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A NEIGHBORHOOD UNION CONDITION FOR FRACTIONAL (k, n', m)-CRITICAL DELETED GRAPHS

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Abstract. A graph G is called a fractional (k, n', m)-critical deleted graph if any n' vertices are removed from G the resulting graph is a fractional (k, m)-deleted graph. In this paper, we prove that for integers $k \geq 2$, $n', m \geq 0$, $n \geq 8k + n' + 4m - 7$, and $\delta(G) \geq k + n' + m$, if

$$|N_G(x) \cup N_G(y)| \geq \frac{n + n'}{2}$$

for each pair of non-adjacent vertices x, y of G, then G is a fractional (k, n', m)-critical deleted graph. The bounds for neighborhood union condition, the order n and the minimum degree $\delta(G)$ of G are all sharp.

1. Introduction

All graphs considered in this paper are finite, loopless, and without multiple edges. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For any $x \in V(G)$, the degree and the neighborhood of x in G are denoted by $d_G(x)$ and $N_G(x)$, respectively. For $S \subseteq V(G)$, we denote by $G[S]$ the subgraph of G induced by S, and $G - S = G[V(G) \setminus S]$. For two vertex-disjoint subsets S and T of G, we use $e_G(S, T)$ to denote the number of edges with one end in S and the other end in T. We denote the minimum degree and the maximum degree of G by $\delta(G)$ and $\Delta(G)$, respectively. The *distance* $d_G(x, y)$ between two vertices x and y is defined to be the length of a shortest path connecting them. The notation and terminology used but undefined in this paper can be found in [1].

Let $k \geq 1$ be an integer. A spanning subgraph F of G is called a *k-factor* if $d_F(x) = k$ for each $x \in V(G)$. Let $h : E(G) \to [0, 1]$ be a function. If $\sum_{x \in e} h(e) = k$ for any $x \in V(G)$, then we call

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$G[F_h]$ a *fractional* k*-factor* of G with indicator function h where $F_h = \{e \in E(G) : h(e) > 0\}$. Zhou [9] introduced the concept of a fractional (k, m)-deleted graph, that is, a graph G is called a *fractional* (k, m)*-deleted graph* if removing any m edges from G, the resulting graph has a fractional k-factor. A fractional (k, m)-deleted graph is simply called a fractional k-deleted graph if $m = 1$. A graph G is called a *fractional* (k, n')*-critical graph* if after deleting any n' vertices from G, the resulting graph still has a fractional k-factor.

The third author of this paper first introduced the concept of a fractional (k, n', m)-critical deleted graph. A graph G is called a *fractional* (k, n', m)*-critical deleted graph* if after deleting any n' vertices from G, the resulting graph is still a fractional (k, m)-deleted graph.

In what follows, we always assume that n is order of G, i.e., $n = |V(G)|$. Yu et al. [7] provided a degree condition for the existence of a fractional k-factor as follows.

Theorem 1.1. (Yu et al. [7]) *Let $k \geq 1$ be an integer, and let G be a connected graph with $n \geq 4k-3$ and $\delta(G) \geq k$. If*

$$\max\{d_G(x), d_G(y)\} \geq \frac{n}{2}$$

for each pair of nonadjacent vertices x, y of G, then G has a fractional k-factor.

Let $\omega(G - S)$ denote the number of components of $G - S$. The *toughness* $t(G)$ of a graph G is defined as follows: $t(G) = +\infty$ if G is a complete graph; otherwise, $t(G) = \min\{\frac{|S|}{\omega(G-S)} : S \subseteq V(G), \omega(G - S) \geq 2\}$.

Liu and Zhang [6] obtained the following toughness condition for a graph to have fractional k-factors.

Theorem 1.2. (Liu and Zhang [6]) *Let $k \geq 2$ be an integer. A graph G of order n with $n \geq k+1$ has a fractional k-factor if $t(G) \geq k - \frac{1}{k}$.*

For fractional (k, m)-deleted graphs, we have the following known results.

Theorem 1.3. (Zhou [9]). *Let $k \geq 2$ and $m \geq 0$ be two integers. Let G be a connected graph with $n \geq 9k - 1 - \sqrt{2(k-1)^2 + 2} + 2(2k+1)m$ and $\delta(G) \geq k + m + \frac{(m+1)^2-1}{4k}$. If*

$$|N_G(x) \cup N_G(y)| \geq \frac{1}{2}(n + k - 2)$$

for each pair of non-adjacent vertices x, y of G, then G is a fractional (k, m)-deleted graph.

Theorem 1.4. (Zhou [8]). *Let $k \geq 1$ and $m \geq 1$ be two integers. Let G be a graph with $n \geq 4k - 5 + 2(2k+1)m$. If $\delta(G) \geq \frac{n}{2}$, then G is a fractional (k, m)-deleted graph.*

More sufficient conditions for graphs to have fractional factors can be found in Gao and Gao [3], Gao et al. [4], Gao and Wang [5], Zhou [9], [10] and [11], and Zhou and Bian [12].

In this paper, we give the following result on the neighborhood union condition for fractional (k, n', m)-critical deleted graphs.

Theorem 1.5. *Let $k \geq 2$ and $n', m \geq 0$ be three integers, and let G be a graph with $n \geq 8k+n'+4m-7$ and $\delta(G) \geq k+n'+m$. If*

$$|N_G(x) \cup N_G(y)| \geq \frac{n+n'}{2}$$

for each pair of non-adjacent vertices x, y of G, then G is a fractional (k, n', m)-critical deleted graph.

Set $n' = 0$ in Theorem 1.5, then it becomes the following necessary condition on fractional (k, m)-deleted graph.

Corollary 1.6. *Let $k \geq 2$ and $m \geq 0$ be two integers, and let G be a graph with $n \geq 8k + 4m - 7$ and $\delta(G) \geq k + m$. If*

$$|N_G(x) \cup N_G(y)| \geq \frac{n}{2}$$

for each pair of non-adjacent vertices x, y of G, then G is a fractional (k, m)-deleted graph.

If $m = 0$ in Theorem 1.5, then we obtain the following corollary.

Corollary 1.7. *Let $k \geq 2$ and $n' \geq 0$ be two integers, and let G be a graph with $n \geq 8k + n' - 7$ and $\delta(G) \geq k + n'$. If*

$$|N_G(x) \cup N_G(y)| \geq \frac{n+n'}{2}$$

for each pair of non-adjacent vertices x, y of G, then G is a fractional (k, n')-critical graph.

We will show that the bounds for neighborhood union condition, the order and the minimum degree of G are all sharp. In order to prove our main result, we need the following Lemma.

Lemma 1.8. (Gao [2]) *Let $k \geq 1$ and $n', m \geq 0$ be three integers, and let G be a graph and H a subgraph of G with m edges. Then G is a fractional (k, n', m)-critical deleted graph if and only if*

$$\delta_G(S,T) = k|S| + \sum_{x \in T} d_{G-S}(x) - k|T| \geq kn' + \sum_{x \in T} d_H(x) - e_H(S,T),$$

for all disjoint subsets S and T of $V(G)$ with $|S| \geq n'$.

2. Proof of Theorem 1.5

Suppose that G satisfies the conditions of Theorem 1.5, but is not a fractional (k, n', m)-critical deleted graph. According to Lemma 1.8 and $\sum_{x \in T} d_H(x) - e_H(S,T) \leq 2m$, there exist disjoint subsets S and T of $V(G)$ with $|S| \geq n'$ such that

$$\delta_G(S,T) = k|S| + \sum_{x \in T} d_{G-S}(x) - k|T| \leq kn' + 2m - 1, \tag{2.1}$$

We choose subsets S and T such that $|T|$ is minimum. Obviously, $T \neq \emptyset$.

Lemma 2.1. $d_{G-S}(x) \leq k - 1$ *for any $x \in T$.*

Proof. If $d_{G-S}(x) \geq k$ for some $x \in T$, then the subsets S and $T \setminus \{x\}$ satisfy (2.1). This contradicts the choice of S and T. □

Let $d_1 = \min\{d_{G-S}(x) : x \in T\}$ and choose $x_1 \in T$ such that $d_{G-S}(x_1) = d_1$. If $T - N_T[x_1] \neq \emptyset$, let $d_2 = \min\{d_{G-S}(x) : x \in T - N_T[x_1]\}$ and choose $x_2 \in T - N_T[x_1]$ such that $d_{G-S}(x_2) = d_2$. So, $d_1 \leq d_2$. Let $|S| = s$, $|T| = t$, $|N_T[x_1]| = p$. Then, $p \leq d_1 + 1$, $d_{G-S}(T) \geq d_1 p + d_2(t-p)$, and $k(s-n') - kt + d_1 p + d_2(t-p) - 2m \leq k|S| - k|T| + d_{G-S}(T) - kn' - 2m < 0$.

We have

$$\begin{aligned} |S| &\leq \frac{k|T| - d_{G-S}(T) + (kn' + \sum_{x\in T} d_H(x) - e_H(S,T) - 1)}{k} \\ &\leq \frac{k|T| + (kn' + 2m - 1)}{k}. \end{aligned}$$

Thus, $|S| \leq |T| + n' + \frac{2m-1}{k}$. If $|S| \leq n' + m$, then $|T| = 0$ by Claim 2.1 and $\delta(G) \geq k + n' + m$, which is a contradiction. So, $m + n' + 1 \leq s \leq t + n' + \frac{2m-1}{k}$. We consider following two cases:

Case 1. $T = N_T[x_1]$. In this case, $t \leq d_1 + 1$ and $d_2 = 0$. If $d_1 = k-1$, then $t \leq k$, $k|S| - k|T| + d_{G-S}(T) - kn' - 2m \geq k(s-n') - kt + d_1 p - 2m = k(s-n') - kt + (k-1)t - 2m \geq k(s-n') - k - 2m \geq k(m+1) - k - 2m \geq 0$. If $0 \leq d_1 \leq k-2$, then $t \leq d_1 + 1 \leq k-1$. By $\delta(G) \geq k + n' + m$ and $d_G(x_1) \leq s + d_1$, we have $s \geq k + n' + m - d_1$. Thus, $k|S| - k|T| + d_{G-S}(T) - kn' - 2m \geq k(s-n') - kt + d_1 p - 2m \geq k(k+m-d_1) + (d_1 - k)t - 2m = (k-d_1)(k-t) + km - 2m > 0$. It is a contradiction.

Case 2. $T - N_T[x_1] \neq \emptyset$. We consider following three subcases.

Case 2.1. $d_1 = d_2 = k-1$. In this subcase, $k|S| - k|T| + d_{G-S}(T) - kn' - 2m \geq k(s-n') - kt + d_1 p + d_2(t-p) - 2m = k(s-n') - kt + (k-1)p + (k-1)(t-p) - 2m = k(s-n') - t - 2m \geq 0$, which is a contradiction. In fact, if $k(s-n') \leq t + 2m - 1$, then $s \leq \frac{t+2m-1}{k} + n'$, $s + k(s-n') - 2m + 1 \leq s + t \leq n$. Thus, $s + 2k - 2 \geq |N_G(x_1) \cup N_G(x_2)| \geq \frac{n+n'}{2} \geq \frac{s+(s-n')k-2m+1+n'}{2}$ since $x_1x_2 \notin E(G)$. Then $4k \geq (k-1)s + 2(2-m) + 1 + n'(1-k) \geq (k-1)s + 2(3+n'-s) + 1 + n'(1-k) = (k-3)s + 7 + n'(3-k)$, i.e., $s \leq \frac{4k-7-n'(3-k)}{k-3}$ if $k \geq 5$. Then $\frac{4k-7-n'(3-k)}{k-3} + 2k - 2 \geq s + 2k - 2 \geq |N_G(x_1) \cup N_G(x_2)| \geq \frac{n+n'}{2}$, i.e., $\frac{4k-7-n'(3-k)}{k-3} + 2k - 2 \geq 4k + 2m + n' - 3$ since $s + 2k - 2$ is an integer. Then we have $\frac{4k-7-n'(3-k)}{k-3} \geq 2k + n' + 2m - 1 \geq 2k + n' - 1$. That is to say $\frac{4k-7}{k-3} \geq 2k - 1$, which contradicts the fact that $k \geq 5$.

If $k = 4$, we have $s \geq \frac{n+n'}{2} - 6$ and $t \leq n - s \leq \frac{n-n'}{2} + 6$ sine $s + 2k - 2 \geq |N_G(x_1) \cup N_G(x_2)| \geq \frac{n+n'}{2}$. Thus, $k|S| - k|T| + d_{G-S}(T) - kn' - 2m \geq k(\frac{n+n'}{2} - 6) - k(\frac{n-n'}{2} + 6) + 3(\frac{n-n'}{2} + 6) - kn' - 2m > 0$, a contradiction.

If $k = 3$, we have $s \geq \frac{n+n'}{2} - 4$ and $t \leq n - s \leq \frac{n-n'}{2} + 4$. Thus, $k|S| - k|T| + d_{G-S}(T) - kn' - 2m \geq k(\frac{n+n'}{2} - 4) - k(\frac{n-n'}{2} + 4) + 2(\frac{n-n'}{2} + 4) - kn' - 2m > 0$, a contradiction.

If $k = 2$, we have $s \geq \frac{n+n'}{2} - 2$ and $t \leq n - s \leq \frac{n-n'}{2} + 2$. If $n \geq 8k + n' + 4m - 4$, then $k|S| - k|T| + d_{G-S}(T) - kn' - 2m \geq k(\frac{n+n'}{2} - 2) - k(\frac{n-n'}{2} + 2) + (\frac{n-n'}{2} + 2) - kn' - 2m > 0$, a contradiction. If $n = 8k + n' + 4m - 5$, then $s \geq \frac{n+n'}{2} - 2$ implies $s \geq 4k + n' + 2m - 4$ and $t \leq 4k + 2m - 1$. Thus, $k|S| - k|T| + d_{G-S}(T) - kn' - 2m \geq k(4k + n' + 2m - 4) - (k-1)(4k + 2m - 1) - kn' - 2m = k - 1 > 0$, which is a contradiction. If $n = 8k + n' + 4m - 6$, then $s \geq \frac{n+n'}{2} - 2$ implies $s \geq 4k + n' + 2m - 5$ and $t \leq 4k + 2m - 1$. If $s \geq 4k + n' + 2m - 4$ or $t \leq 4k + 2m - 2$, then $k|S| - k|T| + d_{G-S}(T) - kn' - 2m \geq 0$. If $s = 4k + n' + 2m - 5$ and $t = 4k + 2m - 1$, then at least one vertex in T has degree at least 2 since $n = s + t$ and t is odd. Thus, $k|S| - k|T| + d_{G-S}(T) - kn' - 2m \geq k(4k + n' + 2m - 5) - k(4k + 2m - 1) + (4k + 2m - 1) + 1 - kn' - 2m = 0$, which is a contradiction. If $n = 8k + n' + 4m - 7$, then $s \geq \frac{n+n'}{2} - 2$ implies $s \geq 4k + n' + 2m - 5$ and $t \leq 4k + 2m - 2$. Thus, $k|S| - k|T| + d_{G-S}(T) - kn' - 2m \geq k(4k + n' + 2m - 5) - k(4k + 2m - 2) + (4k + 2m - 2) - kn' - 2m = k - 2 \geq 0$, which is a contradiction.

Case 2.2. $0 \leq d_1 \leq k - 2$ **and** $d_2 = k - 1$**.** In this subcase, $p \leq d_1 + 1 \leq k - 1$. $s + k - 1 + d_1 \geq |N_G(x_1) \cup N_G(x_2)| \geq \frac{n+n'}{2} \geq 4k + n' + 2m - 3$ since $s + k + d_1 - 1$ is an integer, i.e., $n \leq 2s + 2k - 2 + 2d_1 - n'$ and $s \geq 3k + n' + 2m - d_1 - 2$. Thus,

$$
\begin{aligned}
& k|S| - k|T| + d_{G-S}(T) - kn' - 2m \\
\geq\ & k(s - n') - kt + d_1 p + d_2(t - p) - 2m \\
\geq\ & k(s - n') - k(n - s) + (d_1 - k + 1)(d_1 + 1) + (k - 1)(n - s) - 2m \\
=\ & (k + 1)s - n - kn' - 2m - k + 1 + d_1^2 + (2 - k)d_1 \\
\geq\ & (k + 1)s - (2k + 2s - 2 + 2d_1 - n') - kn' - 2m - k + 1 + d_1^2 + (2 - k)d_1 \\
=\ & (k - 1)s - 3k - 2m + 3 + d_1^2 - kd_1 - (k - 1)n' \\
\geq\ & (k - 1)(3k + n' + 2m - d_1 - 2) - 3k - 2m + 3 + d_1^2 - kd_1 - (k - 1)n' \\
=\ & 3k^2 + (2k - 4)m - 8k + 5 + d_1^2 - (2k - 1)d_1 \\
\geq\ & 3k^2 - 8k + 5 + (k - 2)^2 - (2k - 1)(k - 2) \\
=\ & 2k^2 - 7k + 7 > 0,
\end{aligned}
$$

which is a contradiction.

Case 2.3. $0 \leq d_1 \leq d_2 \leq k-2$**.** In this subcase, $k-1-d_2 \geq 1$, $n-s-t \geq 0$. So, $(k-1-d_2)(n-s-t) > ks - kt + d_1 p + d_2(t - p) - kn' - 2m$. Thus, $(k - d_2)(n - s) - ks > (d_1 - d_2)p + (n - s - t) - kn' - 2m \geq (d_1 - d_2)(d_1 + 1) + (n - s - t) - kn' - 2m \geq (d_1 - d_2)(d_1 + 1) - kn' - 2m$, i.e.,

$$(k - d_2)(n - s) - ks \geq (d_1 - d_2)(d_1 + 1) - kn' - 2m + 1. \tag{2.2}$$

Since $n \geq 8k + n' + 4m - 7$, we have

$$d_2 \frac{n + n'}{2} \geq d_2(4k + n' + 2m - \frac{7}{2}). \tag{2.3}$$

By $s + d_1 + d_2 \geq \frac{n+n'}{2}$, we have

$$(s - \frac{n + n'}{2})(2k - d_2) \geq -(d_1 + d_2)(2k - d_2). \tag{2.4}$$

Adding (2.2),(2.3) and (2.4), we get

$$
\begin{aligned}
0 \ &\geq\ d_1^2 + d_2^2 + 2kd_2 - 2kd_1 + d_1 - \frac{9}{2}d_2 - 2m + 1 + 2md_2 \\
&\geq\ d_1^2 + d_2^2 + d_1 - \frac{9}{2}d_2 - 2m + 1 + 2md_2 \\
&=\ (d_1 + \frac{1}{2})^2 + (d_2 - (\frac{9}{4} - m))^2 - m^2 + \frac{5m}{2} - \frac{69}{16}.
\end{aligned}
$$

So,

$$(d_2 - (\frac{9}{4} - m))^2 \leq m^2 - \frac{5m}{2} + \frac{69}{16} - \frac{1}{4},$$

which implies

$$d_2 \le \sqrt{m^2 - \frac{5m}{2} + \frac{65}{16}} + (\frac{9}{4} - m).$$

Let

$$f(m) = \sqrt{m^2 - \frac{5m}{2} + \frac{65}{16}} + (\frac{9}{4} - m).$$

Then

$$f^{'}(m) = \frac{m - \frac{5}{4}}{\sqrt{m^2 - \frac{5m}{2} + \frac{65}{16}}} - 1 < 0.$$

Namely, $f(m)$ is a monotonically decreasing function, $d_2 \le f(0) = \frac{\sqrt{65}+9}{4} = 4$. Therefore, $0 \le d_1 \le d_2 \le 4$.

If (d_1, d_2) =(0,4), (1,4), (2,4), (3,4), (4,4), (3,3), (2,3), (1,3), (0,3), (2,2), (1,2), (0,2), (0,1), we can check that $d_1^2 + d_2^2 + 2kd_2 - 2kd_1 + d_1 - \frac{9}{2}d_2 - 2m + 1 + 2md_2 > 0$ since $k \ge 2$.

If $d_1 = d_2 = 1$, then a contradiction can be found by a discussion similar to that in Case 2.1 for k=2.

If $d_1 = d_2 = 0$, then $s \ge \frac{n+n'}{2} - d_1 - d_2 = \frac{n+n'}{2}$ and $t \le \frac{n-n'}{2}$. Thus, $k|S| - k|T| + d_{G-S}(T) - (kn' + \sum_{x\in T} d_H(x) - e_H(S,T)) \ge 0$, which is a contradiction.

Thus, we complete the proof of Theorem 1.5.

3. Sharpness

In this section, we construct some graphs to show that the bounds in Theorem 1.5 are best possible.

First, $\delta(G) \ge k + n' + m$ cannot be replaced by $\delta(G) \ge k + n' + m - 1$. Otherwise, choose a vertex v such that $d(v) = k + n' + m - 1$. Delete n' vertices in $N_G(v)$, and then delete m edges incident to v. The resulting graph has $\delta(G) = k - 1$, which has no fractional k-factor by the definition.

Let $G = K_{4k+n'+2m-4} \vee (4k + 2m - 3)K_1$. Then $n = 8k + n' + 4m - 7$, $\delta(G) = 4k + n' + 2m - 4 > k + m$, but $|N_G(x_1) \cup N_G(x_2)| = 4k + n' + 2m - 4 < \frac{n+n'}{2}$ for each non-adjacent vertex x_1 and x_2 in $(4k + 2m - 3)K_1$. Let $S = K_{4k+n'+2m-4}$ and $T = (4k + 2m - 3)K_1$. Then $d_{G-S}(T) = 0$ and $\sum_{x\in T} d_H(x) - e_H(S,T) = 0$. We have $k|S| + \sum_{x\in T} d_{G-S}(x) - k|T| - (kn' + \sum_{x\in T} d_H(x) - e_H(S,T)) = -k < 0$. Thus, G is not a fractional (k, n', m)-critical deleted graph, and the condition $|N_G(x_1) \cup N_G(x_2)| \ge \frac{n+n'}{2}$ is sharp.

Let $G = K_{4k+n'+2m-6} \vee (2k+m-1)K_2$. Then $n = 8k+n'+4m-8$, $\delta(G) = 4k+n'+2m-5 \ge k+m$ and $|N_G(x_1) \cup N_G(x_2)| = 4k + n' + 2m - 4 = \frac{n+n'}{2}$ for any non-adjacent vertices x_1 and x_2 in G. Let $S = K_{4k+n'+2m-6}$ and $T = (2k + m - 1)K_2$. Let H be the set of m edges such that $H \subseteq (2k + m - 1)K_2$, then $\sum_{x\in T} d_H(x) - e_H(S,T) = 2m$ and $\sum_{x\in T} d_{G-S}(x) = 4k + 2m - 2$. We have, $k|S| + \sum_{x\in T} d_{G-S}(x) - k|T| - (kn' + \sum_{x\in T} d_H(x) - e_H(S,T)) = k(4k + n' + 2m - 6) - k(4k + 2m - 2) + (4k + 2m - 2) - kn' - 2m = -2 < 0$. Thus, G is a not fractional (k, n', m)-critical deleted graph. Therefore, the condition $n \ge 8k + n' + 4m - 7$ is sharp.

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