NEW CLASS OF INTEGRAL BIPARTITE GRAPHS WITH LARGE DIAMETER

ALIREZA FIUJ LAALI∗ AND HAMID HAJ SEYYED JAVADI

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Abstract. In this paper, we construct a new class of integral bipartite graphs (not necessarily trees) with large even diameters. In fact, for every finite set $A$ of positive integers of size $k$ we construct an integral bipartite graph $G$ of diameter $2k$ such that the set of positive eigenvalues of $G$ is exactly $A$. This class of integral bipartite graphs has never found before.

1. Introduction

Let $G$ be a simple graph with the vertex set $\{v_1, v_2, \ldots, v_n\}$. The adjacency matrix of $G$ is an $n \times n$ matrix $A(G)$ whose $(i, j)$-entry is 1 if $v_i$ is adjacent to $v_j$ and 0, otherwise. The characteristic polynomial of $G$, denoted by $f_G(x)$, is the characteristic polynomial of $A(G)$. We will write it simply $f_G$ when there is no confusion. The roots of $f_G$ are called the eigenvalues of $G$ and can be ordered as $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. It is important to mention that the set of eigenvalues of a bipartite graph is symmetrical with respect to 0. An integral graph is a graph for which the eigenvalues of its adjacency matrix are all integers [5]. Many different classes of integral graphs have been constructed in the past decades. For a long time it has been an open question whether there exist integral trees of arbitrarily large diameter [6]. Csikvari in [2] constructed integral trees with arbitrary large even diameter. A very elegant and short proof of the integrality of Csikvari’s trees can be found in page 90 of [1]. Ghorbani, Mohammadian, and Tayfeh-Rezaie constructed integral trees with arbitrary large odd diameter [4]. In this paper by a recursive method we construct new class of integral bipartite graphs (not necessarily trees) with arbitrary large even diameter. Let $A = \{n_1, n_2, \ldots, n_k\}$ be a set of positive integers such

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∗Corresponding author.

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that \( n_1 < n_2 < \cdots < n_k \). In Section 2 we give a recursive method to construct integral bipartite graph \( G \) with diameter \( 2k \) which the set of all positive eigenvalues of \( G \) is exactly \( A \). If \( n_1 > 1 \), then the constructed bipartite graph will be not a tree. In Section 3 we prove the integrality of the constructed graph.

2. Recursive Method

We use the notation \( K_{n,m} \) for complete bipartite graph with two partite sets having \( n \) and \( m \) vertices.

**Definition 2.1.** Let \( G_1 \) and \( G_2 \) be rooted graphs with roots \( a \) and \( b \), respectively. Then the graph \( G_1 \sim G_2 \) obtained from \( G_1 \cup G_2 \) by joining the vertices \( a \) and \( b \). We obtain the \( G_1 \sim mG_2 \) by taking \( G_1 \) and \( m \) copies of \( G_2 \) and we join \( a \) to each copies of \( b \). The root of \( G_1 \) is considered to be the root of \( G_1 \sim G_2 \).

Let \( G - a \) be the graph obtained from \( G \) by deleting the vertex \( a \) and all edges containing \( a \). Then we have following lemma.

**Lemma 2.2.** [3] Let \( G_1 \) and \( G_2 \) be rooted graphs with roots \( a \) and \( b \), respectively, and \( m \geq 1 \). Then

\[
f_{G_1 \sim mG_2} = f_{G_2}^{m-1}(f_{G_1}f_{G_2} - mf_{G_1 - a}f_{G_2 - b}).
\]

Let \( A = \{n_1,n_2,\ldots,n_k\} \) be a set of positive integers which \( n_1 < n_2 < \cdots < n_k \). Let a vertex in the part of size \( n_1 + 1 \) be the root of \( K_{n_1,n_1+1} \) and a vertex in the part of size \( 1 \) be the root of \( K_{1,n_1^2} \). Define

\[
G_1 = K_{n_1,n_1+1},
G_2 = G_1 \sim (n_2^2 - n_1^2 - n_1)K_{1,n_1^2},
G_3 = K_{1,n_1^2} \sim (n_3^2 - n_2^2)G_2,
G_4 = G_2 \sim (n_4^2 - n_3^2)G_3,
\vdots
G_k = G_{k-2} \sim (n_k^2 - n_{k-1}^2)G_{k-1}.
\]

It is easy to see that the diameter of \( G_k \) is \( 2k \). Clearly, if \( n_1 = 1 \), then all graphs constructed in the above are trees, and if \( n_1 > 1 \), then none of graphs constructed in the above is a tree. In Section 3 we prove that the set of positive eigenvalues of graph \( G_k \) is exactly \( A \).

**Example 2.3.** Let \( A = \{2, 3, 4\} \). In the following we indicate the graphs \( G_1, G_2, \) and \( G_3 \) constructed in the above manner.

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The set of the positive eigenvalues of $G_1$ is $\{\sqrt{6}\}$.

The set of the positive eigenvalues of $G_2$ is $\{2, 3\}$.

The set of the positive eigenvalues of $G_3$ is $\{2, 3, 4\}$.

3. Proof Of The Integrality Of Graph $G_k$

**Theorem 3.1.** Let $A = \{n_1, n_2, \ldots, n_k\}$ be a set of positive integers, $k \geq 2$, and $n_1 < n_2 < \cdots < n_k$. Then the set of all positive eigenvalues of $G_k$ is exactly $A$.

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Proof. For convenience, we set $f_i(\lambda) = f_{G_i}(\lambda)$, $p_i = n_i^2 - n_i^2 - 1$ and $f'_i(\lambda) = f_{G_i-a_i}(\lambda)$ where $a_i$ is the root of graph $G_i$ for $i = 2, \ldots, k$. By the recursive construction which is given in Section 2, it is easily seen that $f'_i = f'_{i-2}f'_i$ for $i = 4, \ldots, k - 1$. By Lemma 2.2 it follows that

$$ f_k = \prod_{i=2}^{k-1} f_{k-1}^{p_{k-i+1}} (f_2 \lambda^{n_1^2-1}(\lambda^2 - n_1^2) - (n_k^2 - n_2^2) f'_{i-2} \lambda^{n_1^2}) $$

By Lemma 2.2 it is clear that

$$ f_2 = \lambda^{(n_1^2-1)(p_2-n_1-1)+n_1^2+2n_1-2(\lambda^2-n_1^2)p_2-n_1}(\lambda^2-n_2^2), $$

$$ f'_2 = \lambda^{n_1^2p_2-n_1^2-p_2+3n_1-1}(\lambda^2-n_1^2)p_2-n_1+1, $$

$$ f_3 = f_2^{p_3-1}(f_2^2 \lambda^{n_1^2-1}(\lambda^2-n_1^2) - p_3 f'_2 \lambda^{n_1^2}), $$

$$ f'_3 = \lambda^{n_1^2} f'^{p_3}. $$

So we obtain that

$$ f_k = \prod_{i=2}^{k-1} f_{k-1}^{p_{k-i+1}} (f_2 \lambda^{n_1^2-1}(\lambda^2 - n_1^2) - (n_k^2 - n_2^2) f'_{i-2} \lambda^{n_1^2}) $$

Since all the roots of polynomials $f_2$ and $f_3$ are integers, $G_2$ and $G_3$ are integral graphs. By the above relation, we obtain $G_i$ is integral and its distinct positive eigenvalues are $n_1, \ldots, n_i$ for $i = 4, \ldots, k$. This completes the proof.

\[\square\]

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References


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Alireza Fiuj Laali
Department of Mathematics and Computer Science, University of Shahed, P.O.Box 18155-159, Tehran, Iran
Email: a.fiujlaali@shahed.ac.ir

Hamid Haj Seyyed Javadi
Department of Mathematics and Computer Science, University of Shahed, P.O.Box 18155-159, Tehran, Iran
Email: h.s.javadi@shahed.ac.ir

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