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ON THE SKEW SPECTRAL MOMENTS OF GRAPHS

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ABSTRACT. Let G be a simple graph, and G^σ be an oriented graph of G with the orientation σ and skew-adjacency matrix $S(G^\sigma)$. The k -th skew spectral moment of G^σ , denoted by $T_k(G^\sigma)$, is defined as $\sum_{i=1}^n (\lambda_i)^k$, where $\lambda_1, \lambda_2, \dots, \lambda_n$ are the eigenvalues of G^σ . Suppose $G_1^{\sigma_1}$ and $G_2^{\sigma_2}$ are two digraphs. If there exists an integer k , $1 \leq k \leq n-1$, such that for each i , $0 \leq i \leq k-1$, $T_i(G_1^{\sigma_1}) = T_i(G_2^{\sigma_2})$ and $T_k(G_1^{\sigma_1}) < T_k(G_2^{\sigma_2})$ then we write $G_1^{\sigma_1} \prec_T G_2^{\sigma_2}$. In this paper, we determine some of the skew spectral moments of oriented graphs. Also we order some oriented unicyclic graphs with respect to skew spectral moment.

1. Introduction

Let G be a simple graph with vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$, and let G^σ be an oriented graph of G with the orientation σ , which allocates to any edge of G a direction such that the induced graph G^σ becomes an oriented graph. Then G is called the underlying graph of G^σ . The skew-adjacency matrix of G^σ is the $n \times n$ matrix $S(G^\sigma) = [s_{ij}]$, where $s_{ij} = 1$ and $s_{ji} = -1$ if $v_i v_j$ is an arc of G^σ , otherwise $s_{ij} = s_{ji} = 0$. Since S is skew-symmetric, iS is Hermitian and so all of the eigenvalues of iS are real. Thus the eigenvalues of S are 0 or pure imaginary and since characteristic polynomial of S has real coefficients, the eigenvalues occur in complex conjugate pairs, see [2] for details.

In [10] Shader et al. studied the relationship between the spectra of a graph G and the skew-spectra of an oriented graph G^σ of G . In [9] the authors characterized the coefficients of the characteristic polynomial of the skew-adjacency matrix of an oriented graph.

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We use the notations P_n , C_n and S_n to denote a path, a cycle and a star with n vertices, respectively. Let F be a graph. An F -subgraph of G is a subgraph of G which is isomorphic to the graph F . Let $\varphi_G(F)$ be the number of all F -subgraphs of G .

Let G be a simple graph. A walk of length k in G is an alternating sequence $v_1, e_1, v_2, e_2, \dots, v_k, e_k, v_{k+1}$ of vertices and edges such that for any $i = 1, 2, \dots, k$ the vertices v_i and v_{i+1} are distinct end-vertices of the edge e_i . A closed walk is a walk in which the first and the last vertex are the same [5].

Let $S(G^\sigma) = [s_{ij}]$ be the skew adjacency matrix of an oriented graph G^σ and $W = v_1 v_2 \dots v_k$ (perhaps $v_i = v_j$ for $i \neq j$) be a walk from v_1 to v_k . The sign of W is defined as:

$$sgn(W) = \prod_{i=1}^{k-1} s_{v_i v_{i+1}}.$$

Let W' be the inverse walk of W obtained from W by replacing the ordering of vertices by its inverses, i.e., $W' = v_k v_{k-1} \dots v_1$. It is easy to check that $sgn(W') = sgn(W)$ if the length of such a walk is even and $sgn(W') = -sgn(W)$ otherwise.

In an oriented graph G^σ , an even cycle C is called evenly oriented if for either choice of direction of traversing around C , the number of edges of C directed in the direction of traversal is even. Otherwise C is called oddly oriented. Throughout this paper we denote by C_n^+ an evenly oriented even cycle and C_n^- an oddly oriented even cycle with n vertices. Moreover, let $w_{ij}^+(k)$ and $w_{ij}^-(k)$ denote the number of all positive walks and negative walks starting from v_i and terminating at v_j with length k , respectively, see [3, 8] for more details.

Suppose that $A(G)$ is the adjacency matrix of graph G and $\lambda_1, \lambda_2, \dots, \lambda_n$ are the eigenvalues of $A(G)$. For a simple graph G , the k -th spectral moment of G is defined as $\sum_{i=1}^n \lambda_i^k$, see [4, 14] for more information. In [11, 12, 13] the authors calculated the spectral moments of some graphs and they ordered these graphs with respect to the spectral moments. Also Wu and Liu [14] determined the last $\lfloor \frac{d}{2} \rfloor + 1$ graphs, among all n -vertex trees of diameter d ($4 \leq d \leq n - 3$) with respect to spectral moments. For more information on the spectral moments and additional results one can see [1, 6, 7].

In this paper we consider an orientation for the simple graph G and determine the skew spectral moments of oriented graph G^σ . Also we give an ordering for some oriented graphs with respect to the skew spectral moments.

Gong and Xu [8] obtained the following result on the relationship between the entries of S^k and the number of walks between any pair of ordered vertices.

Theorem 1.1. *Let S be the skew-adjacency matrix of an oriented graph G^σ and u and v be two arbitrary vertices of G^σ . Then*

$$(S^k)_{uv} = w_{uv}^+(k) - w_{uv}^-(k).$$

It is easy to see that for each odd k , $(S^k)_{uu} = 0$, see [8].

An oriented graph H is called a basic oriented graph if each component of H is K_2 or a cycle of even length. The characteristic polynomial of the oriented graph G^σ , which is called the skew characteristic polynomial of G^σ and denoted by $\chi(G^\sigma, \lambda)$, is defined as:

$$\chi(G^\sigma, \lambda) = \det(\lambda I_n - S(G^\sigma)) = \sum_{i=0}^n (-1)^i s_i(G^\sigma) \lambda^{n-i}.$$

Hou and Lei in [9] proved the following result for the skew characteristic polynomial of the oriented graph G^σ .

Theorem 1.2. Let G^σ be an oriented graph with n vertices and with skew characteristic polynomial

$$\chi(G^\sigma, \lambda) = \sum_{i=0}^n (-1)^i s_i(G^\sigma) \lambda^{n-i} = \lambda^n - s_1 \lambda^{n-1} + s_2 \lambda^{n-2} + \dots + (-1)^{n-1} s_{n-1} \lambda + (-1)^n s_n.$$

Then $s_i = 0$ if i is odd and

$$s_i = \sum_{\mathcal{H}} (-1)^{c^+(\mathcal{H})} 2^{c(\mathcal{H})},$$

if i is even, where the summation goes over all basic oriented graphs \mathcal{H} of G^σ with i vertices and $c^+(\mathcal{H})$ and $c(\mathcal{H})$ are the number of evenly oriented even cycles and even cycles contained in \mathcal{H} , respectively.

In the following we express the Newton's identity for the coefficients of characteristic polynomial of a graph G and at the end of this paper by using of this Theorem we determine the characteristic polynomial of an oriented graph G^σ .

Theorem 1.3. (Newton's identity) Let $\lambda_1, \dots, \lambda_n$ be the roots of the polynomial

$$\chi(G, \lambda) = \lambda^n - s_1 \lambda^{n-1} + s_2 \lambda^{n-2} + \dots + (-1)^{n-1} s_{n-1} \lambda + (-1)^n s_n$$

with spectral moment T_k . Then the k -th coefficient of characteristic polynomial is equal to

$$s_k = \frac{-1}{k} (T_k + T_{k-1} s_1 + \dots + T_1 s_{k-1}).$$

2. Main Results

In this section, we first define the concept of the spectral moments of oriented graph G^σ and we call the skew spectral moment of G^σ . Then we determine the k -th skew spectral moments of oriented graphs for $k = 0, 1, 2, 3, 4, 5, 6$. Also we consider an ordering of the oriented graphs and we give an ordering for the oriented unicyclic graphs. Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be the eigenvalues of S , where each of the λ_i , ($1 \leq i \leq n$), is 0 or pure imaginary. The number of $\sum_{i=1}^n (\lambda_i)^k$ is called the k -th skew spectral moment of G^σ , denoted by $T_k(G^\sigma)$.

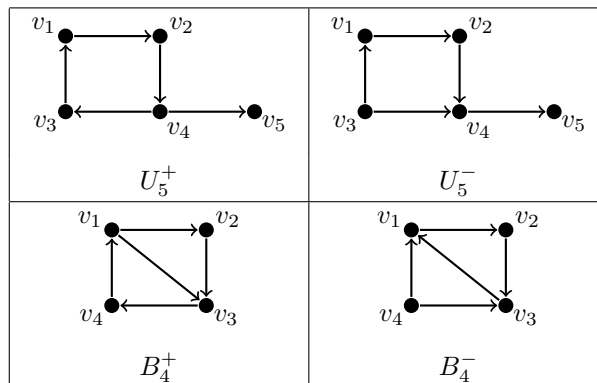


FIGURE 1. The oriented graphs U_5^σ and B_4^σ .

Lemma 2.1. The k -th skew spectral moment of G^σ is the number of closed walks with positive sign of length k minus closed walks with negative sign of length k .

Proof. Since iS is a Hermitian matrix and S is diagonalizable matrix over the complex field \mathbb{C} , $tr(S^k)$ is equal to k -th skew spectral moment of G^σ . On the other hand by Theorem 1.1, $tr(S^k)$ is the number of closed walks with positive sign of length k minus closed walks with negative sign of length k . \square

Let $T(G^\sigma) = (T_0(G^\sigma), T_1(G^\sigma), \dots, T_{n-1}(G^\sigma))$ be the sequence of skew spectral moments of G^σ . For two oriented graphs $G_1^{\sigma_1}$ and $G_2^{\sigma_2}$ we shall write $G_1^{\sigma_1} =_T G_2^{\sigma_2}$ if $T_i(G_1^{\sigma_1}) = T_i(G_2^{\sigma_2})$ for $i = 0, 1, \dots, n-1$. Similarly, we have $G_1^{\sigma_1} \prec_T G_2^{\sigma_2}$ if for some $k, k = 1, 2, \dots, n-1, T_i(G_1^{\sigma_1}) = T_i(G_2^{\sigma_2}) (i = 0, 1, \dots, k-1)$ and $T_k(G_1^{\sigma_1}) < T_k(G_2^{\sigma_2})$. We shall also write $G_1^{\sigma_1} \preceq_T G_2^{\sigma_2}$ if $G_1^{\sigma_1} \prec_T G_2^{\sigma_2}$ or $G_1^{\sigma_1} =_T G_2^{\sigma_2}$.

It is easy to see that $T_0(G^\sigma) = n$ and if k is odd then $T_k(G^\sigma) = 0$. In the following we compute the skew spectral moments of the oriented graph $G^\sigma, T_k(G^\sigma),$ for $k = 2, 4, 6$.

Theorem 2.2. *Let G^σ be an oriented graph. Then we have:*

- (1) $T_2(G^\sigma) = -2m,$
- (2) $T_4(G^\sigma) = 2\varphi(P_2) + 4\varphi(P_3) + 8(\varphi(C_4^+) - \varphi(C_4^-)),$
- (3) $T_6(G^\sigma) = -2\varphi(P_2) - 12\varphi(P_3) - 6\varphi(P_4) - 12\varphi(K_{1,3}) + 12(\varphi(U_5^-) - \varphi(U_5^+)) + 12(\varphi(B_4^-) - \varphi(B_4^+)) - 12\varphi(C_3) + 48(\varphi(C_4^-) - \varphi(C_4^+)) + 12\varphi((C_6^+) - \varphi(C_6^-)),$

where the subgraphs $B_4^+, B_4^-, U_5^+, U_5^-$ is shown in figure 1.

Proof. We have $T_2(G^\sigma) = tr(S^2) =$ the number of closed walks of length 2. For the any adjacent vertex to $v_i,$ there is a closed walk of length 2. It is easy to see that all of such walks are negative and so $w_{ii}^+(2) = 0$. This implies that $T_2(G^\sigma) = -2m$.

Now we compute the 4-th skew spectral moment of G^σ . We have $T_4(G^\sigma) = tr(S^4) =$ the number of positive closed walks minus negative closed walks of length 4. Vertices that belong to a closed walk of length 4 induce in G a subgraph isomorphic to P_2, P_3 or C_4 . For any edge $e = uv$ there are two closed walk with length 4 of the forms $u - v - u - v - u$ and $v - u - v - u - v$. It is easy to see that the sign of such walks is positive. Now we consider a cycle graph C_4 . For any vertex of C_4 there are two closed walks of length 4. If C_4 is oddly oriented, then the sign of each of these closed walks is negative. While if C_4 is evenly oriented, then the sign of each of these closed walks is positive. Thus for any square the number of closed walks of length 4 is equal to $8(\varphi(C_4^+) - \varphi(C_4^-)).$

Now consider a path P_3 . One can see that there are four closed walks with length 4 in P_3 . It is easy to see that the sign such walks is positive, for any orientation of P_3 . Thus the 4-th skew spectral moment of G^σ is equal to $2\varphi(P_2) + 4\varphi(P_3) + 8(\varphi(C_4^+) - \varphi(C_4^-)).$

The vertices that belong to a closed walk of length 6 induce in G a subgraph isomorphic to $P_2, P_3, P_4, S_4, U_5, B_4, B_5, C_3, C_4$ or C_6 . With a simple check one can see that

$$T_6(G^\sigma) = -2\varphi(P_2) - 12\varphi(P_3) - 6\varphi(P_4) - 12\varphi(K_{1,3}) + 12(\varphi(U_5^-) - \varphi(U_5^+)) + 12(\varphi(B_4^-) - \varphi(B_4^+)) - 12\varphi(C_3) + 48(\varphi(C_4^-) - \varphi(C_4^+)) + 12(\varphi(C_6^+) - \varphi(C_6^-)),$$

where the oriented graphs U_5^σ and B_4^σ is depicted in figure 1. In the graph $U_5^+,$ the cycle with length 4, is evenly oriented, while in the graph $U_5^-,$ the cycle with length 4, is oddly oriented. Also in the graph $B_4^+ (B_4^-)$ the cycle with length 4, is evenly (oddly) oriented and the orientation of the edge v_1v_3 is not effective. \square

In the following we consider the set of oriented trees and the set of oriented unicyclic graphs with n vertices and we order these oriented graphs with respect to the skew spectral moments.

Corollary 2.3. *Suppose that T_n^σ is an oriented tree with n vertices. Then in an T -order we have:*

- (1) $T_n^\sigma \preceq_T S_n^\sigma$ and $T_n^\sigma =_T S_n^\sigma$ if and only if $T_n^\sigma \cong S_n^\sigma$.
- (2) $P_n^\sigma \preceq_T T_n^\sigma$ and $T_n^\sigma =_T P_n^\sigma$ if and only if $T_n^\sigma \cong P_n^\sigma$.

Proof. For $i = 0, 1, 2, 3$, we have $T_i(T_n^\sigma) = T_i(S_n^\sigma) = T_i(P_n^\sigma)$. Thus we consider the 4–th skew spectral moment. Theorem 2 of [4] states that, $\varphi(P_3)$ is minimal in P_n and maximal in S_n , and this completes the proof. \square

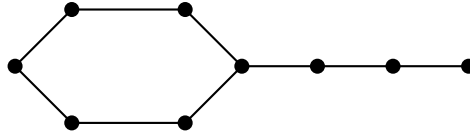


FIGURE 2. The lollipop graph $L(6, 3)$.

Now we consider the set of all unicyclic graphs with an orientation and we give an ordering for these graphs with respect to skew spectral moment. Let $U_{n,k}$ be the set of unicyclic graphs with n vertices ($n \geq 8$) and girth k , $k \geq 5$. Also let $L(k, f)$ be the graph obtained by the coalescence of a cycle C_k of length k with a path P_{f+1} at one of its end vertices. Let $F(k, f)$ be the graph obtained by the coalescence of C_k and a star $K_{1,f}$ at its central vertex. Then we have the following Theorem.

Theorem 2.4. *Let σ be an orientation for $U_{n,k}$. In an T –order of the set of $U_{n,k}^\sigma$, the first oriented graph is $L^\sigma(k, f)$ and the last oriented graph is $F^\sigma(k, f)$.*

Proof. It is easy to see that $T_i(L^\sigma(k, f)) = T_i(F^\sigma(k, f))$, for $i = 0, 1, 2, 3$. Thus we have to calculate the T_4 . In $L^\sigma(k, f)$ we have:

$$\varphi(P_3) = \sum_{i=1}^n \binom{d_i}{2} = f + k + 1$$

and in $F^\sigma(k, f)$ we have:

$$\varphi(P_3) = \sum_{i=1}^n \binom{d_i}{2} = \frac{1}{2}(f^2 + 3f + 2k).$$

It is easy to prove that the minimal and the maximal value of P_3 occur in $L^\sigma(k, f)$ and $F^\sigma(k, f)$, respectively and this completes the proof. \square

The graphs of the form $L(k, f)$ is called a lollipop graph, for example $L(6, 3)$ is shown in figure 2. In the following we consider an orientation σ for the set of all lollipop graphs with n vertices and we find the skew spectral moments of these oriented graphs. In a lollipop graph $L(k, f)$ with n vertices we have $f + k = n$. Since a lollipop graph is a unicyclic graph, then the number of edges in $L(k, f)$ is equal to $f + k = n$. Now let $\mathcal{L}^\sigma(k, f)$ be the set of all lollipop graphs with n vertices. Suppose that σ is an orientation for these graphs. Consider $\mathcal{L}^\sigma(k, f) = \{L^\sigma(k, f) : k, f \geq 3, f + k = n, n \geq 9\}$.

Now we give an ordering for all graphs of $\mathcal{L}^\sigma(k, f)$ with $f + k = n$ vertices.

Theorem 2.5. *For any oriented graph $G^\sigma \in \mathcal{L}^\sigma(k, f) \setminus L^\sigma(4, f)$ we have:*

$$L^-(4, f) \prec_T G^\sigma \prec_T L^+(4, f).$$

Proof. It is easy to see that for $i = 0, 1, 2, 3$, $T_i(L^-(4, f)) = T_i(G^\sigma) = T_i(L^+(4, f))$. Also by using of Theorem (2.2) we have:

$$\begin{aligned} T_4(L^-(4, f)) &= 2n + 4(3 + (n - 2)) - 8 = 6n - 4, \\ T_4(G^\sigma) &= 2n + 4(3 + (n - 2)) = 6n + 4, \\ T_4(L^+(4, f)) &= 2n + 4(3 + (n - 2)) + 8 = 6n + 12. \end{aligned}$$

Thus $T_4(L^-(4, f)) < T_4(G^\sigma) < T_4(L^+(4, f))$. Therefore

$$L^-(4, f) \prec_T G^\sigma \prec_T L^+(4, f).$$

□

Theorem 2.6. For any oriented graph $G^\sigma \in \mathcal{L}^\sigma(k, f) \setminus \{L^\sigma(4, f), L^\sigma(6, f), L^\sigma(3, f)\}$ we have:

$$L^-(6, f) \prec_T G^\sigma \prec_T L^\sigma(3, f) \prec_T L^+(6, f).$$

Proof. It is easy to see that $T_i(G^\sigma) = T_i(L^-(6, f)) = T_i(L^+(6, f)) = T_i(L^\sigma(3, f))$ for $i = 0, 1, 2, 3, 4, 5$. By Theorem (2.2) we have:

$$\begin{aligned} T_6(L^+(6, f)) &= -2n - 12(n + 1) - 6(n + 2) - 12 + 12 = -20n - 24, \\ T_6(L^-(6, f)) &= -2n - 12(n + 1) - 6(n + 2) - 12 - 12 = -20n - 48, \end{aligned}$$

and

$$T_6(L^\sigma(k, f)) = \begin{cases} -20n - 30 & k = 3 \\ -20n - 36 & \text{Otherwise.} \end{cases}$$

Thus $T_6(L^-(6, f)) < T_6(L^\sigma(k, f)) < T_6(L^\sigma(3, f)) < T_6(L^+(6, f))$. Therefore

$$L^-(6, f) \prec_T L^\sigma(k, f) \prec_T L^\sigma(3, f) \prec_T L^+(6, f).$$

□

In the following we consider the skew characteristic polynomial of G^σ and we determine the coefficients of $\chi(G^\sigma, \lambda)$. It is clear that $s_k = 0$ if k is odd.

Theorem 2.7. Let G^σ be an oriented graph with n vertices and m edges and skew characteristic polynomial $\chi(G^\sigma, \lambda)$. Then the k -th coefficient of $\chi(G^\sigma, \lambda)$ for $k = 2, 4, 6$ is as the following:

- (1) $s_2(G^\sigma) = m$,
- (2) $s_4(G^\sigma) = \frac{1}{2}(m^2 - m) - \sum_{i=1}^n \binom{d_i}{2} - 2(\varphi(C_4^+) - \varphi(C_4^-))$,
- (3) $s_6(G^\sigma) = \frac{1}{3}m - \frac{1}{2}m^2 + \frac{1}{6}m^3 - (m - 2) \sum_{i=1}^n \binom{d_i}{2} + \varphi(P_4) + 2 \sum_{i=1}^n \binom{d_i}{3} - 2(\varphi(U_5^-) - \varphi(U_5^+)) - 2(\varphi(B_4^-) - \varphi(B_4^+)) + 2\varphi(C_3) - (2m - 8)(\varphi(C_4^+) - \varphi(C_4^-)) - 2(\varphi(C_6^+) - \varphi(C_6^-))$.

Proof. By using of Theorem 1.3 we have

$$s_k = \frac{-1}{k}(T_k + T_{k-1}s_1 + \dots + T_1s_{k-1}),$$

where T_k is k -th skew spectral moment of G^σ . Notice that if $k = 2$, then

$$s_2(G^\sigma) = \frac{-1}{2}(T_2) = \frac{-1}{2}(-2m) = m.$$

If $k = 4$, then

$$\begin{aligned} s_4(G^\sigma) &= \frac{-1}{4}(T_4 + T_2s_2) = \frac{-1}{4}(2m + 4 \sum_{i=1}^n \binom{d_i}{2}) - 2m^2 + 8(\varphi(C_4^+) - \varphi(C_4^-)) \\ &= \frac{1}{2}(m^2 - m) - \sum_{i=1}^n \binom{d_i}{2} - 2(\varphi(C_4^+) - \varphi(C_4^-)). \end{aligned}$$

Now assume that $k = 6$, then

$$\begin{aligned} s_6(G^\sigma) &= \frac{-1}{6}(T_6 + T_4s_2 + T_2s_4) = \frac{-1}{6}[-2\varphi(P_2) - 12\varphi(P_3) - 6\varphi(P_4) - 12\varphi(K_{1,3}) \\ &+ 12(\varphi(U_5^-) - \varphi(U_5^+)) + 12(\varphi(B_4^-) - \varphi(B_4^+)) - 12\varphi(C_3) + 48(\varphi(C_4^-) - \varphi(C_4^+)) \\ &+ 12(\varphi(C_6^+) - \varphi(C_6^-)) + m(2\varphi(P_2) + 4\varphi(P_3) + 8(\varphi(C_4^+) - \varphi(C_4^-))) - 2m(\frac{1}{2}(m^2 - m) \\ &- \sum_{i=1}^n \binom{d_i}{2} - 2(\varphi(C_4^+) - \varphi(C_4^-)))] \\ &= \frac{-1}{6}[-2m + (6m - 12) \sum_{i=1}^n \binom{d_i}{2} - 6\varphi(P_4) - 12 \sum_{i=1}^n \binom{d_i}{3}] \\ &+ 12(\varphi(U_5^-) - \varphi(U_5^+)) + 12(\varphi(B_4^-) - \varphi(B_4^+)) - 12\varphi(C_3) \\ &+ (12m - 48)(\varphi(C_4^+) - \varphi(C_4^-)) - 12(\varphi(C_6^+) - \varphi(C_6^-)) + 3m^2 - m^3] \\ &= \frac{1}{3}m - \frac{1}{2}m^2 + \frac{1}{6}m^3 - (m - 2) \sum_{i=1}^n \binom{d_i}{2} + \varphi(P_4) + 2 \sum_{i=1}^n \binom{d_i}{3} - 2(\varphi(U_5^-) - \varphi(U_5^+)) \\ &- 2(\varphi(B_4^-) - \varphi(B_4^+)) + 2\varphi(C_3) - (2m - 8)(\varphi(C_4^+) - \varphi(C_4^-)) - 2(\varphi(C_6^+) - \varphi(C_6^-)). \end{aligned}$$

□

Corollary 2.8. *Let G be an r -regular graph with n vertices and m edges and σ be an orientation for G . Then the coefficients of the skew characteristic polynomial of G^σ are:*

- (1) $s_2(G^\sigma) = \frac{nr}{2}$,
- (2) $s_4(G^\sigma) = \frac{1}{8}n^2r^2 + \frac{1}{4}nr - \frac{1}{2}nr^2 + 2(\varphi(C_4^+) - \varphi(C_4^-))$,
- (3) $s_6(G^\sigma) = \frac{1}{2}nr + \frac{1}{8}n^2r^2 + \frac{1}{48}n^3r^3 - \frac{1}{4}n^2r^3 - nr^2 + \frac{2}{3}nr^3 + \varphi(P_4) - 2(\varphi(U_5^-) - \varphi(U_5^+)) - 2(\varphi(B_4^-) - \varphi(B_4^+)) + 2\varphi(C_3) - (nr - 8)(\varphi(C_4^+) - \varphi(C_4^-)) - 2(\varphi(C_6^+) - \varphi(C_6^-))$.

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