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A NEW PROOF OF VALIDITY OF BOUCHET'S CONJECTURE ON EULERIAN BIDIRECTED GRAPHS

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Abstract. Recently, E. Máčajová and M. Škoviera proved that every bidirected Eulerian graph which admits a nowhere zero flow, admits a nowhere zero 4-flow. This result shows the validity of Bouchet's nowhere zero conjecture for Eulerian bidirected graphs. In this paper we prove the same theorem in a different terminology and with a short and simple proof. More precisely, we prove that every Eulerian undirected graph which admits a zero-sum flow, admits a zero-sum 4-flow. As a conclusion we obtain a shorter proof for the previously mentioned result of Máčajová and Škoviera.

1. Introduction

Let $G = (V, E)$ be a simple graph with vertex set V and edge set E. The incidence matrix of G denoted by $W(G)$ and is defined as follows:

$$w_{ij} = \begin{cases} 1 & \text{if } e_j \text{ is incident with } v_i, \\ 0 & \text{otherwise.} \end{cases}$$

The kernel of the incidence matrix of G is denoted by $\ker(G)$. We call a real valued edge function $f : E \to \mathbb{R}$ a *flow* on G, if $\sum_{e;v\in e} f(e) = 0$, for every vertex $v \in V$. A *k-flow* is a flow with values in the set $\{0, \pm1, \pm2, \ldots, \pm(k-1)\}$. A zero-sum k-flow, is a k-flow whose value is nonzero for every edge. In fact a flow corresponds to a vector in $\ker(G)$ and a zero-sum flow corresponds to a vector in $\ker(G)$ with no zero entry. We call an edge function f satisfies the *zero-sum rule* on the vertex v if $\sum_{u:uv\in E(G)} f(uv) = 0$.

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A *bidirected edge* is an edge consisting of two half-edges which receive separate orientations and a *bidirected graph* is a graph whose edges are bidirected. Thus every edge is oriented with one of the four possible orientations:

The first two edges are called ordinary edges and the next two edges are called out-edge and in-edge (or opposite edges), respectively.

An integer valued function f on the edges of a bidirected graph G with the edge set $E(G)$ is a *nowhere-zero bidirected k-flow* if for every $e \in E(G)$, $0 < |f(e)| < k$, and at every vertex v, the sum of values on the half-edges directed to v equals the sum of values on the half-edges directed out of v.

In the case of bidirected graph, Bouchet in [3] have the following conjecture:

Bouchet's Conjecture Every bidirected graph which admits a nowhere-zero bidirected flow admits a nowhere-zero bidirected 6-flow.

This conjecture has been verified for by Zyka if 6 is replaced by 30, see [8] and recently has been proved by DeVos in [4], when 6 replaced with 12.

In [2] we study zero-sum flows in undirected graphs and mentioned the following conjecture:

Zero-Sum Conjecture(ZSC). If G is a graph with a zero-sum flow, then G has a zero-sum 6-flow. We also found the necessary and sufficient condition for a graph to have a zero-sum flow. It is easy to see that ZSC is a special case of Bouchet's conjecture. In [1], it is proved that ZSC is equivalent to Bouchet's conjecture.

In this paper, firstly we prove that every Eulerian graph which admits a zero-sum flow admits a zero-sum 4-flow. As a conclusion we obtain a shorter proof for the result of Máčajová and Škoviera in [6], which state that every Eulerian bidirected graph having nowhere zero flow, admits a nowhere zero 4-flow.

Here is some notations and definitions most of which can be found in [7]. A *walk* in a graph G is a list $v_0e_1v_1 \cdots v_ke_k$ of vertices and edges such that, for $1 \leq i \leq k$, the edge e_i has end points v_{i-1} and v_i. A *trail* is a walk with no repeated edge. A u, v-trail is a trail with first vertex u and last vertex v, and we denote a v, v-trail by v-trail. A trail is called a *closed trail* if its end points are the same. A graph is *Eulerian* if it has a closed trail containing all edges of the graph. We call a closed trail a *circuit* when we do not specify the first vertex but keep the list in the cyclic order. An *Eulerian circuit* or *Eulerian trail* in a graph is a circuit or trail containing all edges.

Let G be a bidirected graph then $S(G)$ is an undirected graph obtained from G by removing all directions of edges of G and replacing the ordinary edges of G with a path of length 2.

2. Zero-sum flows in undirected graphs

The following Lemma shows the relation between nowhere zero k-flow in bidirected graphs and zero-sum k-flow in undirected graphs. For the proof see [1, Theorem 2.1].

Lemma 2.1. *A bidirected graph G has nowhere zero k-flow if and only if the graph $S(G)$ has zero-sum k-flow.*

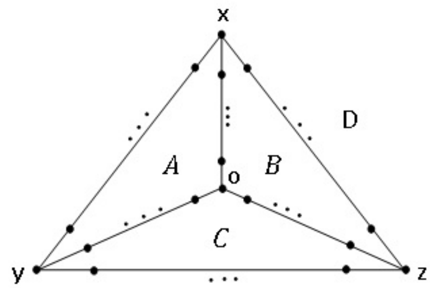

Figure 1. Odd edge-k_4

Definition 1. [5] An *odd edge-bicycle* is a union of two odd cycles joined by a path (the path may have length zero); the cycles and the path are required to be edge-disjoint (but may intersect at vertices). An *edge-*K_4 is a union of six paths as shown in Figure 1, which are again required to be edge-disjoint (but may intersect at vertices). If the cycles A, B, C, D are all odd (D is the boundary of the outer region), we have an *odd edge-*K_4.

Theorem 2.2. [5] *The following statements are equivalent for a connected graph* G*.*

(1) *G does not have an odd edge-bicycle or an odd edge-K_4;*
(2) *There is an edge e such that $G \setminus e$ is bipartite.*

Theorem 2.3. [2] *Suppose G is not a bipartite graph. Then G has a zero-sum flow if and only if for any edge e of G, $G \setminus \{e\}$ has no bipartite component.*

Corollary 2.4. *Let G be a non bipartite graph. If G has a zero-sum flow, then G has an odd edge-bicycle or an odd edge-K_4.*

Lemma 2.5. *Every odd edge-bicycle graph and every odd edge-k_4 graph admits zero-sum* 3*-flow.*

Proof. For an odd edge-bicycle G one can simply assign ± 1 to the edges of the cycles, respectively. And assign ± 2 to the edges of the connecting path, respectively in such a way that the result is a flow on G. So, every odd edge-bicycle graph has a zero-sum 3-flow.

An edge-K_4 graph is an odd edge-K_4, if and only if one of the following cases occur:

(1) All the six paths ox, oy, oz, xy, xz, yz have odd lengths,
(2) Two of the boundary paths and the inner path intersect them (for example the paths xy, ox, xz) have even lengths and the other paths have odd length,
(3) One boundary path and one inner path which do not intersect (for example paths ox, yz) have odd lengths and the other have even lengths,
(4) The three boundary paths have odd length and three inner paths have even lengths.

For example, in Figure 2 it is shown that in case (1) the odd edge-k_4 have zero-sum 3-flow. The other three cases have zero-sum 3-flow, similarly.

□

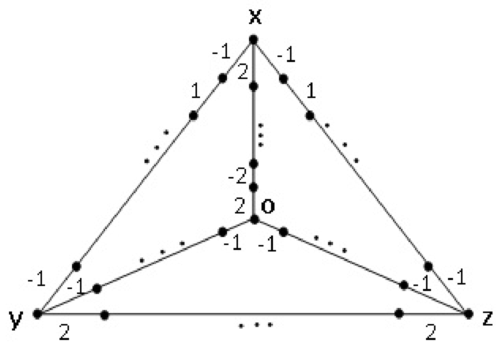

FIGURE 2. zero-sum 3-flow on Odd edge-k_4

3. Eulerian Graphs

Lemma 3.1. *Let G be an Eulerian connected graph with an even number of edges, then G admits a zero-sum 2-flow.*

Proof. Let C be an Eulerian circuit of G. If we traverse the edges of C according to the cyclic ordering and assign ± 1 to the edges, respectively, we yield a zero-sum 2-flow for G. $\square$

Theorem 3.2. *Let G be an Eulerian graph which admits a zero-sum flow. Then G admits a zero-sum 4-flow.*

Proof. If G is an Eulerian graph with an even number of edges, then by Lemma 3.1, G has a zero-sum 2-flow. Now, let G be an Eulerian graph with odd number of edges which admits a zero-sum flow. Then G contains an odd cycle. Otherwise, G is bipartite with parts X and Y. Hence, $|E(G)| = \sum_{v \in X} deg(v)$, since each vertex has even degree, $E(G)$ will be even, which contradicts to the assumption. By Corollary 2.4, G contains an odd edge-bicycle or an odd edge-K_4. Let C be an odd cycle which contained in an odd edge-bicycle or an odd edge-k_4.

We delete the edges of C from G and denote the result graph by $H = G \setminus E(C)$. Clearly each component of H is an Eulerian Graph. The graph H has some components with even number of edges. We denote the union of these components by K. By lemma 3.1, K has a zero-sum 2-flow, we call it f_0. H may also have some components with odd number of edges, suppose H has m of such components and called them $G_1, G_2, \ldots, G_m$. Then m should be an even number, otherwise G will have an even number of edges which is a contradiction.

If $m = 0$, then by Lemma 2.5, there is a 3-flow f_1 whose values are nonzero on the edges of C. Hence, $f = 3f_0 + f_1$ is a zero-sum 4-flow for G.

Now, suppose that $m > 0$. Let $H_1 = C \cup G_m$. Then H_1 is an Eulerian graph with even number of edges, so it has a zero-sum 2-flow called f_1. Let $H_2 = C \cup G_1 \cup \cdots \cup G_{m-1}$. Since m is an even number and the number of edges of each of the graphs $C, G_1, \ldots, G_{m-1}$ is an odd number, the number of edges of H_2 is an even number. Hence, by Lemma 3.1, H_2 has a zero-sum 2-flow called f_2. Now, let $f = 2f_2 + f_1 + f_0$ is a zero-sum 3-flow for G. □

Theorem 3.3. *Every Eulerin bidirected graph G which admits a nowhere zero k-flow, admits a nowhere zero 4-flow.*

Proof. Let G be an Eulerian bidirected graph which admits a nowhere zero k-flow, then $S(G)$ is an Eulerian undirected graph which admits a zero-sum k-flow. By Theorem 3.2, $S(G)$ admits a zero-sum 4-flow. Hence, by Lemma 2.1, the bidirected graph G admits a nowhere zero 4-flow, as desired. □

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