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CONGRUENCES FROM q-CATALAN IDENTITIES

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Abstract. In this paper, by studying three q-Catalan identities given by Andrews, we arrive at a certain number of congruences. These congruences are all modulo $\Phi_n(q)$, the n-th cyclotomic polynomial or the related functions and modulo q-integers.

1. Introduction

The basic notation of this paper is the quantum factorial symbol, which is defined as

$$(x;q)_n := \prod_{k=0}^{n-1}(1-xq^k).$$

Sometimes, we abbreviate $(x;q)_n$ by $(x)_n$.

Since the appearance of Euler's triangulation problem in 1751 [1, pp.184] and Catalan's parenthesization problem in 1838 [2, pp.295], more than 400 articles and problems on Catalan numbers, which are often defined as

$$C_n = \frac{(2n)!}{(n+1)!n!} = \frac{1}{n+1}\binom{2n}{n}, \quad n \geqslant 0,$$

have emerged in different periodicals. All these articles tell us different kinds of beauty and ubiquity of Catalan numbers. Here, we would like to present a few of the many works on Catalan numbers. These famous integers are, according to the aforementioned definition, alight variations on the ubiquitous central binomial coefficients $\binom{2n}{n}$. As Martin Gardner has ever said "They have the same delightful propensity for popping up unexpectedly, particularly in combinatorial problems". Several years ago, T. Koshy published a book [2] titled *Catalan Numbers with Applications*. In R.P. Stanley's books

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[3, 4], one can acquire a much more knowledge on Catalan numbers. H.W. Gould [5] provided an extensive bibliography of Catalan numbers.

With the rapid development of q-series, people began to be interested in various q-analogs polynomials or rational functions that reduce naturally to the Catalan numbers, (see, for example [6]). Finding q-analogs is a form of art. For example, the q-analog of Lagrange's identity for the sum of the squares of the binomial coefficients

$$\sum_{j=0}^{n}\binom{n}{j}^2=\binom{2n}{n}$$

is

$$\sum_{j=0}^{n}q^{j^2}\begin{bmatrix}n\\j\end{bmatrix}_q^2=\begin{bmatrix}2n\\n\end{bmatrix}_q,$$

where $\begin{bmatrix}*\\ *\end{bmatrix}_q$ is Gaussian polynomial (also known as q-binomial coefficient)

$$\begin{bmatrix}n\\j\end{bmatrix}_q=\begin{cases}0, & \text{if } j<0 \text{ or } j>n\\ \dfrac{(q;q)_n}{(q;q)_j(q;q)_{n-j}}, & 0\leqslant j\leqslant n\end{cases}.$$

The leading role of this paper is q-analogs of the Catalan numbers. There are so many q-analogs of the Catalan numbers (cf. [7]) that we cannot deal with all these q-analogs. What we are interested in are the following two. The first is [8]

$$C_n(q)=\frac{(1-q)}{(1-q^{n+1})}\begin{bmatrix}2n\\n\end{bmatrix}_q.$$

And another is [9]

$$\mathcal{C}_n(\lambda,q)=\frac{q^{2n}(-\lambda/q;q^2)_n}{(q^2;q^2)_n}.$$

For some detailed information on these two q-analogs of the Catalan numbers, one can consult [10].

My interest in Catalan numbers has arisen from looking at the three q-Catalan identities, which are found by Andrews in [10], namely,

Theorem 1.1.

$$\frac{(1+q^{n-r+1})}{(1+q^{r+1})}\begin{bmatrix}n+1\\r\end{bmatrix}_{q^2}=-(-q;q)_{n+1}\sum_{j=0}^{r}\begin{bmatrix}n-2j\\r-j\end{bmatrix}_{q^2}\frac{\mathcal{C}_{j+1}(-1;q)}{(-q;q)_{n-2j}}q^{-j-1}, \tag{1.1}$$

$$C_{n+1}(q)=\sum_{r=0}^{n}q^{2r^2+2r}\begin{bmatrix}n\\2r\end{bmatrix}_q C_r(q)\frac{(-q^{r+2};q)_{n-r}}{(-q;q)_r}, \tag{1.2}$$

$$C_n(q)=\sum_{r=1}^{n}(-1)^{r-1}q^{r^2-r}\begin{bmatrix}n-r+1\\r\end{bmatrix}_q C_{n-r}(q)\frac{(-q^{n-r+1};q)_r}{(-q;q)_r}. \tag{1.3}$$

These three q-Catalan identities are q-analogs of the following three identities from [2] respectively, say,

$$\binom{n+1}{r} = \sum_{j=0}^{r} \binom{n-2j}{r-j} C_j, \qquad \text{(Jonah's identity)}$$

$$C_{n+1} = \sum_{r=0}^{n} \binom{n}{2r} 2^{n-2r} C_r, \qquad \text{(Touchard's identity)}$$

$$C_n = \sum_{r=1}^{n} (-1)^{r-1} \binom{n-r+1}{r} C_{n-r}. \qquad \text{(Koshy's identity)}$$

In this paper, we will provide a certain amount of congruences from the three q-Catalan identities. All these congruences are under modulo $\Phi_n(q)$ or functions related to $\Phi_n(q)$ and modulo the q-integer $[n]$. Next, explanations on $\Phi_n(q)$ and $[n]$ need to be given.

Let $\Phi_n(q)$ be the n-cyclotomic polynomial,

$$\Phi_n(q) = \prod_{\substack{0 \leqslant m < n \\ \gcd(m,n)=1}} (q - \mathrm{e}^{2\pi m i/n}).$$

It is well-known that $\Phi_n(q) \in \mathbb{Z}[q]$ is the irreducible polynomial for $\mathrm{e}^{2\pi i/n}$. The polynomial $x^n - 1$ has the following factorization into irreducible polynomials over $\mathbb{Z}$:

$$x^n - 1 = \prod_{j|n} \Phi_j(x).$$

Let $[n]$ denote the q-integer, which is defined as

$$[n] = 1 + q + \cdots + q^{n-1}.$$

This paper is structured as follows: In Section 2, we present part of the main results of our findings, which are just the congruences from (1.1). In Section 3 and Section 4, another interpretation of the Gaussian polynomials will be introduced, which can let us derive much more congruences from (1.2) and (1.3).

2. Congruences from (1.1)

Before giving our main results on (1.1), the following lemma needs to be introduced.

Lemma 2.1. *When* $r = 1$,

$$\begin{bmatrix} n+1 \\ r \end{bmatrix}_{q^2} \equiv 0 \pmod{[2n+2]}.$$

When $2 < 2r \leqslant n$,

$$\begin{bmatrix} n+1 \\ r \end{bmatrix}_{q^2} \equiv 0 \pmod{\Phi_n(q^2)}.$$

Proof. When $r = 1$,

$$\begin{bmatrix} n+1 \\ r \end{bmatrix}_{q^2} = \frac{(q^2;q^2)_{n+1}}{(q^2;q^2)_1(q^2;q^2)_n} = \frac{(1-q^{2n+2})}{(1-q^2)} = \frac{1+q+\cdots+q^{2n+1}}{1+q},$$

which indicates the first congruence.

Next, let us turn to prove the second situation. By [11], we know that for $2 < 2r \leqslant n$,

$$\begin{bmatrix} n+1 \\ r \end{bmatrix}_{q} \equiv 0 \pmod{\Phi_n(q)},$$

which shows $\Phi_n(q) \mid \begin{bmatrix} n+1 \\ r \end{bmatrix}_{q}$. Since $f(q) \mid g(q)$ implies $f(q^2) \mid g(q^2)$, we obtain

$$\Phi_n(q^2) \mid \begin{bmatrix} n+1 \\ r \end{bmatrix}_{q^2},$$

i.e.,

$$\begin{bmatrix} n+1 \\ r \end{bmatrix}_{q^2} \equiv 0 \pmod{\Phi_n(q^2)}.$$

This completes the proof of this lemma. □

According to this lemma, we can derive the following conclusion.

Theorem 2.2. *For $r = 1$,*

$$\sum_{j=0}^{r} \begin{bmatrix} n-2j \\ r-j \end{bmatrix}_{q^2} \frac{\mathcal{C}_{j+1}(-1;q)}{(-q;q)_{n-2j}} q^{-j-1} \equiv 0 \pmod{[2n+2]}.$$

For $2 < 2r \leqslant n$,

$$\sum_{j=0}^{r} \begin{bmatrix} n-2j \\ r-j \end{bmatrix}_{q^2} \frac{\mathcal{C}_{j+1}(-1;q)}{(-q;q)_{n-2j}} q^{-j-1} \equiv 0 \pmod{\Phi_n(q^2)}.$$

Proof. The proof of this theorem is straightforward by combining (1.1) with Lemma 2.1. □

3. Congruences from (1.2)

Theorem 3.1. *For any positive integer n,*

$$\sum_{r=0}^{n} q^{2r^2+2r} \begin{bmatrix} n \\ 2r \end{bmatrix}_{q} C_r(q) \frac{(-q^{r+2};q)_{n-r}}{(-q;q)_r} \equiv \frac{2}{[n+2]} \pmod{\Phi_{n+1}(q)}.$$

Proof. Recall one of the q-analogs of the Catalan numbers given in Section 1, say,

$$C_n(q) = \frac{(1-q)}{(1-q^{n+1})} \begin{bmatrix} 2n \\ n \end{bmatrix}_q .$$

Thus,

$$C_{n+1}(q) = \frac{(1-q)}{(1-q^{n+2})} \begin{bmatrix} 2(n+1) \\ n+1 \end{bmatrix}_q = \frac{1}{[n+2]} \begin{bmatrix} 2(n+1) \\ n+1 \end{bmatrix}_q .$$

By [12], one can check that

$$\begin{bmatrix} an \\ bn \end{bmatrix}_q \equiv \binom{a}{b} \pmod{\Phi_n(q)},$$

which is also indicated in [11]. Accordingly, we have

$$\begin{bmatrix} 2(n+1) \\ n+1 \end{bmatrix}_q \equiv \binom{2}{1} \equiv 2 \pmod{\Phi_{n+1}(q)}.$$

Combining this congruence with (1.2), we reach at this theorem. □

Corollary 3.2. *For any positive integer n,*

$$\begin{bmatrix} 2n \\ n \end{bmatrix}_q \equiv 0 \pmod{[n+1]},$$

i.e.,

$$\frac{1}{(q;q)_n} \equiv 0 \pmod{[n+1]}.$$

Proof. (1.2) tells us that

$$\begin{bmatrix} 2(n+1) \\ n+1 \end{bmatrix}_q = [n+2] \sum_{r=0}^{n} q^{2r^2+2r} \begin{bmatrix} n \\ 2r \end{bmatrix}_q C_r(q) \frac{(-q^{r+2};q)_{n-r}}{(-q;q)_r},$$

which indicates

$$\begin{bmatrix} 2(n+1) \\ n+1 \end{bmatrix}_q \equiv 0 \pmod{[n+2]}.$$

Let $n \to n-1$, we arrive at the conclusion. □

Besides $[n+1]$, it is natural to consider another factor of $\begin{bmatrix} 2n \\ n \end{bmatrix}_q$, namely,

$$\sum_{r=0}^{n} q^{2r^2+2r} \begin{bmatrix} n \\ 2r \end{bmatrix}_q C_r(q) \frac{(-q^{r+2};q)_{n-r}}{(-q;q)_r}.$$

But this summation is a little bit complex in some degree. We would like to derive a simple one. In order to achieve the goal, some new concepts need to be introduced here.

Let $\mathfrak{S}(m,n)$ denote the set of lattice paths [2, Chap.9],[13] π from $(0,0)$ to $(m+n, m-n)$ with m steps diagonally up and n steps diagonally down. Usually, we associate π with a sequence $\pi = \pi_1 \cdots \pi_{m+n}$

with m zeros and n ones where zeros denote up steps and ones denote down steps. For a lattice path $\pi \in \mathfrak{S}(m,n)$, we define the inversion number $\mathrm{inv}(\pi)$ to be

$$\mathrm{inv}(\pi) = \#\{(i,j) : i < j, \pi_i > \pi_j\};$$

descent set $D(\pi)$,

$$D(\pi) = \{i : \pi_i > \pi_{i+1}, 1 \leqslant i \leqslant m+n-1\};$$

the major index $\mathrm{maj}(\pi)$,

$$\mathrm{maj}(\pi) = \sum_{i \in D(\pi)} i;$$

and the descent index $\mathrm{des}(\pi)$,

$$\mathrm{des}(\pi) = \#D(\pi).$$

After defining those notations, we can interpret $\begin{bmatrix} m+n \\ n \end{bmatrix}_q$, which will play an important role in the following discussion, as below:

$$\begin{bmatrix} m+n \\ n \end{bmatrix}_q = \sum_{\pi \in \mathfrak{S}(n,m)} q^{\mathrm{maj}(\pi)},$$

which is due to MacMahon [14].

In 1874, Catalan defined the following number [15]

$$S(m,n) = \frac{\binom{2m}{m}\binom{2n}{n}}{\binom{n+m}{n}} \in \mathbb{Z}.$$

He pointed out that $S(1,n) = 2C_n$, where C_n is Catalan numbers.

In reference [13], Allen gave the natural q-analog of $S(m,n)$, which is defined as

$$S_q(m,n) = \frac{\begin{bmatrix} 2m \\ m \end{bmatrix}_q \begin{bmatrix} 2n \\ n \end{bmatrix}_q}{\begin{bmatrix} m+n \\ n \end{bmatrix}_q} = \frac{(q)_{2m}(q)_{2n}}{(q)_m(q)_n(q)_{m+n}}.$$

$S_q(m,n)$ is a polynomial with symmetric coefficients. For example,

$$S_q(1,3) = 1 + q + q^2 + 2q^3 + 2q^4 + q^5 + q^6 + q^7,$$

$$S_q(2,4) = 1 + q + 2q^2 + 2q^3 + 3q^4 + 3q^5 + 4q^6 + 3q^7 + 3q^8 + 2q^9 + 2q^{10} + q^{11} + q^{12}.$$

What's more, one can check that

$$S_q(0,n) = \begin{bmatrix} 2n \\ n \end{bmatrix}_q.$$

Define

$$U_q(m,n) = \frac{S_q(m,n)}{1+q^m},$$

then we have the following conclusion.

Theorem 3.3. *For any positive integer n,*

$$S_q(0,n) = (1+q+\cdots+q^n)U_q(1,n).$$

Proof. By the very definition, we can deduce that

$$U_q(1,n) = \frac{S_q(1,n)}{1+q} = \frac{(1-q)(q;q)_{2n}}{(q;q)_n(q;q)_{n+1}}. \tag{3.1}$$

Thus,

$$\begin{aligned}(1+q+\cdots+q^n)U_q(1,n) &= (1+q+\cdots+q^n)\frac{(1-q)(q;q)_{2n}}{(q;q)_n(q;q)_n}\cdot\frac{1}{1-q^{n+1}}\\ &= \frac{(q;q)_{2n}}{(q;q)_n(q;q)_n} = \begin{bmatrix}2n\\ n\end{bmatrix}_q = S_q(0,n).\end{aligned}$$

This completes the proof. □

Having developed to this point, another factor of $\begin{bmatrix}2n\\ n\end{bmatrix}_q$ can be taken in everything in a glance, say,

Corollary 3.4. *For any positive integer n, we have*

$$\sum_{r=0}^{n} q^{2r^2+2r}\begin{bmatrix}n\\ 2r\end{bmatrix}_q C_r(q)\frac{(-q^{r+2};q)_{n-r}}{(-q;q)_r} = U_q(1,n+1).$$

By which, we deduce

Theorem 3.5. *For any positive integer n,*

$$\sum_{r=0}^{n} q^{2r^2+2r}\begin{bmatrix}n\\ 2r\end{bmatrix}_q C_r(q)\frac{(-q^{r+2};q)_{n-r}}{(-q;q)_r} \equiv 0 \pmod{1-q}.$$

Proof. This conclusion comes from (3.1). □

Also, we have

Theorem 3.6. *For any positive integer n,*

$$\sum_{r=0}^{n} q^{2r^2+2r}\begin{bmatrix}n\\ 2r\end{bmatrix}_q C_r(q)\frac{(-q^{r+2};q)_{n-r}}{(-q;q)_r} \equiv 0 \pmod{[n+3]}.$$

$$\sum_{r=0}^{n} q^{2r^2+2r}\begin{bmatrix}n\\ 2r\end{bmatrix}_q C_r(q)\frac{(-q^{r+2};q)_{n-r}}{(-q;q)_r} \equiv 0 \pmod{[n+4]}.$$

$$\vdots$$

$$\sum_{r=0}^{n} q^{2r^2+2r}\begin{bmatrix}n\\ 2r\end{bmatrix}_q C_r(q)\frac{(-q^{r+2};q)_{n-r}}{(-q;q)_r} \equiv 0 \pmod{[2n+2]}.$$

Proof. This theorem follows by

$$\begin{aligned}
&U_q(1,n+1)\\
=&\frac{(1-q)(q;q)_{2n+2}}{(q;q)_{n+1}(q;q)_{n+2}}=\frac{(1-q)(1-q)(1-q^2)\cdots(1-q^{n+1})(1-q^{n+2})\cdots(1-q^{2n+2})}{(1-q)\cdots(1-q^{n+1})(1-q)(1-q^2)\cdots(1+q^{n+2})}\\
=&\frac{(1-q^{n+3})\cdots(1-q^{2n+2})}{(1-q^2)\cdots(1-q^{n+1})}=\frac{[n+3]\cdots[2n+2]}{[2]\cdots[n+1]}.
\end{aligned}$$

□

In order to derive more congruences from (1.2), more works on $\begin{bmatrix}2n\\n\end{bmatrix}_q$ need to be done.

Proposition 3.7. *For any positive integer n,*

$$\begin{bmatrix}2n\\n\end{bmatrix}_q=(1+q^n)\begin{bmatrix}2n-1\\n\end{bmatrix}_q=(1+q^n)\begin{bmatrix}2n-1\\n-1\end{bmatrix}_q.$$

Proof. Since

$$1+q^n=\frac{1+q+\cdots+q^{n-1}+q^n(1+q+\cdots+q^{n-1})}{(1+q+\cdots+q^{n-1})}=\frac{1+\cdots+q^{2n-1}}{1+\cdots+q^{n-1}}=\frac{(1-q^{2n})}{(1-q^n)}.$$

Thus,

$$(1+q^n)\begin{bmatrix}2n-1\\n\end{bmatrix}_q=\frac{(1-q^{2n})}{(1-q^n)}\frac{(q;q)_{2n-1}}{(q;q)_n(q;q)_{n-1}}=\frac{(q;q)_{2n}}{(q;q)_n^2}=\begin{bmatrix}2n\\n\end{bmatrix}_q.$$

As for $\begin{bmatrix}2n-1\\n\end{bmatrix}_q=\begin{bmatrix}2n-1\\n-1\end{bmatrix}_q$, it just comes from the very definition of Gaussian polynomials. □

By this proposition, we can get the following by-product.

Corollary 3.8. *For any positive integer n,*

$$\begin{bmatrix}2n\\n\end{bmatrix}_q\equiv 0 \pmod{(1+q^n)}.$$

According to this corollary, two congruences can be derived, namely,

Theorem 3.9. *For any positive integer n,*

$$\sum_{r=0}^{n}q^{2r^2+2r}\begin{bmatrix}n\\2r\end{bmatrix}_q C_r(q)\frac{(-q^{r+2};q)_{n-r}}{(-q;q)_r}\equiv 0 \pmod{1+q^{n+1}}. \tag{3.2}$$

$$\sum_{r=0}^{n}q^{2r^2+2r}\begin{bmatrix}n\\2r\end{bmatrix}_q C_r(q)\frac{(-q^{r+2};q)_{n-r}}{(-q;q)_r}\equiv 0 \pmod{[n+2]}. \tag{3.3}$$

Proof. Corollary 3.8 indicates that

$$[n+1]\sum_{r=0}^{n} q^{2r^2+2r}\begin{bmatrix} n \\ 2r \end{bmatrix}_q C_r(q)\frac{(-q^{r+2};q)_{n-r}}{(-q;q)_r} \equiv 0 \pmod{1+q^{n+1}}.$$

While $(1+q^{n+1}) \nmid [n+1]$, which shows that (3.2) holds true.

$$\begin{aligned}\begin{bmatrix} 2(n+1) \\ n+1 \end{bmatrix}_q &= (1+q^{n+1})\begin{bmatrix} 2n+1 \\ n+1 \end{bmatrix}_q \\ &= \frac{(1+q^{n+1})[n+2]\cdots[2n+1]}{[2]\cdots[n]},\end{aligned}$$

which tells us that

$$[n+1]\sum_{r=0}^{n} q^{2r^2+2r}\begin{bmatrix} n \\ 2r \end{bmatrix}_q C_r(q)\frac{(-q^{r+2};q)_{n-r}}{(-q;q)_r} \equiv 0 \pmod{[n+2]}.$$

While $[n+2] \nmid [n+1]$, which shows that (3.3) holds true. □

4. Congruences from (1.3)

The structure of this section is similar to the previous one in some degree.

Theorem 4.1. *For any positive integer n,*

$$\sum_{r=1}^{n}(-1)^{r-1}q^{r^2-r}\begin{bmatrix} n-r+1 \\ r \end{bmatrix}_q C_{n-r}(q)\frac{(-q^{n-r+1};q)_r}{(-q;q)_r} \equiv \frac{2}{[n+1]} \pmod{\Phi_n(q)}.$$

Proof. Similar to the proof of theorem 3.1, we get

$$\begin{aligned}C_n(q) &= \frac{1}{[n+1]}\begin{bmatrix} 2n \\ n \end{bmatrix}_q \\ &\equiv \binom{2}{1}\Big/[n+1] \equiv \frac{2}{[n+1]} \pmod{\Phi_n(q)},\end{aligned}$$

which reveals this theorem holds true. □

Imitating corollary 3.4 and theorem 3.5 shown in section 3, we have

Theorem 4.2. *For any positive integer n,*

$$\sum_{r=1}^{n}(-1)^{r-1}q^{r^2-r}\begin{bmatrix} n-r+1 \\ r \end{bmatrix}_q C_{n-r}(q)\frac{(-q^{n-r+1};q)_r}{(-q;q)_r} = U_q(1,n).$$

Theorem 4.3. *For any positive integer n,*

$$\sum_{r=1}^{n}(-1)^{r-1}q^{r^2-r}\begin{bmatrix} n-r+1 \\ r \end{bmatrix}_q C_{n-r}(q)\frac{(-q^{n-r+1};q)_r}{(-q;q)_r} \equiv 0 \pmod{(1-q)}.$$

What's more, we have

Theorem 4.4. *For any positive integer n,*

$$\sum_{r=1}^{n}(-1)^{r-1}q^{r^2-r}\begin{bmatrix}n-r+1\\ r\end{bmatrix}_q C_{n-r}(q)\frac{(-q^{n-r+1};q)_r}{(-q;q)_r}\equiv 0 \pmod{[n+2]}.$$

$$\sum_{r=1}^{n}(-1)^{r-1}q^{r^2-r}\begin{bmatrix}n-r+1\\ r\end{bmatrix}_q C_{n-r}(q)\frac{(-q^{n-r+1};q)_r}{(-q;q)_r}\equiv 0 \pmod{[n+3]}.$$

$$\vdots$$

$$\sum_{r=1}^{n}(-1)^{r-1}q^{r^2-r}\begin{bmatrix}n-r+1\\ r\end{bmatrix}_q C_{n-r}(q)\frac{(-q^{n-r+1};q)_r}{(-q;q)_r}\equiv 0 \pmod{[2n]}.$$

Proof. This theorem follows by

$$\begin{aligned}&U_q(1,n)\\ =&\frac{(1-q)(q;q)_{2n}}{(q;q)_n(q;q)_{n+!}}=\frac{(1-q)(1-q)(1-q^2)\cdots(1-q^n)(1-q^{n+!})\cdots(1-q^{2n})}{(1-q)\cdots(1-q^n)(1-q)(1-q^2)\cdots(1+q^{n+!})}\\ =&\frac{(1-q^{n+2})\cdots(1-q^{2n})}{(1-q^2)\cdots(1-q^n)}=\frac{[n+2]\cdots[2n]}{[2]\cdots[n]}.\end{aligned}$$

□

Since $\begin{bmatrix}2n\\ n\end{bmatrix}_q \equiv 0 \pmod{(1+q^n)}$, which shows

$$C_n(q)=\frac{1}{[n+1]}\begin{bmatrix}2n\\ n\end{bmatrix}_q\equiv 0 \pmod{(1+q^n)}.$$

This congruence indicates

Theorem 4.5. *For any positive integer n,*

$$\sum_{r=1}^{n}(-1)^{r-1}q^{r^2-r}\begin{bmatrix}n-r+1\\ r\end{bmatrix}_q C_{n-r}(q)\frac{(-q^{n-r+1};q)_r}{(-q;q)_r}\equiv 0 \pmod{(1+q^n)}.$$

5. **Conclusion**

In this paper, by studying three q-Catalan identities given by Andrews, we arrive at a certain amount of congruences. Actually, we can do a little more on this topic. For example, we have the following combinatorial identities, like

$$(2^n-1)C_n=\sum_{r=0}^{\lfloor\frac{n-1}{2}\rfloor}(-1)^r\frac{4r+3}{2n+1}\binom{2n+1}{n-2r-1}2^{n-2r-1}C_r,$$

$$C_n=\sum_{r=0}^{2n}(-1)^r\binom{2n}{r}C_{r+1}2^{2n-r},$$

$$C_{n+1} = \sum_{r=0}^{n} (-1)^r \binom{n}{r} C_{r+1} 4^{n-r}.$$

Following the procedure given by Andrews in [16, Sec.5], one can arrive at the q-Catalan identities like Andrews gave in [10]. Then, similar to the congruences given in this paper, we can get some new congruences. While these works are just mundane tasks, so we omit here.

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