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DETERMINANTS OF ADJACENCY MATRICES OF GRAPHS

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Abstract. We study the set of all determinants of adjacency matrices of graphs with a given number of vertices. Using Brendan McKay's data base of small graphs, determinants of graphs with at most 9 vertices are computed so that the number of non-isomorphic graphs with given vertices whose determinants are all equal to a number is exhibited in a table. Using an idea of M. Newman, it is proved that if G is a graph with n vertices, m edges and $\{d_1, \dots, d_n\}$ is the set of vertex degrees of G, then $\gcd(2m, d^2)$ divides the determinant of the adjacency matrix of G, where $d = \gcd(d_1, \dots, d_n)$. Possible determinants of adjacency matrices of graphs with exactly two cycles are obtained.

1. Introduction

Let G be a simple graph with finite number of vertices. We denote by $\det(G)$ the determinant of the adjacency matrix of G. This number $\det(G)$ is an integer and is an invariant of G so that its value is independent of the labeling of the vertices.

It is a famous result due to Hadamard [6] that if $A = [a_{ij}]$ is an $n \times n$ complex matrix such that $||a_{ij}|| \leq \mu$ for all i, j, then $|| \det(A)|| \leq \mu^n n^{\frac{n}{2}}$.
In [4], Fallat and van den Driessche studied, among others things, the maximum $W(n,k)$ and minimum $w(n,k)$ of non-zero absolute values of determinants of k-regular graphs with n vertices and they determined $W(n,2)$, $W(n,n-3)$ and $w(n,2)$, $w(n,n-3)$.
In [9], Ryser found an upper bound for the absolute value of a $(0,1)$ square matrix of size n with t non-zero entries. It follows from Theorem 3 of [9] that

$$|\det(G)| \leq \big(\frac{2m}{n}\big)^n \big(1 - \frac{2m-n}{n(n-1)}\big)^{n-1},$$

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for any graph G with $n \geq 2$ vertices and m edges.
Newman [8] proved that $k \cdot \gcd(k, n)$ divides the determinants of $n \times n$ $(0,1)$ matrices all of whose row and column sums equal to k. A similar result for the determinants of graphs is proved in Theorem 2.7, below.

In this paper, we study the distributions of $\det(G)$, whenever G runs over graphs with n vertices for a given integer $n \geq 1$. We denote by $\mathcal{G}_n$ the set of all non-isomorphic graphs with n vertices and

$$\mathcal{DG}_n = \left\{ \det(G) \mid G \in \mathcal{G}_n \right\}, \quad \alpha_n = \max \mathcal{DG}_n \text{ and } \beta_n = \min \mathcal{DG}_n.$$

Using Brendan McKay's data base of small graphs on http://cs.anu.edu.au/∼ bdm/data/, we have computed $\mathcal{DG}_n$ for all $n \leq 9$ (see Proposition 2.5, below).

Hu [7] has determined the determinant of graphs with exactly one cycle. Here we obtain the possible determinants of graphs with exactly two cycles (see Proposition 2.11, below).

2. **Results**

For a graph G with adjacency matrix A, we will denote its characteristic polynomial $|\lambda \mathbf{I} - A|$ by $P_G(\lambda)$. We use the following results in the sequel.

Lemma 2.1 (Theorem 1.3, p. 32 of [3]). *Let*

$$P_G(\lambda) = |\lambda \mathbf{I} - A| = \lambda^n + a_1 \lambda^{n-1} + \cdots + a_n$$

be the characteristic polynomial of an arbitrary undirected multigraph G.
Call an "elementary figure"
a) the graph K_2, or
b) every graph C_q $(q \geq 1)$ (loops being included with $q = 1$),
call a "basic figure" U every graph all of whose components are elementary figures;
let $p(U)$, $c(U)$ be the number of components and the number of circuits contained in U, respectively, and let $\mathcal{U}_i$ denote the set all basic figures contained in G having exactly i vertices.
Then

$$a_i = \sum_{U \in \mathcal{U}_i} (-1)^{p(U)} \cdot 2^{c(U)} \quad (i = 1, 2, \ldots, n).$$

Lemma 2.2 (Theorem 2.11, p. 59 of [3]). *Let x_1 be a vertex of degree 1 in the graph G and let x_2 be the graph adjacent to x_1. Let G_1 be the induced subgraph obtained from G be deleting the vertex x_1. If the induced subgraph obtained by x_1 and x_2 from G is denoted by G_2, then*

$$P_G(\lambda) = \lambda \cdot P_{G_1}(\lambda) - P_{G_2}(\lambda).$$

Lemma 2.3 (Theorem 2.12, p. 59 of [3]). *Let G be the graph obtained by joining the vertex x of the graph G_1 to the vertex y of the graph G_2 by an edge. Let G_1' (G_2', respectively) be the induced subgraph of G_1 (G_2, respectively) obtained by deleting the vertex x (y, respectively) from G_1 (G_2, respectively). Then*

$$P_G(\lambda) = P_{G_1}(\lambda) P_{G_2}(\lambda) - P_{G_1'}(\lambda) P_{G_2'}(\lambda).$$

Let us to compute the determinants of some of famous classes of graphs which will be used in the sequel.

Proposition 2.4. *Let K_n and P_n be the complete graph and the path with n vertices, respectively, and C_m be the cycle with $m \geq 3$ vertices.*

(1) $\det(K_n) = (-1)^{n-1}(n-1)$.

(2) $$\det(P_n) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ 1 & \text{if } n = 2k,\ k \text{ is even} \\ -1 & \text{if } n = 2k,\ k \text{ is odd.} \end{cases}$$

(3) $$\det(C_m) = \begin{cases} 2 & \text{if } m \text{ is odd} \\ 0 & \text{if } m = 2k,\ k \text{ is even} \\ -4 & \text{if } m = 2k,\ k \text{ is odd.} \end{cases}$$

Proof. (1) It is well-known.
(2) It follows from the fact that $\det(P_n) = -\det(P_{n-2})$.
(3) It follows from Lemma 2.1. □

Proposition 2.5. *The following table gives the set of all determinants of graphs with at most 9 vertices. By i^j in the n-th row of the table, we mean that there are exactly j non-isomorphic graphs with n vertices whose determinants are all equal to i, and a single number i (with no exponent) shows that there is a unique graph with n vertices whose determinant is equal to i.*

Number of Vertices	*Determinants of graphs with multiplicities*
1	0
2	$0,-1$
3	$0^3,2$
4	$-3,0^7,1^3$
5	$-4,0^{25},2^6,4$
6	$-5^3,-4^5,-1^{32},0^{99},3^{10},4^2,7^2$
7	$-12^2,-10^2,-6^{13},-4^{21},-2^{20},0^{690},2^{204},4^{40},6^{17},8^{25},10^5,12^5$
8	$-28^2,-27^2,-24^5,-23^5,-20^7,-19^{21},-16^{51},-15^{43},-12^{90},-11^{79},-8^{128},$ $-7^{251}, -4^{581},-3^{813},0^{6551},1^{2416},4^{758},5^{240},8^{73},9^{139},12^{24},13^{23},16^{32},17^8,20^1,21^3$
9	$-128^2, -96^3,-72^{12},-64^7,-60^5,-56^{17},-54^{12},-50^{27},-48^{13}, -46^{20},-44^{39},-42^{47},$ $-40^{103},-38^{52},-36^{110},-34^{128},-32^{593},-30^{199},-28^{295},-26^{392},-24^{765},-22^{579},$ $-20^{869},-18^{2747},-16^{2247},-14^{1805},-12^{3062},-10^{4290},-8^{17582},-6^{8531},-4^{14901},$ $-2^{57065},0^{133174},2^{6767},4^{6950},6^{4669},8^{1566},10^{1349},12^{1156},14^{695},16^{606},$ $18^{106},20^{297}, 22^{173},24^{240},26^{95},28^{91},30^{61},32^{46},34^5,36^{32},38^{28},40^3, 42^{17},44^{16},54^3,60^3,64$

Proof. We use Brendan McKay's data base of small graphs on http://cs.anu.edu.au/∼ bdm/data/, and then by GAP for example, it is easy to find the determinants. □

For two graphs G_1 and G_2, $G_1 \dot{+} G_2$ denotes the disjoint union of G_1 and G_2 which is the graph G having exactly two connected components, one is isomorphic to G_1 and the other is isomorphic to G_2.

Proposition 2.6. (1) *$\beta_n \neq 0$ if and only if $\beta_n < 0$ if and only if $n \notin \{1, 3\}$.*

(2) *$\alpha_n \neq 0$ if and only if $\alpha_n > 0$ if and only if $n \notin \{1, 2\}$.*

(3) *If G is a graph with odd number of vertices, then $\det(G)$ is even.*

(4) *The determinant of every bipartite graph with odd vertices is zero.*

Proof. (1) It is clear that if $n = 1$ or 3, then $\beta_n = 0$ and if $n = 2$, then $\beta_n = -1$. Thus assume that $n \geq 4$. It is enough to show that $\beta_n < 0$. If n is odd, then we have $\beta_n \leq \det(C_{n-2} \dot{+} P_2) = -2$, by Proposition 2.4. If n is even, then we have $\beta_n \leq \det(K_n) = -(n-1)$, by Proposition 2.4. This proves (1).

(2) If $n = 1$ or 2, then clearly $\alpha_n = 0$. Thus it is enough to prove that $\alpha_n > 0$ whenever $n \geq 3$. If n is odd, then $\alpha_n \geq \det(K_n) = n - 1$, by Proposition 2.4. If n is even, then $\alpha_n \geq \det(K_{n-2} \dot{+} P_2) = n - 3$. This completes the proof of (2).

(3) Let $n = |V(G)|$ be odd and let $\mathcal{F}$ be an elementary figure (if exists) of G with n vertices. Then, by definition, $\mathcal{F}$ is a disjoint union of s number of edges and t number of cycles. If $t = 0$, then $n = 2s$ which is not possible. It follows that any elementary figure of G with n vertices have at least one cycle. Now it follows from Lemma 2.1 that $\det(G)$ is an even number.

(4) Let G be a bipartite graph with odd n vertices. Since every bipartite graph has no odd cycles, it follows that G has no elementary figure with n vertices. Again Lemma 2.1 implies that $\det(G) = 0$. □

The proof of the following proposition we use the same ideas as the proof of Theorem 2 of Newman's paper [8].

Proposition 2.7. *Let G be a graph with n vertices and let $\{d_1, \ldots, d_n\}$ be the set of vertex degrees of G. If $d = \gcd(d_1, \ldots, d_n)$, then $\gcd(2m, d^2)$ divides $\det(G)$.*

Proof. Let $A = \begin{bmatrix} A_1 & \cdots & A_n \end{bmatrix}$ be the adjacency matrix of G, where $A_1, \ldots, A_n$ are its columns. Then

$$\det(G) = \begin{vmatrix} A_1 + \cdots + A_n & A_2 & \cdots & A_n \end{vmatrix} = \begin{vmatrix} \mathbf{d} & A_2 & \cdots & A_n \end{vmatrix},$$

where $\mathbf{d} = \begin{bmatrix} d_1 & \cdots & d_n \end{bmatrix}^T$. Now sum up all the rows with the nth one. Then $\det(G)$ is equal to the determinant of a matrix whose first column is

$$\begin{bmatrix} d_1 & \cdots & d_{n-1} & \sum_{i=1}^{n} d_i \end{bmatrix}^T$$

and its nth row is

$$\begin{bmatrix} \sum_{i=1}^{n} d_i & d_2 & \cdots & d_n \end{bmatrix}.$$

Since $\sum_{i=1}^{n} d_i = 2m$, by factoring d from the first column and $\gcd(\frac{2m}{d}, d_2, \ldots, d_n)$ from the last column, we have that $d \cdot \gcd(\frac{2m}{d}, d_2, \ldots, d_n)$ divides $\det(G)$. Since $d_1 = 2m - \sum_{i=2}^{n} d_i$, we have $\gcd(\frac{2m}{d}, d_2, \ldots, d_n) = \gcd(\frac{2m}{d}, d)$. This completes the proof. □

Theorem 2.8. (Hu [7]) *Let G be a connected graph with n vertices having a unique cycle C with $k < n$ vertices. Then*

$$\det(G) = \begin{cases} 1 & G \text{ has a perfect matchig, } k \equiv 1 \mod 2,\ n \equiv 0 \mod 4 \\ -1 & G \text{ has a perfect matchig, } k \equiv 1 \mod 2,\ n \equiv 2 \mod 4 \\ 4 & G \text{ has a perfect matchig, } k \equiv 2 \mod 4,\ n \equiv 0 \mod 4 \\ -4 & G \text{ has a perfect matchig, } k \equiv 2 \mod 4,\ n \equiv 2 \mod 4 \\ 0 & G \text{ has a perfect matchig, } k \equiv 0 \mod 4 \\ 2 & G \text{ has no perfect matchig, } G\backslash C \text{ has a perfect matching}, n-k \equiv 0 \mod 4 \\ -2 & G \text{ has no perfect matchig, } G\backslash C \text{ has a perfect matching}, n-k \equiv 2 \mod 4 \\ 0 & \text{both } G \text{ and } G\backslash C \text{ have no perfect matching.} \end{cases}$$

Lemma 2.9. *Let G be a graph containing exactly two cycles C_1 and C_2 of orders k and ℓ, respectively such that $|V(C_1) \cap V(C_2)| = 1$. If $|V(G)| = k + \ell - 1$, then*

$$\det(G) = \begin{cases} 0 & \text{if both } k, \ell \text{ are even} \\ 2 \cdot (-1)^{\ell+k-1} \cdot \left((-1)^{\frac{\ell-1}{2}+1} + (-1)^{\frac{k-1}{2}+1}\right) & \text{if both } k, \ell \text{ are odd} \\ 2 \cdot (-1)^{\ell+k-1} \cdot \left((-1)^{\frac{\ell}{2}+\frac{k-1}{2}} + (-1)^{\frac{k-1}{2}+1}\right) & \text{if } k \text{ is odd, } \ell \text{ is even} \\ 2 \cdot (-1)^{\ell+k-1} \cdot \left((-1)^{\frac{k}{2}+\frac{\ell-1}{2}} + (-1)^{\frac{\ell-1}{2}+1}\right) & \text{if } \ell \text{ is odd, } k \text{ is even.} \end{cases}$$

Proof. Let $C_1 := x_1 \cdots x_k x_1$ and $C_2 := y_1 \cdots y_\ell y_1$. Suppose $V(C_1) \cap V(C_2) = \{x_k\} = \{y_\ell\}$. We want to apply Lemma 2.1 and so we need to find all elementary figures of G, that are all spanning subgraphs of G with exactly $k + \ell - 1$ vertices whose connected components are either an edge or a cycle of G. If k and ℓ are both even, then G has no elementary figure. If k and ℓ are both odd, then we have only two elementary figures

$$F_1 = \{C_1, y_1y_2, \ldots, y_{\ell-2}y_{\ell-1}\} \text{ and } F_2 = \{C_2, x_1x_2, \ldots, x_{k-2}x_{k-1}\}.$$

If k is odd and ℓ is even, then we have exactly 3 elementary figures as follows:

$$\{y_1y_2, \ldots, y_{\ell-1}y_\ell, x_1x_2, \ldots, x_{k-2}x_{k-1}\}, \{y_1y_\ell, y_2y_3, \ldots, y_{\ell-2}y_{\ell-1}, x_{k-1}x_{k-2}, \ldots, x_2x_1\},$$
$$\{C_2, x_1x_2, \ldots, x_{k-2}x_{k-1}\}.$$

Similarly, for the case k even and ℓ odd, we have exactly 3 elementary figures (interchange k and ℓ in the latter elementary figures). Now we can apply Lemma 2.1 and this completes the proof. ☐

Lemma 2.10. *Let G be a connected graph containing exactly two cycles C_1 and C_2 of orders k and ℓ, respectively such that $V(C_1) \cap V(C_2) = \varnothing$. If $V(G) = V(C_1) \cup V(C_2)$, then* $\det(G) \in \{-8, 0, 3, 5, 16\}$. *If C_1 and C_2 are connected by a path P of length $t - 1 \geq 2$ and $V(G) = V(C_1) \cup V(C_2) \cup V(P)$, then*

$$\det(G) \in \begin{cases} \{0, \pm 4\} & \textit{if } t = 3 \\ \{-16, -8, -5, -3, 0\} & \text{if } t = 4 \\ \{0, \pm 3, \pm 4, \pm 5, \pm 8, \pm 16\} & \text{if } t > 4. \end{cases}$$

Proof. Let $C_1 = x_1 \cdots x_k x_1$, $C_2 = y_1 \cdots y_\ell y_1$. By hypothesis, in any case, there exists a path $P = x_k - z_1 z_2 \cdots z_t - y_\ell$ connecting C_1 and C_2 such that $V(G) = V(C_1) \cup V(C_2) \cup V(P)$. It is clear that $t = 2$ if and only if $V(G) = V(C_1) \cup V(C_2)$. In this case, by Lemma 2.3, we have $\det(G) = \det(C_1)\det(C_2) - \det(P_{k-1})\det(P_{\ell-1})$. Now it follows from Proposition 2.4, $\det(G) \in \{-8, 0, 3, 5, 16\}$. Now assume that $t \geq 3$, that is the length of P is at least 3, and let H be the induced subgraph of G on the vertices $V = V(C_1) \cup \big(V(P)\backslash\{z_t\}\big)$. Note that, by Lemma 2.2, we have

$$\det(H) = \begin{cases} -\det(P_{k-1}) & \text{if } t = 3 \\ \det(C_k)\det(P_{t-2}) - \det(P_{k-1})\det(P_{t-3}) & \text{if } t \geq 4. \end{cases}$$

Thus it follows from Lemma 2.3 that

$$\det(G) = \det(H)\det(C_\ell) - \det(H\backslash\{z_{t-1}\})\det(P_{\ell-1}).$$

Note that $\det(H\backslash\{z_{t-1}\})$ can be similarly compute by a formulae as given for $\det(H)$. Now it is easy to complete the proof. □

Proposition 2.11. *Let G be a graph with exactly two cycles. Then* $\det(G) \in \{0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \pm 8, \pm 16\}$.

Proof. By using Lemma 2.2 on the (possible) vertices of G with degree 1, we find that there exist a (possibly empty) forest F and either a connected graph H as of the form in Lemmas 2.9 or 2.10 or two connected graphs H_1 and H_2 each of which contains exactly one cycles such that $\det(G) = \pm\det(F)\cdot\det(H)$ or $\det(G) = \pm\det(F)\cdot\det(H_1)\cdot\det(H_2)$, respectively, where $\det(F) = 1$ if F is empty. Now Lemmas 2.9 and 2.10 and Theorem 2.8 complete the proof. □

3. Questions

Let us end the paper by the following questions mainly arising from the data table given in Proposition 2.5 and some other investigations.

In an earlier version of the paper, we had further questions. However, thanks to the referee, we learnt that they have known answers. The first one is the following:

Question 3.1. *Is it true that* $|\beta_n| > \alpha_n$ *for all* $n > 7$*?*

The answer of Question 3.1 is negative. For, in [1] the following is proved:

Theorem 3.2. *Suppose that there exists a graphical Hadamard matrix of order* $n + 1 = 4k^2$*. Then if k is odd, then* $\alpha_n = 2^{-n}(n+1)^{(n+1)/2}$ *and* $\beta_n \geq -2^{-n}(n+1)^{(n+1)/2} + 1$*, and if k is even, then* $\beta_n = -2^{-n}(n+1)^{(n+1)/2}$ *and* $\alpha_n \leq 2^{-n}(n+1)^{(n+1)/2} - 1$.

Theorem 3.2 together with a result of [5] which shows the existence of graphical Hadamard matrices of order $4k^4$ for all positive integers k, gives a negative answer to Question 3.1 for infinitely many values of n. Besides, the first counterexample to Question 3.1 is $n = 10$: in [1], by computation, it is shown that $\alpha_{10} = 256$ and $\beta_{10} = -192$.

The second question that we have already proposed and the referee informed us that its answer is known, was the following:

Question 3.3. *Does* $\lim\limits_{n\to\infty} \frac{|\{G\in\mathcal{G}_n \mid \det(G)=0\}|}{|\mathcal{G}_n|}$ *exist?*

The proportion of the number of singular graphs (i.e. with zero determinants) of order n to the number of all graphs of order n goes to 0 as $n \to \infty$, see [2].

The third question was the following:

Question 3.4. *Is it possible to determine graphs* $G, H \in \mathcal{G}_n$ *such that* $\det(G) = \alpha_n$ *and* $\det(H) = \beta_n$*? Do they have some distinguished properties from other graphs with* n *vertices? For example, must they always be connected?*

The referee gave us the following information concerning Question 3.4: in [1] it is proved that:

Theorem 3.5. *For any graph* G *there exists a connected graph with the same order and the same determinant as* G*.*

This shows that α_n and β_n are attained by connected graphs. So we replace Question 3.4 by Question 4, below.

Finally the referee informed us concerning Questions 6 and 7, (see below), that in [1] it is proved the sequence $\{\theta_n\}_{n=1}^{\infty}$ is increasing, where $\theta_n = \max(\alpha_n, -\beta_n)$.

The questions are as follows:

1) Describe $\mathcal{DG}_n = \big\{ \det(G) \mid G \in \mathcal{G}_n \big\}$.
2) What are α_n and β_n?
3) For given n, which integers can never belong to $\mathcal{D}_n$?
4) Are α_n and β_n attained *only by* connected graphs?
5) Find relations between $\mathcal{DG}_n$ and $\mathcal{DG}_{n+1}$.
6) Is it true that $\alpha_n < \alpha_{n+1}$ for all $n > 3$?
7) Is it true that $\beta_{n+1} < \beta_n$?

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