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STEINER WIENER INDEX OF GRAPH PRODUCTS

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Abstract. The Wiener index $W(G)$ of a connected graph G is defined as $W(G) = \sum_{u,v \in V(G)} d_G(u,v)$ where $d_G(u,v)$ is the distance between the vertices u and v of G. For $S \subseteq V(G)$, the *Steiner distance* $d(S)$ of the vertices of S is the minimum size of a connected subgraph of G whose vertex set is S. The *k-th Steiner Wiener index* $SW_k(G)$ of G is defined as $SW_k(G) = \sum_{\substack{S \subseteq V(G) \\ |S|=k}} d(S)$. We establish expressions for the k-th Steiner Wiener index on the join, corona, cluster, lexicographical product, and Cartesian product of graphs.

1. Introduction

All graphs in this paper are assumed to be undirected, finite and simple. We refer to [1] for graph theoretical notation and terminology not specified here. For a graph G, let $V(G)$, $E(G)$ and $e(G) = |E(G)|$ denote the set of vertices, the set of edges and the size of G, respectively. We divide our introduction into the following three subsections, in order to state the motivations and results of this paper.

1.1. Distance and its generalizations.

Distance is one of the basic concepts of graph theory [2]. If G is a connected graph and $u, v \in V(G)$, then the *distance* $d(u,v)$ between u and v is the length of a shortest path connecting u and v.

The distance between two vertices u and v in a connected graph G also equals the minimum size of a connected subgraph of G containing both u and v. This observation suggests a generalization of the distance concept. The Steiner distance of a graph, introduced by Chartrand, Oellermann, Tian and Zou in 1989 [4], is a natural and consequent generalization of the classical graph distance. For a graph

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$G(V, E)$ and a set $S \subseteq V(G)$ of at least two vertices, *an S-Steiner tree* or *a Steiner tree connecting S* (or simply, *an S-tree*) is a such subgraph $T(V', E')$ of G that is a tree with $S \subseteq V'$. Let G be a connected graph of order at least 2 and let S be a nonempty set of vertices of G. Then the *Steiner distance* $d_G(S)$ among the vertices of S (or simply the distance of S) is the minimum size among all connected subgraphs whose vertex sets contain S. Note that if H is a connected subgraph of G such that $S \subseteq V(H)$ and $|E(H)| = d_G(S)$, then H is a tree. Observe that $d_G(S) = \min\{e(T) \,|\, S \subseteq V(T)\}$, where T is subtree of G. Furthermore, if $S = \{u, v\}$, then $d_G(S) = d(u, v)$ coincides with the classical distance between u and v. Set $d_G(S) = \infty$ when there is no S-Steiner tree in G.

Observation 1.1. *Let G be a graph of order n and k an integer, $2 \leq k \leq n$. If $S \subseteq V(G)$ and $|S| = k$, then $k - 1 \leq d_G(S) \leq n - 1$.*

The *average Steiner distance* $\mu_k(G)$ of a graph G, introduced by Dankelmann, Oellermann and Swart in [6], is defined as the average of the Steiner distances of all k-subsets of $V(G)$, i.e.,

$$\mu_k(G) = \binom{n}{k}^{-1} \sum_{\substack{S \subseteq V(G) \\ |S|=k}} d_G(S)\,.$$

For more details on average Steiner distance, we refer to [5, 6].

1.2. **Wiener index and its generalizations.** The first investigation of the sum of distance between all pairs of vertices of a (connected) graph was done by Harold Wiener in 1947, who realized that there exists a correlation between the boiling points of paraffins and this sum [15]. Eventually, the distance–based graph invariant,

$$W(G) = \sum_{\{u,v\} \subseteq V(G)} d(u, v)$$

named *Wiener index*, became the topic of countless studies. Mathematical researches of the Wiener index started the 1970s [8]; for details see the surveys [7, 16], the books [14, 9, 10], and the references cited therein.

Li et al. [12] generalized the concept of Wiener index by employing Steiner distance [12]. The *Steiner k-Wiener index* $SW_k(G)$ of graph G is defined by

$$SW_k(G) = \sum_{\substack{S \subseteq V(G) \\ |S|=k}} d(S)\,.$$

For $k = 2$, the Steiner Wiener index coincides with the ordinary Wiener index $W(G)$. It is usual to consider SW_k for $2 \leq k \leq n - 1$, but the above definition implies $SW_1(G) = 0$ and $SW_n(G) = n - 1$.

Expressions for SW_k for some special graphs were obtained in [12], where also gave sharp upper and lower bounds of SW_k of a connected graph are established, as well as some of its properties in the case of trees. A chemical application of the Steiner Wiener index is reported in [11].

1.3. **Our main results.** The join, Cartesian and direct products are defined as follows.

The *join* or *complete product* $G+H$ of two disjoint graphs G and H, is the graph with vertex set $V(G)\cup V(H)$ and edge set $E(G)\cup E(H)\cup\{uv\,|\,u\in V(G), v\in V(H)\}$.

The *Cartesian product* $G\Box H$ of two graphs G and H, is the graph with vertex set $V(G)\times V(H)$, in which two vertices (u,v) and (u',v') are adjacent if and only if $u=u'$ and $(v,v')\in E(H)$, or $v=v'$ and $(u,u')\in E(G)$.

The *lexicographic product* $G[H]$ of graphs G and H has the vertex set $V(G[H])=V(G)\times V(H)$, and two vertices $(u,v),(u',v')$ are adjacent if $uu'\in E(G)$, or if $u=u'$ and $vv'\in E(H)$.

The *corona* $G\ominus H$ is obtained by taking one copy of G and $|V(G)|$ copies of H, and by joining each vertex of the i-th copy of H with the i-th vertex of G, where $i=1,2,\ldots,|V(G)|$.

The *cluster* $G\odot H$ is obtained by taking one copy of G and $|V(G)|$ copies of a rooted graph H, and by identifying the root of the i-th copy of H with the i-th vertex of G, where $i=1,2,\ldots,|V(G)|$.

Yeh and one of the present authors [17] investigated the Wiener index of graph products and obtained the following results.

Theorem 1.2. [17] *Let G be a connected graph with n vertices, and let H be a connected graph with m vertices. Then*

$$W(G+H)=e(G)+e(H)+mn+2\left[\binom{n}{2}-e(G)+\binom{m}{2}-e(H)\right].$$

Theorem 1.3. [17] *Let G be a connected graph with n vertices, and let H be a connected graph with m vertices. Then*

$$W(G[H])=m^2\left[(W(G)+n)-n(e(H)+m)\right].$$

Theorem 1.4. [17] *Let G be a connected graph with n vertices, and let H be a connected graph with m vertices. Then*

$$W(G\Box H)=m^2\,W(G)+n^2\,W(H)\,.$$

Theorem 1.5. [17] *Let G be a connected graph with n vertices, and let H be a connected graph with m vertices. Then*

$$W(G\odot H)=m^2\,W(G)+n\,W(H)+m(n^2-n)\,d(v|H)$$

where v is the root-vertex of H and

$$d(v|H)=\sum_{u\in V(H)}d(u,v)\,.$$

Theorem 1.6. [17] *Let G be a connected graph with n vertices, and let H be a connected graph with m vertices. Then*

$$W(G\ominus H)=(m+1)^2\,W(G)+n\big[m^2-e(H)\big]+mn(m+1)(n-1)\,.$$

In this paper, we study the k-th Steiner Wiener index of the above specified graph products. Each of these is treated in one of the following subsections.

2. Main results

In this section, let G and H be two connected graphs with $V(G) = \{u_1, u_2, \ldots, u_n\}$ and $V(H) = \{v_1, v_2, \ldots, v_m\}$, respectively. Then $V(G * H) = \{(u_i, v_j) \,|\, 1 \leq i \leq n,\ 1 \leq j \leq m\}$, where $*$ denotes the Cartesian product operation or lexicographical product operation. For $v \in V(H)$, we use $G(v)$ to denote the subgraph of $G * H$ induced by the vertex set $\{(u_i, v) \,|\, 1 \leq i \leq n\}$. Similarly, for $u \in V(G)$, we use $H(u)$ to denote the subgraph of $G * H$ induced by the vertex set $\{(u, v_j) \,|\, 1 \leq j \leq m\}$.

2.1. Join.

Theorem 2.1. *Let G be a connected graph with n vertices, and let H be a connected graph with m $(n \geq m)$ vertices. Let k be an integer, $3 \leq k \leq n + m$.*

(1) *If $k > n$, then*

$$SW_k(G + H) = (k-1)\binom{n+m}{k}.$$

(2) *If $k \leq m$, then*

$$SW_k(G + H) = (k-1)\binom{n+m}{k} + \binom{n}{k} + \binom{m}{k} - x - y\,.$$

where x is the number of the k-subsets of $V(G)$ such that the subgraph induced by each k-subset is connected, and y is the number of the k-subsets of $V(H)$ such that the subgraph induced by each k-subset is connected.

(3) *If $m < k \leq n$, then*

$$SW_k(G + H) = (k-1)\binom{n+m}{k} + \binom{n}{k} + (k-1)\binom{m}{k} - x\,.$$

Proof. (1) Since $k > n$, it follows that $S \cap V(G) \neq \emptyset$ and $S \cap V(H) \neq \emptyset$ for any $S \subseteq V(G + H)$. Set $S \cap V(G) = \{u_1, u_2, \ldots, u_x\}$ and $S \cap V(H) = \{v_1, v_2, \ldots, v_{k-x}\}$. Then the tree induced by the edges in $\{u_1, v_1\} \cup \{u_1 v_i \,|\, 2 \leq i \leq k - x\} \cup \{u_i v_1 \,|\, 2 \leq i \leq x\}$ is an S-Steiner tree, and so $d_{G+H}(S) \leq k - 1$. From Observation 1.1, $d_{G+H}(S) = k - 1$. So $SW_k(G + H) = (k-1)\binom{n+m}{k}$, as desired.

(2) For any $S \subseteq V(G + H)$, we have $S \subseteq V(G)$, or $S \subseteq V(H)$, or $S \cap V(G) \neq \emptyset$ and $S \cap V(H) \neq \emptyset$. Suppose that $S \subseteq V(G)$. If $G[S]$ is connected, then $G[S]$ contains a spanning tree, which is an S-Steiner tree. Therefore, $d_{G+H}(S) = d_G(S) = k - 1$. If $G[S]$ is not connected, then $d_{G+H}(S) \geq k$. Set $S = \{u_1, u_2, \ldots, u_k\}$. Clearly, the tree induced by the edges in $\{u_i v \,|\, 1 \leq i \leq k\}$ is an S-Steiner tree, where $v \in V(H)$. Therefore, $d_{G+H}(S) \leq k$. So $d_{G+H}(S) = k$. Since x is the number of the k-subsets of $V(G)$ such that the subgraph induced by each k-subset is connected, it follows that the contribution to $SW_k(G + H)$ is

$$(k-1)x + k\left[\binom{n}{k} - x\right] = k\binom{n}{k} - x\,.$$

Suppose that $S \subseteq V(H)$. In a similar manner, since y is the number of the k-subsets of $V(H)$ such that the subgraph induced by each k-subset is connected, it follows that in this case, the contribution to $SW_k(G + H)$ is

$$(k-1)y + k\left[\binom{m}{k} - y\right] = k\binom{n}{k} - y\,.$$

Suppose that $S \cap V(G) \neq \emptyset$ and $S \cap V(H) \neq \emptyset$. Clearly, $d_{G+H}(S) = k-1$. So, in this case, the contribution to $SW_k(G+H)$ is

$$(k-1)\left[\sum_{i=1}^{k-1}\binom{n}{i}\binom{m}{k-i}\right].$$

From the above arguments, we get

$$\begin{aligned} SW_k(G+H) &= (k-1)\left[\sum_{i=1}^{k-1}\binom{n}{i}\binom{m}{k-i}\right]+k\binom{n}{k}+k\binom{m}{k}-x-y \\ &= (k-1)\left[\binom{n+m}{k}-\binom{n}{k}-\binom{m}{k}\right]+k\binom{n}{k}+k\binom{m}{k}-x-y \\ &= (k-1)\binom{n+m}{k}+\binom{n}{k}+\binom{m}{k}-x-y\,. \end{aligned}$$

(3) Since $m < k \leq n$, it follows that for any $S \subseteq V(G+H)$, either $S \subseteq V(G)$ or $S \cap V(G) \neq \emptyset$ and $S \cap V(H) \neq \emptyset$. Suppose that $S \subseteq V(G)$. Similarly to the proof of (2), we conclude that in this case, the contribution to $SW_k(G+H)$ is

$$(k-1)x + k\left[\binom{n}{k}-x\right] = k\binom{n}{k}-x\,.$$

Suppose that $S \cap V(G) \neq \emptyset$ and $S \cap V(H) \neq \emptyset$. Then $d_{G+H}(S) = k-1$, and thus the contribution to $SW_k(G+H)$ is

$$(k-1)\left[\sum_{i=1}^{k-1}\binom{n}{i}\binom{m}{k-i}\right].$$

From the above arguments it follows

$$\begin{aligned} SW_k(G+H) &= (k-1)\left[\sum_{i=1}^{k-1}\binom{n}{i}\binom{m}{k-i}\right]+k\binom{n}{k}-x \\ &= (k-1)\left[\binom{n+m}{k}-\binom{n}{k}-\binom{m}{k}\right]+k\binom{n}{k}-x \\ &= (k-1)\binom{n+m}{k}+\binom{n}{k}+(k-1)\binom{m}{k}-x\,. \end{aligned}$$

□

2.2. **Lexicographical product.**

Theorem 2.2. *Let G be a connected graph with n vertices, and let H be a connected graph with m vertices. Let k be an integer, $2 \leq k \leq nm$. Then*

$$\begin{aligned} SW_k(G[H]) &= nk\binom{m}{k} - nx + \sum_{\ell=2}^{k}\binom{m}{r_1}\binom{m}{r_2}\cdots\binom{m}{r_\ell}SW_\ell(G) \\ &+ \sum_{\ell=2}^{k}(k-\ell)\binom{n}{\ell}\binom{m\ell-\ell}{k-\ell} \end{aligned} \tag{2.1}$$

where $\sum_{i=1}^{\ell} r_i = k$, $r_i \geq 1$ *and* x *is the number of the* k*-subsets of* $V(H)$ *such that the subgraph induced by each* k*-subset is connected in* H.

Proof. For any $S \subseteq V(G)$ and $|S| = k$, if there exists some $H(u_j)$ such that $S \subseteq V(H(u_j))$, then the contribution to $SW_k(G[H])$ is

$$(k-1)x + k\left[\binom{m}{k} - x\right] = k\binom{m}{k} - x\,.$$

Since in $G[H]$ there are n copies of H, i.e., $H(u_j)$ $(1 \leq i \leq n)$, it follows that the contribution to $SW_k(G[H])$ is

$$nk\binom{m}{k} - nx\,. \tag{2.2}$$

From now on, we assume that the vertices in S belong to at least two copies of H in $G[H]$. Without loss of generality, we assume that $H(u_1), H(u_2), \ldots, H(u_\ell)$ satisfy $S \cap V(H(u_i)) \neq \emptyset$ for each u_i $(1 \leq i \leq \ell)$, and $S \cap V(H(u_i)) = \emptyset$ for each u_i $(\ell+1 \leq i \leq n)$. Set $|S \cap V(H(u_i))| = r_i$. Then $\sum_{i=1}^{\ell} r_i = k$. Without loss of generality, let $S_i = S \cap V(H(u_i)) = \{(u_i, v_j) \,|\, 1 \leq j \leq r_i\}$ for each u_i $(1 \leq i \leq \ell)$. Pick up one vertex, say (u_i, v_{j_i}), from $H(u_i)$, where $1 \leq i \leq \ell$ and $v_{j_i} \in \{v_1, v_2, \ldots, v_{r_i}\}$. Set $S' = \{(u_i, v_{j_i}) \,|\, 1 \leq i \leq \ell\}$. Then $|S'| = \ell$, $d_{G[H]}(S') = d_G(\{u_1, u_2, \ldots, u_\ell\})$. Since in G there exists a Steiner tree T_G connecting $\{u_1, u_2, \ldots, u_\ell\}$ of size $d_G(\{u_1, u_2, \ldots, u_\ell\})$, it follows that in $G[H]$ there is a Steiner tree T' connecting S' of size $d_G(\{u_1, u_2, \ldots, u_\ell\})$. Note that the S'-Steiner tree T' is a subtree of the S-Steiner tree T, where $S' \subseteq S$.

We first consider the contribution of these subtrees. Note that there are $\binom{m}{r_1}\binom{m}{r_2}\cdots\binom{m}{r_\ell}$ ways to choose $S_1, S_2, \ldots, S_\ell$. For fixed $S_1, S_2, \ldots, S_\ell$, there is a Steiner tree T' connecting S' of size $d_G(\{u_1, u_2, \ldots, u_\ell\})$. When $\{u_1, u_2, \ldots, u_\ell\}$ takes over all ℓ-subset of $V(G)$, the contribution of these subtrees to $SW_k(G[H])$ is

$$\sum_{\ell=2}^{k} \binom{m}{r_1}\binom{m}{r_2}\cdots\binom{m}{r_\ell} SW_\ell(G)\,. \tag{2.3}$$

We next consider the remaining contribution to $SW_k(G[H])$. For given $S_1, S_2, \ldots, S_\ell$, there is a Steiner tree T' connecting S' of size $d_G(\{u_1, u_2, \ldots, u_\ell\})$. We now extend this subtree T' to a Steiner tree connecting S in $G \circ H$. For each vertex (u, v) in $(\bigcup_{i=1}^{\ell} S_i) \setminus S'$, there exists a vertex in $V(T')$, say (u', v'), such that $(u, v)(u', v') \in E(G[H])$. By adding all these edges to $E(T)$, we can obtain an S-Steiner tree, say T. The total number of edges adding to T is $k - \ell$. Observe that there are $\binom{n}{\ell}\binom{m\ell-\ell}{k-\ell}$ ways to choose the vertices in $(\bigcup_{i=1}^{\ell} S_i) \setminus S'$. So, in this case, the contribution to $SW_k(G[H])$ is

$$\sum_{\ell=2}^{k} (k-\ell)\binom{n}{\ell}\binom{m\ell-\ell}{k-\ell}\,. \tag{2.4}$$

Combining (2.2), (2.3), and (2.4), we arrive at Eq. (2.1). □

2.3. **Cartesian product.**

Theorem 2.3. *Let G be a connected graph with n vertices, and let H be a connected graph with m vertices. Let k be an integer with $2 \leq k \leq nm$. Then*

$$\sum_{x=2}^{k}\binom{m}{r_1}\binom{m}{r_2}\cdots\binom{m}{r_x}SW_x(G)+\sum_{y=2}^{k}\binom{n}{s_1}\binom{n}{s_2}\cdots\binom{n}{s_y}SW_y(G)$$

$$\leq \quad SW_k(G\Box H) \leq \frac{k}{2}\left[\sum_{x=2}^{k}\binom{m}{r_1}\binom{m}{r_2}\cdots\binom{m}{r_x}SW_x(G)+\sum_{x=2}^{k}\binom{n}{s_1}\binom{n}{s_2}\cdots\binom{n}{s_y}SW_y(G)\right]$$

where $\sum_{i=1}^{x} r_i = k$ and $r_i \geq 1$, and $\sum_{i=1}^{y} s_i = k$ and $s_i \geq 1$.

In [13], the following results was obtained, which will be used in the proof of the above theorem.

Lemma 2.4. [13] *Let $k \geq 2$. Let $S = \{(u_1, v_1), (u_2, v_2), \ldots, (u_k, v_k)\}$ be a set of distinct vertices of $G\Box H$. Let $S_G = \{u_1, u_2, \ldots, u_k\}$ and $S_H = \{v_1, v_2, \ldots, v_k\}$. Then*

$$d_{G\Box H}(S) \geq d_G(S_G) + d_H(S_H).$$

In order to prove Theorem 2.3 we need the following two lemmas.

Lemma 2.5. *Let G be a connected graph with n vertices, and let H be a connected graph with m vertices. Let k be an integer, $2 \leq k \leq nm$. Then*

$$SW_k(G\Box H) \geq \sum_{x=2}^{k}\binom{m}{r_1}\binom{m}{r_2}\cdots\binom{m}{r_x}SW_x(G)+\sum_{y=2}^{k}\binom{n}{s_1}\binom{n}{s_2}\cdots\binom{n}{s_y}SW_y(G)$$

where $\sum_{i=1}^{x} r_i = k$ and $r_i \geq 1$, and $\sum_{i=1}^{y} s_i = k$ and $s_i \geq 1$.

Proof. For any $S \subseteq V(G\Box H)$ and $|S| = k$, all the vertices in S belong to some copies of G and some copies of H. Without loss of generality, let $H(u_1), H(u_2), \ldots, H(u_x)$ be all the copies of H such that $S \cap V(H(u_i)) \neq \emptyset$ $(1 \leq i \leq x)$ and $S \cap V(H(u_i)) = \emptyset$ $(x+1 \leq i \leq n)$, and let $G(v_1), G(v_2), \ldots, G(v_x)$ be all the copies of G such that $S \cap V(G(v_j)) \neq \emptyset$ $(1 \leq j \leq y)$ and $S \cap V(G(v_j)) = \emptyset$ $(y+1 \leq i \leq m)$. Observe that $1 \leq x \leq k$ and $1 \leq y \leq k$. Let $S = \{(u_1, v_1), (u_2, v_2), \ldots, (u_k, v_k)\}$ be a set of distinct vertices of $G\Box H$. Then $S_G = \{u_1, u_2, \ldots, u_k\} \subseteq \bigcup_{i=1}^{x} V(H(u_i))$ and $S_H = \{v_1, v_2, \ldots, v_k\} \subseteq \bigcup_{i=1}^{y} V(G(v_i))$. Note that u_i and u_j are not necessarily different for $1 \leq i \neq j \leq k$, and v_i and v_j are not necessarily different for $1 \leq i \neq j \leq k$. Without loss of generality, let $|S_G \cap V(H(u_i))| = r_i$ and $|S_H \cap V(G(v_j))| = s_j$, where $1 \leq i \leq x$ and $1 \leq j \leq y$. It is clear that $\sum_{i=1}^{x} r_i = k$ and $r_i \geq 1$, and $\sum_{i=1}^{y} s_i = k$ and $s_i \geq 1$. From Lemma 2.4, $d_{G\Box H}(S) \geq d_G(S_G) + d_H(S_H)$ for any $S \subseteq V(G\Box H)$.

Note that we have $\sum_{x=1}^{k}\binom{m}{r_1}\binom{m}{r_2}\cdots\binom{m}{r_x}$ ways to determine $S_G = \{u_1, u_2, \ldots, u_k\}$. From the definition, the Steiner distance of S_G in G is at least $d_G(S_G)$. So, in this case, the contribution to $SW_k(G\Box H)$ is at least

$$\sum_{x=1}^{k}\binom{m}{r_1}\binom{m}{r_2}\cdots\binom{m}{r_x}SW_x(G)\,.$$

Similarly, since we have $\sum_{y=1}^{k}\binom{n}{s_1}\binom{n}{s_2}\cdots\binom{n}{s_y}$ ways to determine $S_H=\{v_1,v_2,\ldots,v_k\}$, it follows that the respective contribution to $SW_k(G\Box H)$ is at least

$$\sum_{y=1}^{k}\binom{n}{s_1}\binom{n}{s_2}\cdots\binom{n}{s_y}SW_y(G)\,.$$

If $x=1$, then $SW_1(G)=0$. If $y=1$, then $SW_1(H)=0$. Therefore,

$$SW_k(G\Box H)\geq\sum_{x=2}^{k}\binom{m}{r_1}\binom{m}{r_2}\cdots\binom{m}{r_x}SW_x(G)+\sum_{y=2}^{k}\binom{n}{s_1}\binom{n}{s_2}\cdots\binom{n}{s_y}SW_y(G)$$

where $\sum_{i=1}^{x}r_i=k$ and $r_i\geq 1$, and $\sum_{i=1}^{y}s_i=k$ and $s_i\geq 1$. □

Lemma 2.6. *Let G be a connected graph with n vertices, and let H be a connected graph with m vertices. Let k be an integer, $2\leq k\leq nm$. Then*

$$SW_k(G\Box H)\leq\frac{k}{2}\left[\sum_{x=2}^{k}\binom{m}{r_1}\binom{m}{r_2}\cdots\binom{m}{r_x}SW_x(G)+\sum_{x=2}^{k}\binom{n}{s_1}\binom{n}{s_2}\cdots\binom{n}{s_y}SW_y(G)\right]$$

where $\sum_{i=1}^{x}r_i=k$ and $r_i\geq 1$, and $\sum_{i=1}^{y}s_i=k$ and $s_i\geq 1$.

Proof. For any $S\subseteq V(G\Box H)$ and $|S|=k$, all the vertices in S belong to some copies of G and some copies of H. Without loss of generality, let $H(u_1),H(u_2),\ldots,H(u_x)$ be all the copies of H such that $S\cap V(H(u_i))\neq\emptyset$ $(1\leq i\leq x)$ and $S\cap V(H(u_i))=\emptyset$ $(x+1\leq i\leq n)$, and let $G(v_1),G(v_2),\ldots,G(v_x)$ be all the copies of G such that $S\cap V(G(v_j))\neq\emptyset$ $(1\leq j\leq y)$ and $S\cap V(G(v_j))=\emptyset$ $(y+1\leq i\leq m)$. Observe that $1\leq x\leq k$ and $1\leq y\leq k$. Let $S=\{(u_1,v_1),(u_2,v_2),\ldots,(u_k,v_k)\}$ be a set of distinct vertices of $G\Box H$. Then $S_G=\{u_1,u_2,\ldots,u_k\}\subseteq\bigcup_{i=1}^{x}V(H(u_i))$ and $S_H=\{v_1,v_2,\ldots,v_k\}\subseteq\bigcup_{i=1}^{y}V(G(v_i))$. Without loss of generality, let $|S_G\cap V(H(u_i))|=r_i$ and $|S_H\cap V(G(v_j))|=s_j$, where $1\leq i\leq x$ and $1\leq j\leq y$. It is clear that $\sum_{i=1}^{x}r_i=k$ and $r_i\geq 1$, and $\sum_{i=1}^{y}s_i=k$ and $s_i\geq 1$.

Suppose that $x\leq k-1$. For each i $(1\leq i\leq x)$, there is an S_H-Steiner tree in $H(u_i)$, say T_i'. Similarly, there is an S_G-Steiner tree in $G(v_1)$, say T. Then the tree induced by the edges in $E(\bigcup_{i=1}^{x}T_i')\cup E(T)$ is an S-Steiner tree in $G\Box H$, and thus

$$d_{G\Box H}(S)\leq d_G(S_G)+xd_H(S_H)\leq d_G(S_G)+(k-1)d_H(S_H)\,.$$

Suppose that $x=k$ and $y=k$. In this case, all the vertices in $S=\{(u_1,v_1),(u_2,v_2),\ldots,(u_k,v_k)\}$ belong to different copies of G and different copies of H. For each i $(1\leq i\leq x-1)$, there is an S_H-Steiner tree in $H(u_i)$, say T_i'. Similarly, there is an S_G-Steiner tree in $G(v_1)$, say T. Then the tree induced by the edges in $E(\bigcup_{i=1}^{x-1}T_i')\cup E(T)$ is an S-Steiner tree in $G\Box H$, and therefore

$$d_{G\Box H}(S)\leq d_G(S_G)+(x-1)d_H(S_H)\leq d_G(S_G)+(k-1)d_H(S_H)\,.$$

Suppose that $x=k$ and $y\leq k-1$. Since $x=k$, it follows that each $H(u_i)$ contains exactly one vertex of S. Since $y\leq k-1$, it follows that there exists some $G(v_j)$ such that $G(v_j)$ contains at least two vertices of S. For each i $(1\leq i\leq x-1)$, there is an S_H-Steiner tree in $H(u_i)$, say T_i'. Similarly, there is an S_G-Steiner tree in $G(v_1)$, say T. Then the tree induced by the edges in $E(\bigcup_{i=1}^{x-1} T_i') \cup E(T)$ is an S-Steiner tree in $G\Box H$, and thus

$$d_{G\Box H}(S) \le d_G(S_G) + (x-1)d_H(S_H) \le d_G(S_G) + (k-1)d_H(S_H)\,.$$

Since there are $\sum_{x=1}^{k} \binom{m}{r_1}\binom{m}{r_2}\cdots\binom{m}{r_x}$ ways to determine $S_G = \{u_1, u_2, \ldots, u_k\}$, it follows that in this case, the contribution to $SW_k(G\Box H)$ is at least

$$\left[\sum_{x=1}^{k} \binom{m}{r_1}\binom{m}{r_2}\cdots\binom{m}{r_x}\right] SW_x(G)\,.$$

Since there are $\sum_{y=1}^{k} \binom{n}{s_1}\binom{n}{s_2}\cdots\binom{n}{s_y}$ ways to determine $S_H = \{v_1, v_2, \ldots, v_k\}$, the respective contribution to $SW_k(G\Box H)$ is at least

$$(k-1)\left[\sum_{y=1}^{k} \binom{n}{s_1}\binom{n}{s_2}\cdots\binom{n}{s_y}\right] SW_y(G)\,.$$

Combining these results, we have

$$SW_k(G\Box H) \le \left[\sum_{x=1}^{k} \binom{m}{r_1}\binom{m}{r_2}\cdots\binom{m}{r_x}\right] SW_x(G) + (k-1)\left[\sum_{y=1}^{k} \binom{n}{s_1}\binom{n}{s_2}\cdots\binom{n}{s_y}\right] SW_y(G)$$

and similarly,

$$SW_k(G\Box H) \le (k-1)\left[\sum_{x=1}^{k} \binom{m}{r_1}\binom{m}{r_2}\cdots\binom{m}{r_x}\right] SW_x(G) + \left[\sum_{y=1}^{k} \binom{n}{s_1}\binom{n}{s_2}\cdots\binom{n}{s_y}\right] SW_y(G)\,.$$

This yields

$$SW_k(G\Box H) \le \frac{k}{2}\left[\sum_{x=2}^{k} \binom{m}{r_1}\binom{m}{r_2}\cdots\binom{m}{r_x} SW_x(G) + \sum_{x=2}^{k} \binom{n}{s_1}\binom{n}{s_2}\cdots\binom{n}{s_y} SW_y(G)\right]$$

where $\sum_{i=1}^{x} r_i = k$ and $r_i \ge 1$, and $\sum_{i=1}^{y} s_i = k$ and $s_i \ge 1$. □

Theorem 2.3 follows from Lemmas 2.5 and 2.6.

Remark 2.7. *Suppose that $k = 2$. Then $x = y = 2$, $r_1 = r_2 = \cdots = r_x = 1$, $\sum_{i=1}^{x} r_i = 2$, $s_1 = s_2 = \cdots = s_y = 1$, $\sum_{i=1}^{y} s_i = 2$. Therefore,*

$$SW_2(G\Box H) = m^2\, SW_2(G) + n^2\, SW_2(H)\,.$$

Thus, the upper and lower bounds in Theorem 2.3 are sharp.

2.4. **Cluster and corona.** Let v is the root vertex of H and

$$d(v, k|H) = \sum_{\substack{v\in V(H),\, S\subseteq V(H)\\ |S|=k}} d(S)\,.$$

Theorem 2.8. *Let G be a connected graph with n vertices, and let H be a connected graph with m vertices. Let k be an integer, $2 \leq k \leq nm$. Then*

$$\begin{aligned} SW_k(G \odot H) &= n\,SW_k(H) + \sum_{\ell=2}^{k} \binom{m}{r_1}\binom{m}{r_2}\cdots\binom{m}{r_\ell} SW_\ell(G) \\ &+ \sum_{\ell=2}^{k} \binom{n}{\ell} \left[\sum_{j=1}^{\ell} \prod_{\substack{x=1 \\ x\neq j}}^{\ell} \binom{m}{r_x} d(v,k|H) \right] \end{aligned}$$

where $\sum_{x=1}^{\ell} r_x = k$, $r_x \geq 1$ and v is the root-vertex of H.

Proof. For any $S \subseteq V(G \odot H)$ and $|S| = k$, if all the vertices in S belong to some copy of H, then $d_{G\odot H}(S) = d_H(S)$ and hence the contribution to $SW_k(G \odot H)$ is $SW_k(H)$. Since there are n copies of H, the contribution of for $SW_k(G \odot H)$ is $n\,SW_k(H)$.

From now on, we assume that all the vertices in S belong to different copies of H. Set $V(G) = \{u_1, u_2, \ldots, u_n\}$. Then $H(u_1), H(u_2), \ldots, H(u_n)$ are all copies of H in $G \odot H$. We now assume that the vertices in S belong to at least two copies of H. Without loss of generality, let $H(u_1), H(u_2), \ldots, H(u_\ell)$ be all the copies of H such that $S \cap V(H(u_i)) \neq \emptyset$ $(1 \leq i \leq \ell)$ and $S \cap V(H(u_i)) = \emptyset$ $(\ell+1 \leq i \leq n)$. Observe that $1 \leq \ell \leq k$. Let $|S \cap V(H(u_x))| = r_x$, where $1 \leq x \leq \ell$. If we find a Steiner tree T_G connecting $\{u_1, u_2, \ldots, u_\ell\}$ in G, and a Steiner tree T_i connecting the vertices in $S_i = S \cap V(H(u_i))$ for each $H(u_i)$, then the tree induced by the edges in $E(T_G) \cup E(\bigcup_{x=1}^{\ell} T_x)$ is a Steiner tree connecting S in $G \odot H$.

For $H(u_1), H(u_2), \ldots, H(u_\ell)$, there are $\binom{m}{r_1}\binom{m}{r_2}\cdots\binom{m}{r_\ell}$ ways to determine all the vertices of $S_i = S \cap V(H(u_i))$ for $1 \leq i \leq \ell$, and there are $\binom{m}{r_1}\binom{m}{r_2}\cdots\binom{m}{r_\ell}$ ways to choose $T_1, T_2, \ldots, T_\ell$. So we need $\binom{m}{r_1}\binom{m}{r_2}\cdots\binom{m}{r_\ell}$ Steiner trees connecting $\{u_1, u_2, \ldots, u_\ell\}$ in G. Therefore, their contribution to $SW_k(G \odot H)$ is

$$\sum_{\ell=2}^{k} \binom{m}{r_1}\binom{m}{r_2}\cdots\binom{m}{r_\ell} SW_\ell(G)$$

where $\sum_{x=1}^{\ell} r_x = k$ and $r_x \geq 1$.

For $T_1, T_2, \ldots, T_\ell$, if we change some S_j and T_j, then we have $\binom{m}{r_1}\binom{m}{r_2}\cdots\binom{m}{r_{j-1}}\binom{m}{r_{j+1}}\cdots\binom{m}{r_\ell}$ ways to determine such $S_1, S_2, \ldots, S_\ell$ or $T_1, T_2, \ldots, T_\ell$. Therefore, the respective contribution to $SW_k(G \odot H)$ is

$$\sum_{\ell=2}^{k} \binom{n}{\ell}\binom{m}{r_1}\binom{m}{r_2}\cdots\binom{m}{r_{j-1}}\binom{m}{r_{j+1}}\cdots\binom{m}{r_\ell} d(u_j, k|H(u_j))$$

where $\sum_{x=1}^{\ell} r_x = k$, $r_x \geq 1$, and u_j is the root vertex of $H(u_j)$. From the definition of $G \odot H$, in this case, the contribution to $SW_k(G \odot H)$ is

$$\sum_{\ell=2}^{k} \binom{n}{\ell}\binom{m}{r_1}\binom{m}{r_2}\cdots\binom{m}{r_{j-1}}\binom{m}{r_{j+1}}\cdots\binom{m}{r_\ell} d(v, k|H),$$

where $\sum_{x=1}^{\ell} r_x = k$, $r_x \geq 1$, and v is the root-vertex of H.

Combining the above specified contributions, we obtain

$$\begin{aligned} SW_k(G \odot H) &= nSW_k(H) + \sum_{\ell=2}^{k} \binom{m}{r_1}\binom{m}{r_2}\cdots\binom{m}{r_\ell} SW_\ell(G) \\ &+ \sum_{\ell=2}^{k} \binom{n}{\ell} \left[\sum_{j=1}^{\ell} \prod_{\substack{x=1 \\ x\neq j}}^{\ell} \binom{m}{r_x} d(u_j|H) \right] \end{aligned}$$

where $\sum_{x=1}^{\ell} r_x = k$ and $r_x \geq 1$. Theorem 2.8 follows. □

Theorem 2.9. *Let G be a connected graph with n vertices, and let H be a connected graph with m vertices. Let k be an integer, $2 \leq k \leq nm$. Then*

$$\begin{aligned} SW_k(G \ominus H) &= \sum_{\ell=2}^{k} \binom{m+1}{r_1}\binom{m+1}{r_2}\cdots\binom{m+1}{r_\ell} SW_\ell(G) + \binom{m}{k-1}(k-1)n + kn\binom{m}{k} - xn \\ &+ \sum_{\ell=2}^{k} \binom{n}{\ell} \left[\sum_{j=1}^{\ell} \prod_{\substack{x=1 \\ x\neq j}}^{\ell} \binom{m+1}{r_x} \left[\binom{m}{r_j-1}(r_j-1) + r_j\binom{m}{r_j} - x_j \right] \right] \end{aligned} \tag{2.5}$$

where $\sum_{x=1}^{\ell} r_x = k$, $r_x \geq 1$, x is the number of the k-subsets of $V(H)$ such that the subgraph induced by each k-subset is connected in H, and x_j is the number of the r_j-subsets of $V(H)$ such that the subgraph induced by each r_j-subset is connected in H.

Proof. Note that $SW_k(G \ominus H) = SW_k(G \odot (H + K_1))$, where $K_1 = \{v\}$. It follows that

$$SW_k(H + K_1) = \binom{m}{k-1}(k-1) + x(k-1) + k(\binom{m}{k} - x) = \binom{m}{k-1}(k-1) + k\binom{m}{k} - x\,.$$

Since v is the root vertex of $H + K_1$, it follows that

$$d(v, k|H) = \binom{m}{r_j-1}(r_j-1) + x_j(r_j-1) + r_j\left[\binom{m}{r_j} - x_j\right] = \binom{m}{r_j-1}(r_j-1) + r_j\binom{m}{r_j} - x_j\,.$$

This implies Eq. (2.5). □

Remark 2.10. *One can see that Theorems 2.1, 2.2, 2.8 and 2.9 are extensions of Theorems 1.2, 1.3, 1.5 and 1.6, respectively. In all considered case for $k = 2$ the new results can be reduced to already known ones.*

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