 1, 1, 1)) = n^2 + 2n + 27$,*
(3) *If $r = 4$ then $M_1(G) \leq M_1(D_4(n - 7, 1, 1, 1, 1, 1, 1, 1)) = n^2 - 5n + 46$ and $M_2(G) \leq M_2(D_4(n - 7, 1, 1, 1, 1, 1, 1, 1)) = n^2 + 2n + 25$,*
(4) *If $r = 5$ then $M_1(G) \leq M_1(D_5(n - 7, 1, 1, 1, 1, 1, 1, 1)) = n^2 - n + 26$ and $M_2(G) \leq M_2(D_5(n - 7, 1, 1, 1, 1, 1, 1, 1)) = n^2 + 6n + 13$,*
(5) *If $r = 6$ then $M_1(G) \leq M_1(D_6(n - 7, 1, 1, 1, 1, 1, 1, 1)) = n^2 - 3n + 34$ and $M_2(G) \leq M_2(D_6(n - 7, 1, 1, 1, 1, 1, 1, 1)) = n^2 + 4n + 17$,*

(6) *If* $r = 7$ *then* $M_1(G) \leq M_1(D_7(n-7,1,1,1,1,1,1,1)) = n^2 - 7n + 58$ *and* $M_2(G) \leq M_2(D_7(n-7,1,1,1,1,1,1,1)) = n^2 + 33$.

Proof. The proof is similar to Lemma 6.2. □

Theorem 6.5. *Among all graphs in* $TG(n, n-7)$, $D_7(n-6,1,1,1,1,1,1)$ *has maximum Zagreb indices.*

Proof. Apply Table 2. □

G	$D_1(n-7,1,\ldots,1)$	$D_2(n-7,1,\ldots,1)$	$D_3(n-7,1,\ldots,1)$	$D_4(n-7,1,\ldots,1)$
M_1	$n^2-5n+50$	$n^2-7n+58$	$n^2-5n+46$	$n^2-5n+46$
M_2	$n^2+2n+33$	$n^2-2n+49$	$n^2+2n+27$	$n^2+2n+25$
G	$D_5(n-7,1,\ldots,1)$	$D_6(n-7,1,\ldots,1)$	$D_7(n-7,1,\ldots,1)$	
M_1	n^2-n+26	$n^2-3n+34$	$n^2-7n+58$	
M_2	$n^2+6n+13$	$n^2+4n+17$	n^2+33	

TABLE 2. The Zagreb Indices of $D_r(n-7,1,\ldots,1)$, $1 \leq r \leq 7$.

7. Tetracyclic Graphs with Nine Non−Pendant Vertices

In this section, we investigate the tetracyclic graphs with exactly nine non−pendant vertices. It is not so difficult to prove that these graphs are cactus, Figure 12. In a similar way as Section 6, we choose a tetracyclic graph with nine non−pendant vertices and add $n_i - 1$ pendant vertices to each vertex v_i, $1 \leq i \leq 9$ such that $n_1 \geq n_2 \geq \cdots \geq n_9 \geq 1$ and $\sum_{i=1}^{9} n_i = n$. So, we find new tetracyclic graphs with nine non−pendant vertices. Let $C_{n,p}$ be the set of all cactus graphs with $p \geq 0$ pendant vertices.

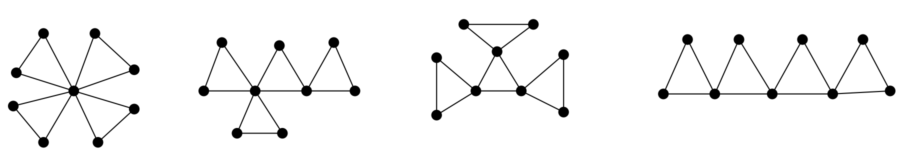


FIGURE 12. Tetracyclic Graphs with Nine Non−Pendant Vertices.

Lemma 7.1. *Suppose* $G \in C_{n,p}$, *Figure 13. Then,*

1) *If* $n - p \equiv 1 \ (mod 2)$, *then* $M_1(G) \leq n^2 + 2n - 3p - 3$. *The equality holds if and only if* $G \cong C^1(n,p)$.
2) *If* $n - p \equiv 1 (mod 2)$, *then* $M_2(G) \leq 2n^2 - (p+2)n - p$. *The equality holds if and only if* $G \cong C^1(n,p)$.

Proof. We refer to [17, Theorem 3.1], for a proof of first part and [17, Theorem 3.6], for a proof of second part. □

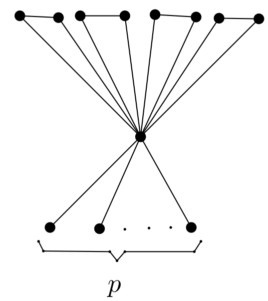


FIGURE 13. The Tetracyclic Graph $C^1(n,p)$.

Corollary 7.2. *If $G \in TG(n, n-9)$, then $M_1(G) \leq n^2 - n + 24$ and $M_2(G) \leq n^2 + 6n + 9$. Equality holds if and only if $G \cong C^1(n, n-9)$.*

8. Extremal Values of Zagreb Indices in the Class of Tetracyclic Graphs

We are now ready to determine the first, second and third maximum values of first Zagreb index and the first and second maximum values of second Zagreb index in the class of all tetracyclic graphs.

Theorem 8.1. *The graph $Q(n-5,1,1,1,1,1)$ attains the maximum value of first Zagreb index among all $n-$vertex tetracyclic graphs. Moreover, $M_1(Q(n-5,1,1,1,1,1)) = n^2 - n + 36$.*

Proof. By Theorems 3.3, 4.16, 5.5, 6.5 and Corollary 7.2, the graphs $F_n(n-4,1,1,1,1)$, $Q(n-5,1,1,1,1,1)$, $B_7(n-6,1,1,1,1,1,1)$, $D_7(n-7,1,1,1,1,1,1,1)$ and $C(n,n-9)$ have the maximum value of the first Zagreb index. On the other hand, a simple calculation on the first Zagreb values of these graphs completes the proof. □

Theorem 8.2. *Among all of graphs in $TG(n)$, $n \geq 6$, the graphs with second and third maximum of Zagreb indices are as follows:*

a) *The graph $F_n(n-4,1,1,1,1)$ with $M_1(F_n(n-4,1,1,1,1)) = n^2 - n + 34$, where $n \geq 6$ and $n \neq 8$.*
b) *The graph $F_8(4,1,1,1,1)$ and $Q(2,2,1,1,1,1)$ with the first Zagreb index 90.*
c) *The graphs $W_7(3,1,1,1,1)$ and $F_7(2,2,1,1,1)$ with the first Zagreb index value 74.*
d) *The graph $W_9(5,1,1,1,1)$ and $Q_9(3,2,1,1,1)$ with the first Zagreb index 104.*
e) *The graph $W_n(n-4,1,1,1,1)$ with $M_1(W_n(n-4,1,1,1,1)) = n^2 - n + 32$, where $n = 8$ or $n \geq 10$.*

Proof. The proof follows from Theorems 3.3, 4.16, 5.5, 6.5 and Corollary 7.2. □

Theorem 8.3. *Among all of graphs in $TG(n)$, $n \geq 6$, the graph $F_n(n-4,1,1,1,1)$ has the first maximum of second Zagreb index with Zagreb value $M_2(F_n(n-4,1,1,1,1)) = n^2 + 6n + 34$. The second maximum are as follows:*

a) *The graph $Q(n-5,1,1,1,1,1)$ with second Zagreb index $n^2 + 6n + 33$, where $n \geq 6$ and $n \neq 7$,*
b) *The graphs $F_7(2,2,1,1,1)$ and $Q(2,1,1,1,1,1)$ with second Zagreb index 124.*

Proof. The case of $n = 7$ is obtained by a simple calculation on 7−vertex graphs. Suppose $n \geq 6$ and $n \neq 7$. On the other hand,

$$\begin{aligned} M_2(Q(n-5,1,1,1,1,1)) &- M_2(F_n(n-4,1,1,1,1)) \\ &= n^2+6n+33-n^2-6n-34=-1<0. \end{aligned}$$

Therefore, $Q(n-5,1,1,1,1,1)$ with second Zagreb index $n^2+6n+33$ attain the second maximum of M_2, as desired. □

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References

[1] V. Andova and M. Petrusevski, Variable Zagreb indices and Karamata's inequality, *MATCH Commun. Math. Comput. Chem.*, **65** no. 3 (2011) 685–690.

[2] V. Andova, N. Cohen and R. Škrekovski, Graph classes (dis)satisfying the Zagreb indices inequality, *MATCH Commun. Math. Comput. Chem.*, **65** no. 3 (2011) 647–658.

[3] A. R. Ashrafi, T. Došlić and A. Hamzeh, The Zagreb coindices of graph operations, *Discrete Appl. Math.*, **158** no. 15 (2010) 1571–1578.

[4] A. R. Ashrafi, T. Dehghan-Zadeh, N. Habibi and P. E. John, Maximum values of atom−bond connectivity index in the class of tricyclic graphs, *J. Appl. Math. Comput.*, **50** no. 1-2 (2016) 511–527.

[5] K. Ch. Das and I. Gutman, Some properties of the second Zagreb index, *MATCH Commun. Math. Comput. Chem.*, **52** (2004) 103–112.

[6] K. Ch. Das, I. Gutman and B. Zhou, New upper bounds on Zagreb indices, *J. Math. Chem.*, **46** no. 2 (2009) 514–521.

[7] T. Dehghan-Zadeh, H. Hua, A. R. Ashrafi and N. Habibi, Extremal tri-cyclic graphs with respect to the first and second Zagreb indices, *Note Mat.*, **33** no. 2 (2013) 107–121.

[8] T. Dehghan-Zadeh, A. R. Ashrafi and N. Habibi, Maximum values of atom−bond connectivity index in the class of tetracyclic graphs, *J. Appl. Math. Comput.*, **46** no. 1-2 (2014) 285–303.

[9] T. Dehghan-Zadeh, A. R. Ashrafi and N. Habibi, Tetracyclic graphs with extremal values of Randic index, *Boll. Unione Mat. Ital.*, **8** no. 1 (2015) 9–16.

[10] I. Gutman and K. Ch. Das, The first Zagreb index 30 years after, *MATCH Commun. Math. Comput. Chem.*, **50** (2004) 83–92.

[11] I. Gutman and N. Trinajstić, Graph theory and molecular orbitals, Total electron energy of alternant hydrocarbons, *Chem. Phys. Lett.*, **17** (1972) 535–538.

[12] A. Ilić, M. Ilić and B. Liu, On the upper bounds for the first Zagreb index, *Kragujevac J. Math.*, **35** no. 1 (2011) 173–182.

[13] M. H. Khalifeh, H. Yousefi-Azari and A. R. Ashrafi, The first and second Zagreb indices of some graph operations, *Discrete Appl. Math*, **157** no. 4 (2009) 804–811.

[14] S. Li and H. Zhou, On the maximum and minimum Zagreb indices of graphs with connectivity at most k, *Appl. Math. Lett.*, **23** no. 2 (2010) 128–132.

[15] S. Li and M. Zhang, Sharp upper bounds for Zagreb indices of bipartite graphs with a given diameter, *Appl. Math. Lett.*, **24** no. 2 (2011) 131–137.

[16] S. Li and Q. Zhao, Sharp upper bounds on Zagreb indices of bicyclic graphs with a given matching number, *Math. Comput. Modelling*, **54** no. 11-12 (2011) 2869–2879.

[17] S. Li, H. Yang and Q. Zhao, Sharp bound on Zagreb indices of cacti with k pendant vertices, *Filomat*, **26** no. 6 (2012) 1189–1200.

[18] S. N. Qiao, On The Zagreb index of quasi-tree graphs, *Appl. Math. E-Notes*, **10** (2010) 147–150.

[19] P. S. Ranjini, V. Lokesha and I. N. Cangül, On the Zagreb indices of the line graphs of the subdivision graphs, *Appl. Math. Comput.*, **218** no. 3 (2011) 699–702.

[20] D. Stevanović, Comparing the Zagreb indices of the NEPS of graphs, *Appl. Math. Comput.*, **219** no. 3 (2012) 1082–1086.

[21] K. Xu, The Zagreb indices of graphs with a given clique number, *Appl. Math. Lett.*, **24** no. 6 (2011) 1026–1030.

[22] Z. Yarahmadi, A. R. Ashrafi and S. Moradi, Extremal polyomino chains with respect to Zagreb indices, *Appl. Math. Lett.*, **25** no. 2 (2012) 166–171.

[23] Q. Zhao and S. Li, Sharp bounds for the Zagreb indices of bicyclic graphs with k-pendant vertices, *Discrete Appl. Math.*, **158** no. 17 (2010) 1953–1962.

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