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THE CENTRAL VERTICES AND RADIUS OF THE REGULAR GRAPH OF IDEALS

FARZAD SHAVEISI

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ABSTRACT. The regular graph of ideals of the commutative ring R, denoted by $\Gamma_{reg}(R)$, is a graph whose vertex set is the set of all non-trivial ideals of R and two distinct vertices I and J are adjacent if and only if either I contains a J-regular element or J contains an I-regular element. In this paper, it is proved that the radius of $\Gamma_{reg}(R)$ equals 3. The central vertices of $\Gamma_{reg}(R)$ are determined, too.

1. **Introduction**

We begin with recalling some definitions and notations on graphs. Let G be a simple graph. The *distance* between two vertices x and y of G is denoted by $d(x, y)$. A graph is said to be *connected* if there exists a path between any two distinct vertices. The *diameter* of a connected graph G, denoted by $diam(G)$, is the maximum distance between any pair of vertices of G. For any vertex x of a connected graph G, the *eccentricity* of x, denoted by $e(x)$, is the maximum of the distances from x to the other vertices of G. The set of vertices with minimal eccentricity is called the *center* of the graph. This minimum eccentricity value is the *radius* of G and it is denoted by $r(G)$. Let Γ be a digraph. An arc from a vertex x to another vertex y of Γ is denoted by $x \longrightarrow y$. Also we distinguish the *out-degree* $d^+_\Gamma(v)$, the number of edges leaving the vertex v, and the *in-degree* $d^-_\Gamma(v)$, the number of edges entering the vertex v. For more details about the standard terminology of graphs, see [4].

Unless otherwise stated, throughout this paper, all rings are assumed to be commutative Artinian rings with identity. We denote by $Max(R)$ and $Nil(R)$, the set of all maximal ideals and the set of all nilpotent elements of R, respectively. The ring R is said to be *reduced* if $Nil(R) = (0)$. Also, the

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set of all zero-divisors of an R-module M is denoted by $Z(M)$. An element r in the ring R is called *M-regular* if $r \notin Z(M)$. For every ideal I of R, the *annihilator* of I is denoted by $Ann(I)$.

Graphs from commutative rings have gained attention of many researchers for the past three decades. Several graphs have been constructed from commutative rings. In many of these graph constructions, the vertex set of the graph is either the elements of the corresponding commutative ring or ideals of the same. Ideal based constructions are in a sense generalizations of element based constructions. To see an instance of these graphs, the reader is referred to [1, 5, 6, 8, 9, 10]. This paper deals about the regular graph of ideals of a commutative ring.The *regular digraph of ideals* of a ring R, denoted by $\overrightarrow{\Gamma_{reg}}(R)$, is a digraph whose vertex set is the set of all non-trivial ideals of R and for every two distinct vertices I and J, there is an arc from I to J if and only if I contains a J-regular element. The underlying graph of $\overrightarrow{\Gamma_{reg}}(R)$, denoted by $\Gamma_{reg}(R)$, is called the *regular graph* of ideals of R. For more information about this graph, see [2, 7]. The main aim of this paper is to prove that $r(\Gamma_{reg}(R)) = 3$.

2. Preliminary Results and Notations

In this section, the distance between any pair of vertices of $\Gamma_{reg}(R)$ is determined, when R is an Artinian non-reduced ring and $\Gamma_{reg}(R)$ is a connected graph.

Remark 2.1. *Let I, J and K be three distinct vertices of $\overrightarrow{\Gamma_{reg}}(R)$ and $I \longrightarrow J \longrightarrow K$ be a directed path in $\overrightarrow{\Gamma_{reg}}(R)$. Then by using the definition, one can show that there is an arc from I to K in $\overrightarrow{\Gamma_{reg}}(R)$.*

Notation. Let I and K be (not necessarily distinct) ideals of R. We denote by $\mathcal{C}^-(I,K)$, the set of all non-trivial ideals J of R such that J contains an I-regular element and J contains a K-regular element. Also, we denote by $\mathcal{C}^+(I,K)$, the set of all non-trivial ideals J of R such that I contains a J-regular element and K contains a J-regular element. For simplicity, we denote $\mathcal{C}^-(I,(0))$ and $\mathcal{C}^+(I,R)$ by $\mathcal{C}^-(I)$ and $\mathcal{C}^+(I)$, respectively.

Remark 2.2. *Let $R_1,\ldots,R_n$ be rings, $R \cong R_1 \times \cdots \times R_n$ and $I \cong I_1 \times \cdots \times I_n$ and $J \cong J_1 \times \cdots \times J_n$ be two distinct vertices of $\overrightarrow{\Gamma_{reg}}(R)$. Then there is an arc from I to J if and only if $I_i \in \mathcal{C}^-(J_i) \cup \{R_i\}$, for every i, if and only if $J_i \in \mathcal{C}^+(I_i) \cup \{(0)\}$, for every i.*

Assume that R is an Artinian ring such that $\Gamma_{reg}(R)$ is a connected graph. Then by [7, Theorem 2.3], $|\mathrm{Max}(R)| \geq 3$ and R contains a field as its direct summand. So, [3, Theorem 8.7] implies that $R \cong F_1 \times R_2 \times R_3$, where F_1 is a field, R_2 is an Artinian local ring and R_3 is an Artinian ring. Moreover, if R is non-reduced, then we also can suppose that R_3 is not a field. Thus, any vertex of $\Gamma_{reg}(R)$ belongs to one of the following subsets:

$$\mathfrak{A} = \{\mathfrak{a} = F_1 \times R_2 \times (0), \mathfrak{b} = F_1 \times (0) \times R_3, \mathfrak{d} = (0) \times R_2 \times R_3, \mathfrak{u} = (0) \times R_2 \times (0), \mathfrak{v} = (0) \times (0) \times R_3\};$$
$$\mathfrak{C} = \{\mathfrak{c} = F_1 \times I_2 \times I_3 |\ I_2 \times I_3 \neq R_2 \times R_3\} \setminus \{\mathfrak{a}, \mathfrak{b}\};$$
$$\mathfrak{W} = \{\mathfrak{w} = (0) \times K_2 \times K_3 |\ K_2 \times K_3 \neq (0)\} \setminus \{\mathfrak{d}, \mathfrak{u}, \mathfrak{v}\},$$

where I_2 and K_2 are ideals of R_2 and I_3 and K_3 are ideals of R_3.

Proposition 2.3. *Let R be an Artinian non-reduced ring. If $\Gamma_{reg}(R)$ is a connected graph, then the following statements hold:*

(1) $d(\mathfrak{a},\mathfrak{b}) = d(\mathfrak{u},\mathfrak{v}) = d(\mathfrak{a},\mathfrak{d}) = d(\mathfrak{b},\mathfrak{d}) = 2$

(2) $d(\mathfrak{a},\mathfrak{u}) = d(\mathfrak{b},\mathfrak{v}) = d(\mathfrak{d},\mathfrak{u}) = d(\mathfrak{d},\mathfrak{v}) = d(\mathfrak{d},\mathfrak{w}) = 1$, *where* $\mathfrak{w} \in \mathfrak{W}$.

(3) $d(\mathfrak{a},\mathfrak{v}) = d(\mathfrak{b},\mathfrak{u}) = 3$.

(4) *For every vertex $\mathfrak{c} = F_1 \times I_2 \times I_3 \in \mathfrak{C}$, $d(\mathfrak{a},\mathfrak{c}) \leq 2$. Moreover, the equality holds if and only if $I_2 \neq R_2$ and $I_3 \neq (0)$.*

(5) *For every vertex $\mathfrak{c} = F_1 \times I_2 \times I_3 \in \mathfrak{C}$, $d(\mathfrak{b},\mathfrak{c}) \leq 2$. Moreover, the equality holds if and only if $I_2 \neq (0)$ and $I_3 \neq R_3$.*

(6) *For every vertex $\mathfrak{w} = (0) \times K_2 \times K_3 \in \mathfrak{W}$, $d(\mathfrak{u},\mathfrak{w}) \leq 2$. Moreover, the equality holds if and only if $K_2 \neq R_2$ and $K_3 \neq (0)$.*

(7) *For every vertex $\mathfrak{w} = (0) \times K_2 \times K_3 \in \mathfrak{W}$, $d(\mathfrak{v},\mathfrak{w}) \leq 2$. Moreover, the equality holds if and only if $K_2 \neq (0)$ and $K_3 \neq R_3$.*

Proof. The assertions (1), (2) and (3) are clear and follow from the definition. Choose $\mathfrak{c} = F_1 \times I_2 \times I_3 \in \mathfrak{C}$. Clearly, $\mathfrak{a}$ and $\mathfrak{c}$ are adjacent if and only if either $I_3 = (0)$ or $I_2 = R_2$. So, we can suppose that $I_3 \neq (0)$ and $I_2 \neq R_2$. Since both of vertices $\mathfrak{a}$ and $\mathfrak{c}$ are adjacent to $F_1 \times (0) \times (0)$, we deduce that

$$d(\mathfrak{a},\mathfrak{c}) = \begin{cases} 1; & \text{either } I_3 = (0) \text{ or } I_2 = R_2 \\ 2; & I_3 \neq (0) \text{ and } I_2 \neq R_2. \end{cases}$$

So, (4) follows. Also, the proofs of (5), (6) and (7) are similar to that of (4). □

Proposition 2.4. *Let R be an Artinian non-reduced ring. If $\Gamma_{reg}(R)$ is a connected graph, then the following statements hold:*

(1) *For every two distinct vertices $\mathfrak{c} = F_1 \times I_2 \times I_3$ and $\mathfrak{c}' = F_1 \times J_2 \times J_3$ in $\mathfrak{C}$, we have $d(\mathfrak{c},\mathfrak{c}') \leq 2$. Moreover, $d(\mathfrak{c},\mathfrak{c}') = 1$ if and only if $I_2 = I_3 = (0)$, $J_2 = J_3 = (0)$ or $I_2 \times I_3$ and $J_2 \times J_3$ are adjacent vertices in $\Gamma_{reg}(R_2 \times R_3)$.*

(2) *For every two distinct vertices $\mathfrak{w} = (0) \times K_2 \times K_3$ and $\mathfrak{w}' = (0) \times L_2 \times L_3$ in $\mathfrak{W}$, we have $d(\mathfrak{w},\mathfrak{w}') \leq 2$. Moreover, $d(\mathfrak{w},\mathfrak{w}') = 1$ if and only if $K_2 \times K_3 = R_2 \times R_3$, $L_2 \times L_3 = R_2 \times R_3$ or $K_2 \times K_3$ and $L_2 \times L_3$ are adjacent vertices in $\Gamma_{reg}(R_2 \times R_3)$.*

Proof. This is a direct consequence of the definition. □

Proposition 2.5. *Let R be an Artinian non-reduced ring. If $\Gamma_{reg}(R)$ is a connected graph, then for every vertex $\mathfrak{c} = F_1 \times I_2 \times I_3 \in \mathfrak{C}$, we have:*

(1) $$d(\mathfrak{c},\mathfrak{u}) = \begin{cases} 1; & I_2 = R_2 \\ 2; & I_2 \neq R_2 \text{ and } \mathcal{C}^{-}(I_3) \neq \varnothing. \\ 3; & \text{Otherwise.} \end{cases}$$

$$(2)\ \ d(\mathfrak{c},\mathfrak{v})=\begin{cases}1; & I_3=R_3\\ 2; & I_3\neq R_3 \text{ and either } I_2=(0) \text{ or } \mathcal{C}^+(I_3)\neq\varnothing\\ 3; & \text{otherwise.}\end{cases}$$

$$(3)\ \ d(\mathfrak{c},\mathfrak{d})=\begin{cases}2; & I_2=R_2, I_3=R_3 \text{ or } \mathcal{C}^+(I_3)\neq\varnothing\\ 3; & d(\mathfrak{c},\mathfrak{d})\neq 2 \text{ and } (I_2=(0), I_3=(0) \text{ or } \mathcal{C}^-(I_3)\neq\varnothing)\\ 4; & \text{Otherwise.}\end{cases}$$

$$(4)\ \ d(\mathfrak{a},\mathfrak{w})=\begin{cases}1; & K_3=(0)\\ 2; & K_3\neq(0) \text{ and either } K_2=R_2 \text{ or } \mathcal{C}^-(K_3)\neq\varnothing\\ 3; & \text{Otherwise.}\end{cases}$$

$$(5)\ \ d(\mathfrak{b},\mathfrak{w})=\begin{cases}1; & K_2=(0)\\ 2; & K_2\neq(0) \text{ and } \mathcal{C}^+(K_3)\neq\varnothing\\ 3; & K_2\neq(0) \text{ and } \mathcal{C}^+(K_3)=\varnothing.\end{cases}$$

Proof. We only prove (3). The other assertions are proved, similarly. It is clear that $\mathfrak{c}$ and $\mathfrak{d}$ are not adjacent. On the other hand, since $\mathfrak{u}$ and $\mathfrak{d}$ are adjacent, (1) implies that $2\leq d(\mathfrak{c},\mathfrak{d})\leq 4$. Now, we follow the proof in the following three steps:

Step 1. $d(\mathfrak{c},\mathfrak{d})=2$ if and only if $I_2=R_2$, $I_3=R_3$ or $\mathcal{C}^+(I_3)\neq\varnothing$:
If $I_2=R_2(I_3=R_3)$, then both $\mathfrak{c}$ and $\mathfrak{d}$ are adjacent with $\mathfrak{u}(\mathfrak{v})$. Assume that $J_3\in\mathcal{C}^+(I_3)$. Then $\mathfrak{c}$ and $\mathfrak{d}$ are adjacent to $(0)\times(0)\times J_3$. Therefore, in any case, we have $d(\mathfrak{c},\mathfrak{d})=2$. Conversely, let $d(\mathfrak{c},\mathfrak{d})=2$. Then by Remark 2.1, there exists a vertex $J=J_1\times J_2\times J_3$ such that one of the paths $\mathfrak{c}\longleftarrow J\longrightarrow\mathfrak{d}$ and $\mathfrak{c}\longrightarrow J\longleftarrow\mathfrak{d}$ exists. By using Remark 2.2, one can deduce that the existence of the first path is impossible. So, we can suppose that only the second path exists. By Remark 2.2, $J_1=(0)$. Hence either $J_2\neq(0)$ or $J_3\neq(0)$. Therefore, Remark 2.2 implies that $I_2=R_2$, $I_3=R_3$ or $J_3\in\mathcal{C}^+(I_3)\neq\varnothing$.

From now, suppose that $I_2\neq R_2$, $I_3\neq R_3$ and $\mathcal{C}^+_{R_3}(I_3)=\varnothing$. This assumption and the proof in the previous step imply that $d(\mathfrak{c},\mathfrak{d})\geq 3$.

Step 2. $d(\mathfrak{c},\mathfrak{d})=3$ if and only if either $I_2=(0)$ or $\mathcal{C}^-(I_3)\neq\varnothing$:
By Remark 2.1, $d(\mathfrak{c},\mathfrak{d})=3$ if and only if there exist two vertices, say $J=J_1\times J_2\times J_3$ and $L=L_1\times L_2\times L_3$, such that one of the following paths exists:

$$(2.1)\qquad \mathfrak{c}=F_1\times I_2\times I_3\longleftarrow J_1\times J_2\times J_3\longrightarrow L_1\times L_2\times L_3\longleftarrow (0)\times R_2\times R_3=\mathfrak{d};$$

$$(2.2)\qquad \mathfrak{c}=F_1\times I_2\times I_3\longrightarrow J_1\times J_2\times J_3\longleftarrow L_1\times L_2\times L_3\longrightarrow (0)\times R_2\times R_3=\mathfrak{d};$$

By Remark 2.2, Path (2.1) exists if and only if $J_1=F_1$ and this is equivalent to either $J_2\neq R_2$ or $J_3\neq R_3$. On the other hand, from Remark 2.2, we deduce that $J_2\neq R_2$ if and only if $I_2=(0)$ and $J_3\neq R_3$ if and only if either $I_3=(0)$ or $\mathcal{C}^-(I_3)\neq\varnothing$. To complete the proof, it is enough to show that Path (2.2) does not exist. Since $I_2\neq R_2$ and $\mathcal{C}^+(I_3)=\varnothing$, we deduce that $J_1=J_2=(0)$. Thus

Remark 2.2 implies that $J = F_1 \times (0) \times (0)$ and $L = (0) \times R_2 \times R_3$ and this contradicts the adjacency of J and L. Hence Path (2.2) does not exist and so, we are done. $\square$

Proposition 2.6. *Let R be an Artinian non-reduced ring. If $\Gamma_{reg}(R)$ is a connected graph, then for every $\mathfrak{c} = F_1 \times I_2 \times I_3 \in \mathfrak{C}$ and every $\mathfrak{w} = (0) \times K_2 \times K_3 \in \mathfrak{W}$, we have:*

$$d(\mathfrak{c}, \mathfrak{w}) = \begin{cases} 1; & I_2 \times I_3 \text{ contains a } K_2 \times K_3\text{-regular element} \\ 2; & I_2 = K_2 = (0), I_2 = K_2 = R_2 \text{ or } \mathcal{C}^-(I_3, K_3) \cup \mathcal{C}^+(I_3, K_3) \neq \varnothing \\ 5; & I_2, K_2 \text{ are nontrivial, } \mathcal{C}^+(I_3) \cup \mathcal{C}^-(I_3) \cup \mathcal{C}^+(K_3) \cup \mathcal{C}^-(K_3) = \varnothing \\ 3 \text{ or } 4; & \text{Otherwise.} \end{cases}$$

Proof. It is clear that $\mathfrak{w}$ contains no $\mathfrak{c}$-regular element. Hence $d(\mathfrak{c}, \mathfrak{w}) = 1$ if and only if $\mathfrak{c}$ contains a $\mathfrak{w}$-regular element, say (x_1, x_2, x_3), if and only if $(x_2, x_3) \in I_2 \times I_3$ is a $K_2 \times K_3$-regular element. Assume that $I_2 \times I_3$ contains no $K_2 \times K_3$-regular element. We claim that $d(\mathfrak{c}, \mathfrak{w}) = 2$ if and only if $I_2 = K_2 = (0)$, $I_2 = K_2 = R_2$ or $\mathcal{C}^-(I_3, K_3) \cup \mathcal{C}^+(I_3, K_3) \neq \varnothing$. By Remark 2.1, $d(\mathfrak{c}, \mathfrak{w}) = 2$ if and only if there exists a vertex, say $J = J_1 \times J_2 \times J_3$, such that one of the following paths exists:

$$\mathfrak{c} = F_1 \times I_2 \times I_3 \longleftarrow J_1 \times J_2 \times J_3 \longrightarrow (0) \times K_2 \times K_3 = \mathfrak{w}; \tag{2.3}$$

$$\mathfrak{c} = F_1 \times I_2 \times I_3 \longrightarrow J_1 \times J_2 \times J_3 \longleftarrow (0) \times K_2 \times K_3 = \mathfrak{w}; \tag{2.4}$$

By Remark 2.2, Path (2.3) exists if and only if $J_1 = F_1$ and this is equivalent to either $J_2 \neq R_2$ or $J_3 \neq R_3$. On the other hand, from Remark 2.2, we deduce that $J_2 \neq R_2$ if and only if $I_2 = K_2 = (0)$ and $J_3 \neq R_3$ if and only if $\mathcal{C}^-(I_3, K_3) \neq \varnothing$. Also, Path (2.4) exists if and only if $J_1 = (0)$ if and only if either $J_2 \neq (0)$ or $J_3 \neq (0)$. Moreover, Remark 2.2 implies that $J_2 \neq (0)$ if and only if $I_2 = K_2 = R_2$ and $J_3 \neq (0)$ if and only if $\mathcal{C}^+(I_3, K_3) \neq \varnothing$. So, the claim is proved. Finally, assume that $d(\mathfrak{c}, \mathfrak{w}) \geq 3$, I_2 and K_2 are non-trivial ideals of R_2 and $\mathcal{C}^-(I_3) \cup \mathcal{C}^+(I_3) \cup \mathcal{C}^-(K_3) \cup \mathcal{C}^+(K_3) = \varnothing$. Then [7, Theorem 2.1] and Remark 2.1 imply that $d^+_{\overrightarrow{\Gamma_{reg}(R)}}(\mathfrak{w}) = d^-_{\overrightarrow{\Gamma_{reg}(R)}}(\mathfrak{c}) = 0$. We show that $d(\mathfrak{c}, \mathfrak{w}) = 5$. Suppose to the contrary, $d(\mathfrak{c}, \mathfrak{w}) = 3$ or 4. Then by Remark 2.1, there exist three non-trivial ideals of R, say J, L and P such that one of the following paths exists:

$$F_1 \times I_2 \times I_3 \longrightarrow J \longleftarrow L \longrightarrow (0) \times K_2 \times K_3; \tag{2.5}$$

$$F_1 \times I_2 \times I_3 \longleftarrow J \longrightarrow L \longleftarrow (0) \times K_2 \times K_3; \tag{2.6}$$

$$F_1 \times I_2 \times I_3 \longrightarrow J \longleftarrow L \longrightarrow P \longleftarrow (0) \times K_2 \times K_3; \tag{2.7}$$

$$F_1 \times I_2 \times I_3 \longleftarrow J \longrightarrow L \longleftarrow P \longrightarrow (0) \times K_2 \times K_3; \tag{2.8}$$

Since $d^+_{\overrightarrow{\Gamma_{reg}(R)}}(\mathfrak{w}) = d^-_{\overrightarrow{\Gamma_{reg}(R)}}(\mathfrak{c}) = 0$, Paths (2.6), (2.7) and (2.8) don't exist. Thus we can assume that Path (2.5) exists. Then Remark 2.2 implies that $J = F_1 \times (0) \times (0)$ and $L = (0) \times R_2 \times R_3$ which contradicts the adjacency of J and L. Therefore, $d(\mathfrak{c}, \mathfrak{w}) = 5$ and the proof is complete. $\square$

3. Main Results

In this section, it is proved that the radius of $\Gamma_{reg}(R)$ equals 3. The central vertices are characterized, too. For every Artinian ring R, we denote by $n_F(R)$, the number of fields appeared in the decomposition of R to a direct product of Artinian local rings.

Lemma 3.1. *Let R be an Artinian ring which is not field. If $n_F(R) \geq 1$, then for every ideal I of R, $\mathcal{C}^+(I) \cup \mathcal{C}^-(I) \neq \varnothing$.*

Proof. From the hypothesis and [3, Theorem 8.7], we deduce that $R \cong F_1 \times R_2$, where F_1 is a field and R_2 is an Artinian ring. For every ideal $I = I_1 \times I_2$ of R, either $I_1 = (0)$ or $I_1 = F_1$. If $I_1 = (0)$, then $(0) \times R_2 \in \mathcal{C}^-(I)$. Also, if $I_1 = F_1$, then $F_1 \times (0) \in \mathcal{C}^+(I)$. Thus in any case, $\mathcal{C}^+(I) \cup \mathcal{C}^-(I) \neq \varnothing$. □

Let $R = R_1 \times R_2 \times \cdots \times R_n$ be an Artinian ring, where every R_i is an Artinian local ring. For every ideal $I = I_1 \times I_2 \times \cdots \times I_n$ of R, setting

$$I_i^c = \begin{cases} R_i; & I_i = (0) \\ (0); & I_i = R_i \\ I_i; & I_i \text{ is a non} - \text{trivial ideal of } R_i, \end{cases}$$

we define the *complement* of I to be $I^c = I_1^c \times I_2^c \times \cdots \times I_n^c$. Also, for every subset X of ideals of R, by X^c, we mean the set $\{I^c |\ I \in X\}$.

Lemma 3.2. *Let R be an Artinian ring such that $\Gamma_{reg}(R)$ is a connected graph. Then for every vertex I of $\Gamma_{reg}(R)$, $e(I) \geq 3$.*

Proof. Let R be an Artinian ring. Since $\Gamma_{reg}(R)$ is connected, [7, Theorem 3.2] and [3, Theorem 8.7] imply that $R \cong F_1 \times R_2 \times \cdots \times R_n$, where $n \geq 3$, F_1 is a field and every R_i, $2 \leq i \leq n$, is an Artinian local ring. So, $I = I_1 \times I_2 \times \cdots \times I_n$, where every I_i is an ideal of R_i. We show that $d(I, I^c) \geq 3$. Suppose to the contrary, $d(I, I^c) \leq 2$. It is clear that I and I^c are not adjacent. Thus by Remark 2.1, there exists a vertex, say $J = J_1 \times J_2 \times \cdots \times J_n$ such that one of the following paths exists:

$$I = I_1 \times I_2 \times \cdots \times I_n \longleftarrow J_1 \times J_2 \times \cdots \times J_n \longrightarrow I_1^c \times I_2^c \times \cdots \times I_n^c = I^c$$

$$I = I_1 \times I_2 \times \cdots \times I_n \longrightarrow J_1 \times J_2 \times \cdots \times J_n \longleftarrow I_1^c \times I_2^c \times \cdots \times I_n^c = I^c$$

If the first path exists, then Remark 2.2 implies that for every i, J_i contains an I_i-regular element and J_i contains an I_i^c-regular element. Thus $J_i = R_i$, for every i, and hence $J = R$, a contradiction. Similarly, it is seen that the existence of the second path leads to a contradiction. □

Theorem 3.3. *Let R be an Artinian non-reduced ring such that $\Gamma_{reg}(R)$ is a connected graph. Then the following statements hold:*
(i) $e(\mathfrak{a}) = e(\mathfrak{b}) = e(\mathfrak{u}) = e(\mathfrak{v}) = 3$.
(ii) $e(\mathfrak{d}) = 3$ *if and only if* $n_F(R) \geq 2$.

Proof. (i) This follows from Propositions 2.3, 2.5 and Lemma 3.2.
(ii) By Proposition 2.3 and Lemma 3.2, it is enough to check $d(\mathfrak{d}, \mathfrak{c})$, where $\mathfrak{c} = F_1 \times I_2 \times I_3 \in \mathfrak{C}$. First suppose that $n_F(R) \geq 2$. If R_3 is not field, then Lemma 3.1 yields that $\mathcal{C}^+(I_3) \cup \mathcal{C}^-(I_3) \neq \varnothing$, for every ideal I_3 of R_3. Thus by Proposition 2.5 (3), $d(\mathfrak{d}, \mathfrak{c}) \leq 3$. Also, if R_3 is a field, then I_3 is a trivial ideal of R_3 and so again by Proposition 2.5 (3), $d(\mathfrak{d}, \mathfrak{c}) \leq 3$. Hence $e(\mathfrak{d}) = 3$. Now, assume that $n_F(R) = 1$. Then Proposition 2.5 (3) implies that $d(\mathfrak{d}, F_1 \times I_2 \times \mathrm{Nil}(R_3)) = 4$ and so we are done. □

Lemma 3.4. *Let R be an Artinian ring such that $\Gamma_{reg}(R)$ is connected. Then the following statements hold:*
(i) $\mathrm{diam}(\Gamma_{reg}(R)) = 5$ *if and only if* $n_F(R) = 1$.
(ii) $\mathrm{diam}(\Gamma_{reg}(R)) = 4$ *if and only if* $n_F(R) = 2$.
(iii) $\mathrm{diam}(\Gamma_{reg}(R)) = 3$ *if and only if* $n_F(R) \geq 3$.

Proof. Since $\Gamma_{reg}(R)$ is connected, [7, Theorem 2.1] implies that $R \cong F_1 \times R_2 \times R_3$, where F_1 is a field, R_2 is an Artinian local ring which is not field and R_3 is an Artinian ring. Thus the assertion follows from [2, Theorem 2.10] and this fact that in any Artinian ring S, $Z(\mathrm{Nil}(S)) = Z(S)$ if and only if S contains no field as its direct summand. □

From Lemmas 3.2 and 3.4, we have the following immediate corollary.

Corollary 3.5. *Let R be an Artinian non-reduced ring and $\Gamma_{reg}(R)$ be a connected graph. If $n_F(R) \geq 3$, then for every vertex I of $\Gamma_{reg}(R)$, $e(I) = 3$.*

Lemma 3.6. *Let I be an ideal of the Artinian ring R. Then $\mathcal{C}^+(I) = \varnothing$ if and only if $I \subseteq \mathrm{Nil}(R)$.*

Proof. First suppose that $I \subseteq \mathrm{Nil}(R)$. We show that $\mathcal{C}^+(I) = \varnothing$. Suppose to the contrary $J \in \mathcal{C}^+(I) \neq \varnothing$. Then I contains a J-regular element, say x. Choose a non-zero element $y \in J$. Then there exists a positive integer n such that $x^n y = 0$ and $x^{n-1} y \neq 0$. Since $x^{n-1} y \in J$, we deduce that x is not J-regular, a contradiction. Conversely, suppose that $\mathcal{C}^+(I) = \varnothing$. By [3, Theorem 8.7], $R \cong R_1 \times R_2 \times \cdots \times R_n$, where n is a positive integer and every R_i is an Artinian local ring. Thus $I = I_1 \times I_2 \times \cdots \times I_n$, where every I_i is an ideal of R_i. Since $\mathcal{C}^+(I) = \varnothing$, we deduce that every I_i is a proper ideal of R_i. Hence $I \subseteq \mathrm{Nil}(R)$. □

According to Corollary 3.5, we only need to calculate the eccentricity of the vertices of R, when $n_F(R) \leq 2$. So from now, we focuss on a ring R which contains at most two fields as its direct summands.

Theorem 3.7. *Let R be an Artinian non-reduced ring and $\Gamma_{reg}(R)$ be a connected graph. If $n_F(R) \leq 2$ and $\mathfrak{c} = F_1 \times I_2 \times I_3 \in \mathfrak{C}$ is a vertex of $\Gamma_{reg}(R)$, then the following statements hold:*
(i) *If $I_2 = R_2$, then $e(\mathfrak{c}) = 3$.*
(ii) *If I_2 is a proper ideal of R_2 and $I_3 = (0)$, then*

$$e(\mathfrak{c}) = \begin{cases} 3; & n_F(R) = 2 \text{ and either } (R_3 \text{ is not a field}) \text{ or } (R_3 \text{ is a field and } I_2 = (0)) \\ 4; & \text{Otherwise.} \end{cases}$$

(iii) *If $I_2 = (0)$ and I_3 is a non-trivial ideal of R_3, then*

$$e(\mathfrak{c}) = \begin{cases} 3; & \text{either } n_F(R) = 2 \text{ or } I_3 \nsubseteq \mathrm{Nil}(R_3) \\ 4; & n_F(R) = 1 \text{ and } I_3 \subseteq \mathrm{Nil}(R_3). \end{cases}$$

(iv) *If I_2 is a non-trivial ideal of R_2 and $I_3 = R_3$, then $e(\mathfrak{c}) = 3$.*

(v) *Let I_2 and I_3 be non-trivial ideals of R_2 and R_3, respectively. Then*

(a) *If $n_F(R) = 1$, then $e(\mathfrak{c}) = 3$ if and only if $\mathcal{C}^+(I_3) \neq \varnothing$.*

(b) *If $n_F(R) = 2$, then $R_3 \cong T_3 \times T_4 \times \cdots \times T_n$, where T_3 is a field and every T_i, $i \neq 4$, is an Artinian local ring which is not field. Moreover, $e(\mathfrak{c}) \neq 3$ if and only if $I_3 = (0) \times Q_4 \times \cdots \times Q_n$, where every Q_i is a non-trivial ideal of T_i.*

Proof. (i) Let $\mathfrak{c} = F_1 \times R_2 \times I_3$, where I_3 is a non-trivial ideal of R_3. If $x \in \{\mathfrak{a}, \mathfrak{b}, \mathfrak{d}, \mathfrak{u}, \mathfrak{v}\} \cup \mathcal{C}$, then Propositions 2.3, 2.4 and 2.5 imply that $d(\mathfrak{c}, x) \leq 3$. Also, the existence of the path

$$\mathfrak{c} = F_1 \times R_2 \times I_3 \longrightarrow (0) \times R_2 \times (0) \longleftarrow (0) \times R_2 \times R_3 \longrightarrow (0) \times K_2 \times K_3 = \mathfrak{w}$$

shows that $d(\mathfrak{c}, \mathfrak{w}) \leq 3$, for every vertex $\mathfrak{w} \in \mathfrak{W}$. Thus by Lemma 3.2, $e(\mathfrak{c}) = 3$.

(ii) Let $\mathfrak{c} = F_1 \times I_2 \times (0)$, where I_2 is a proper ideal of R_2. Then by Propositions 2.3, 2.4 and 2.5, we have $d(\mathfrak{c}, x) \leq 3$, for every vertex $x \in \{\mathfrak{a}, \mathfrak{b}, \mathfrak{d}, \mathfrak{u}, \mathfrak{v}\} \cup \mathfrak{C}$. Therefore, from Proposition 2.6, we deduce that $3 \leq e(\mathfrak{c}) \leq 4$. Now, we follow the proof in the following cases:

Case 1. $n_F(R) = 1$. In this case, $R \cong F_1 \times R_2 \times \cdots \times R_n$, where $n \geq 3$ and for every $2 \leq i \leq n$, R_i is an Artinian local ring which is not a field. We prove that $d(\mathfrak{c}, \mathrm{Nil}(R)) = 4$. Suppose to the contrary, $d(\mathfrak{c}, \mathrm{Nil}(R)) \neq 4$. By Proposition 2.6, $d(\mathfrak{c}, \mathrm{Nil}(R)) = 3$. Thus Remark 2.1 implies that there exist two vertices, say $J = J_1 \times J_2 \times J_3$ and $L = L_1 \times L_2 \times L_3$, such that one of the following paths exists:

$$\mathfrak{c} = F_1 \times I_2 \times (0) \longleftarrow J_1 \times J_2 \times J_3 \longrightarrow L_1 \times L_2 \times L_3 \longleftarrow \mathrm{Nil}(R) \tag{3.1}$$

$$\mathfrak{c} = F_1 \times I_2 \times (0) \longrightarrow J_1 \times J_2 \times J_3 \longleftarrow L_1 \times L_2 \times L_3 \longrightarrow \mathrm{Nil}(R) \tag{3.2}$$

By Lemma 3.6, Path (3.1) does not exist. So, we can assume that Path (3.2) exists. Thus Remark 2.2 implies that $J = F_1 \times (0) \times (0)$ and $L = (0) \times R_2 \times R_3$ which contradicts the adjacency of J and L. Therefore, $e(\mathfrak{c}) = 4$.

Case 2. $n_F(R) = 2$, R_3 is a field and $I_2 \neq (0)$. In this case, a similar proof to that of case 1 shows that $d(\mathfrak{c}, (0) \times I_2 \times R_3) = 4$. Thus, in this case, $e(\mathfrak{c}) = 4$.

Case 3. $n_F(R) = 2$, R_3 is a field and $I_2 = (0)$. Choose a vertex $\mathfrak{w} \in \mathfrak{W}$. Then there exists a non-trivial ideal K_2 of R_2 such that either $\mathfrak{w} = (0) \times K_2 \times (0)$ or $\mathfrak{w} = (0) \times K_2 \times R_3$. If $\mathfrak{w} = (0) \times K_2 \times (0)$, then the vertex $F_1 \times R_2 \times (0)$ is adjacent to both $\mathfrak{c}$ and $\mathfrak{w}$ and so $d(\mathfrak{c}, \mathfrak{w}) = 2$. Also, the existence of the path

$$\mathfrak{c} = F_1 \times (0) \times (0) \longleftarrow F_1 \times (0) \times R_3 \longrightarrow (0) \times (0) \times R_3 \longleftarrow (0) \times K_2 \times R_3 = \mathfrak{w}$$

implies that $d(\mathfrak{c}, \mathfrak{w}) \leq 3$. Thus, by Lemma 3.2, $e(\mathfrak{c}) = 3$.

Case 4. $n_F(R) = 2$ and R_3 is not a field. In this case, $R_3 \cong T_3 \times T_4 \times \cdots \times T_n$, where every T_i, $i \neq 3$, is an Artinian local ring which is not field and T_3 is a field. So, every vertex $\mathfrak{w} \in \mathfrak{W}$ is of the form $(0) \times Q_2 \times Q_3 \times \cdots \times Q_n$. Now, in the following two subcases, we prove that $d(\mathfrak{c}, \mathfrak{w}) \leq 3$:

Subcase 1. There exists $3 \leq j \leq n$ such that $Q_j = T_j$. With no loss of generality, one can assume that $Q_3 = T_3$ and so, by Remark 2.2, the path

$$\mathfrak{c} \longleftarrow F_1 \times R_2 \times T_3 \times (0) \times \cdots \times (0) \longrightarrow (0) \times (0) \times T_3 \times (0) \times \cdots \times (0) \longleftarrow \mathfrak{w}$$

exists and hence $d(\mathfrak{c}, \mathfrak{w}) \leq 3$.

Subcase 2. For every $3 \leq j \leq n$, $Q_j \neq T_j$. In this subcase, $Q_3 = (0)$ and the existence of the path

$$\mathfrak{c} \longrightarrow F_1 \times (0) \times (0) \times \cdots \times (0) \longleftarrow F_1 \times R_2 \times (0) \times T_4 \times \cdots \times T_n \longrightarrow \mathfrak{w}$$

shows that $d(\mathfrak{c}, \mathfrak{w}) \leq 3$.

Therefore, in this case, $e(\mathfrak{c}) = 3$ and this completes the proof of (ii).

(iii) Let $\mathfrak{c} = F_1 \times (0) \times I_3$, where I_3 is a non-trivial ideal of R_3. Then by Propositions 2.3,2.4 and 2.5, we only need to check $d(\mathfrak{c}, \mathfrak{w})$, where $\mathfrak{w} \in \mathfrak{W}$. Now, consider the following cases:

Case 1. $n_F(R) = 1$ and $I_3 \nsubseteq \mathrm{Nil}(R_3)$. In this case, Lemma 3.6 implies that $\mathcal{C}^+(I_3) \neq \varnothing$. Choose $J_3 \in \mathcal{C}^+(I_3)$. Then for every $\mathfrak{w} = (0) \times K_2 \times K_3 \in \mathfrak{W}$, the path

$$\mathfrak{c} = F_1 \times (0) \times I_3 \longrightarrow (0) \times (0) \times J_3 \longleftarrow (0) \times R_2 \times R_3 \longrightarrow \mathfrak{w}$$

exists and so $d(\mathfrak{c}, \mathfrak{w}) \leq 3$, for every $\mathfrak{w} \in \mathfrak{W}$. Therefore, in this case, $e(\mathfrak{c}) = 3$

Case 2. $n_F(R) = 1$ and $I_3 \subseteq \mathrm{Nil}(R_3)$. Since $I_3 \subseteq \mathrm{Nil}(R_3)$, Lemma 3.6 implies that $\mathcal{C}^+(I_3) = \varnothing$. By [7, Theorem 8.7], $R \cong F_1 \times R_2 \times \cdots \times R_n$, where $n \geq 3$ and for every $2 \leq i \leq n$, R_i is an Artinian local ring which is not a field. Thus by [7, Theorem 2.1], $\mathcal{C}^+(\mathrm{Nil}(R_2 \times R_3 \times \cdots \times R_n)) \cup \mathcal{C}^-(\mathrm{Nil}(R_2 \times R_3 \times \cdots \times R_n)) = \varnothing$. We prove that $d(\mathfrak{c}, \mathrm{Nil}(R)) = 4$. Suppose to the contrary, $d(\mathfrak{c}, \mathrm{Nil}(R)) \neq 4$. By Proposition 2.6, $d(\mathfrak{c}, \mathrm{Nil}(R)) = 3$. Thus Remark 2.1 implies that there exist two vertices, say $J = J_1 \times J_2 \times J_3$ and $L = L_1 \times L_2 \times L_3$, such that one of the following paths exists:

$$\mathfrak{c} = F_1 \times (0) \times I_3 \longleftarrow J_1 \times J_2 \times J_3 \longrightarrow L_1 \times L_2 \times L_3 \longleftarrow \mathrm{Nil}(R) \tag{3.3}$$

$$\mathfrak{c} = F_1 \times (0) \times I_3 \longrightarrow J_1 \times J_2 \times J_3 \longleftarrow L_1 \times L_2 \times L_3 \longrightarrow \mathrm{Nil}(R) \tag{3.4}$$

By Lemma 3.6, Path (3.3) does not exist. So, we can assume that Path (3.4) exists. Thus Remark 2.2 implies that $J = F_1 \times (0) \times (0)$. So, $L = F_1 \times R_2 \times R_3$, a contradiction. Thus $d(\mathfrak{c}, \mathrm{Nil}(R)) = 4$ and hence $e(\mathfrak{c}) = 4$.

Case 3. $n_F(R) = 2$ and R_3 is not a field. In this case, $R_3 \cong T_3 \times T_4 \times \cdots \times T_n$, where every T_i, $i \neq 3$, is an Artinian local ring which is not field and T_3 is a field. So, every vertex $\mathfrak{w} \in \mathfrak{W}$ is of the form $(0) \times Q_2 \times Q_3 \times \cdots \times Q_n$. Now, in the following two subcases, we prove that $d(\mathfrak{c}, \mathfrak{w}) \leq 3$:

Subcase 1. There exists $3 \leq j \leq n$ such that $Q_j = T_j$. With no loss of generality, one can assume that $Q_3 = T_3$ and so, by Remark 2.2, the path

$$\mathfrak{c} \longleftarrow F_1 \times (0) \times T_3 \times T_4 \times \cdots \times T_n \longrightarrow (0) \times (0) \times T_3 \times (0) \times \cdots \times (0) \longleftarrow \mathfrak{w}$$

exists and hence $d(\mathfrak{c},\mathfrak{w}) \leq 3$.

Subcase 2. For every $3 \leq j \leq n$, $Q_j \neq T_j$. In this subcase, the existence of the path

$$\mathfrak{c} \longrightarrow F_1 \times (0) \times (0) \times \cdots \times (0) \longleftarrow F_1 \times (0) \times T_3 \times \cdots \times T_n \longrightarrow \mathfrak{w}$$

shows that $d(\mathfrak{c},\mathfrak{w}) \leq 3$.

Therefore, in this case, Lemma 3.2 implies that $e(\mathfrak{c}) = 3$.

(iv) Assume that $\mathfrak{c} = F_1 \times I_2 \times R_3$, where I_2 is a non-trivial ideal of R_2. By Propositions 2.3,2.4 and 2.5, we have $d(\mathfrak{c}, x) \leq 3$, for every vertex $x \in \{\mathfrak{a},\mathfrak{b},\mathfrak{d},\mathfrak{u},\mathfrak{v}\} \cup \mathfrak{C}$. Also, for every vertex $\mathfrak{w} = (0) \times K_2 \times K_3 \in \mathfrak{W}$, the path

$$\mathfrak{c} \longrightarrow (0) \times (0) \times R_3 \longleftarrow (0) \times R_2 \times R_3 \longrightarrow \mathfrak{w}$$

exists and hence by Lemma 3.2, $e(\mathfrak{c}) = 3$.

(v) Let $\mathfrak{c} = F_1 \times I_2 \times I_3$, where I_2 and I_3 are non-trivial ideals of R_2 and R_3, respectively. Then by Propositions 2.3,2.4 and 2.5, we have $d(\mathfrak{c}, x) \leq 3$, for every vertex $x \in \{\mathfrak{a},\mathfrak{b},\mathfrak{u},\mathfrak{v}\} \cup \mathfrak{C}$. Hence by Lemma 3.2, $e(\mathfrak{c})$ depends only on $d(\mathfrak{c},\mathfrak{d})$ and $d(\mathfrak{c},\mathfrak{w})$, where $\mathfrak{w} \in \mathfrak{W}$. On the other hand, Proposition 2.5(2) implies that $d(\mathfrak{c},\mathfrak{d}) \leq 3$ if and only if $\mathcal{C}^+(I_3) \cup \mathcal{C}^-(I_3) \neq \varnothing$. Therefore, by Lemma 3.2, $e(\mathfrak{c}) = 3$ if and only if $\mathcal{C}^+(I_3) \cup \mathcal{C}^-(I_3) \neq \varnothing$ and $d(\mathfrak{c},\mathfrak{w}) \leq 3$, for every $\mathfrak{w} \in \mathfrak{W}$. To complete the proof, we consider the following three cases:

Case 1. $\mathcal{C}^+(I_3) \neq \varnothing$. In this case, choose $J_3 \in \mathcal{C}^+(I_3)$. Then for every $\mathfrak{w} = (0) \times K_2 \times K_3 \in \mathfrak{W}$, the path

$$\mathfrak{c} = F_1 \times I_2 \times I_3 \longrightarrow (0) \times (0) \times J_3 \longleftarrow (0) \times R_2 \times R_3 \longrightarrow (0) \times K_2 \times K_3 = \mathfrak{w}$$

exists and so $e(\mathfrak{c}) = 3$.

Case 2. $\mathcal{C}^+(I_3) = \varnothing$ and $n_F(R) = 1$. In this case, we prove that $d(\mathfrak{c}, (0) \times I_2 \times \mathrm{Nil}(R_3)) \geq 4$. Suppose to the contrary, $d(\mathfrak{c},\mathfrak{w}) \leq 3$. Then Remark 2.1 implies that there are two vertices, say $J = J_1 \times J_2 \times J_3$ and $L = L_1 \times L_2 \times L_3$, such that one of the following paths exists:

$$\mathfrak{c} = F_1 \times I_2 \times I_3 \longleftarrow J_1 \times J_2 \times J_3 \longrightarrow L_1 \times L_2 \times L_3 \longleftarrow (0) \times I_2 \times \mathrm{Nil}(R_3) \tag{3.5}$$

$$\mathfrak{c} = F_1 \times I_2 \times I_3 \longrightarrow J_1 \times J_2 \times J_3 \longleftarrow L_1 \times L_2 \times L_3 \longrightarrow (0) \times I_2 \times \mathrm{Nil}(R_3) \tag{3.6}$$

By Lemma 3.6, Path (3.5) does not exist. So, we can assume that Path (3.6) exists. Thus Remark 2.2 implies that $J = F_1 \times (0) \times (0)$ and $L = (0) \times R_2 \times R_3$ which contradicts the adjacency of J and L. Therefore, $e(\mathfrak{c}) = 4$.

Case 3. $\mathcal{C}^+(I_3) = \varnothing$ and $n_F(R) = 2$. In this case, Lemma 3.1 implies that $\mathcal{C}^-(I_3) \neq \varnothing$. Since I_3 is a non-trivial ideal of R_3 and $n_F(R) = 2$, we deduce that $R_3 \cong T_3 \times T_4 \times \cdots \times T_n$, where T_3 is a field and for every $4 \leq i \leq n$, T_i is an Artinian local ring which is not field. So, $I_3 = (0) \times Q_4 \times \cdots \times Q_n$, where every Q_i is a proper ideal of T_i. To complete the proof, we consider the following two subcases:

Subcase 1. Every Q_j, $4 \leq j \leq n$, is a non-trivial ideal of T_j. In this subcase, we prove that $d(\mathfrak{c}, (0) \times I_2 \times I_3^c) \geq 4$ (Note that $I_3^c = T_3 \times Q_4 \times \cdots \times Q_n$). Suppose to the contrary, $d(\mathfrak{c}, (0) \times I_2 \times I_3^c) \leq 3$.

Then by Remark 2.1, there are two vertices, say $J = J_1 \times J_2 \times J_3$ and $L = L_1 \times L_2 \times L_3$, such that one of the following paths exists:

$$\mathfrak{c} = F_1 \times I_2 \times I_3 \longleftarrow J_1 \times J_2 \times J_3 \longrightarrow L_1 \times L_2 \times L_3 \longleftarrow (0) \times I_2 \times I_3^c \tag{3.7}$$

$$\mathfrak{c} = F_1 \times I_2 \times I_3 \longrightarrow J_1 \times J_2 \times J_3 \longleftarrow L_1 \times L_2 \times L_3 \longrightarrow (0) \times I_2 \times I_3^c \tag{3.8}$$

If Path (3.7) exists, then by Remark 2.2, $J = F_1 \times R_2 \times (0) \times T_4 \times \cdots \times T_n$ and $L = (0) \times (0) \times T_3 \times (0) \times \cdots \times (0)$ and this contradicts the adjacency of J and L. Also, since $\mathcal{C}^+(I_3) = \varnothing$, by Remark 2.2, the existence of the Path (3.8) implies that $J = F_1 \times (0) \times (0)$ and $L = (0) \times R_2 \times R_3$, a contradiction. Hence $e(\mathfrak{c}) \geq 4$

Subcase 2. There exists $4 \leq j \leq n$ such that $Q_j = (0)$. With no loss of generality, one can assume that $Q_4 = (0)$. We show that for every $\mathfrak{w} = (0) \times K_2 \times K_3 \in \mathfrak{W}$, $d(\mathfrak{c}, \mathfrak{w}) \leq 3$. From Lemma 3.1, we deduce that $\mathcal{C}^+(K_3) \cup \mathcal{C}^-(K_3) \neq \varnothing$. If $\mathcal{C}^-(K_3) \neq \varnothing$, then there exists a non-trivial ideal $L_3 \in \mathcal{C}^-(K_3)$. Thus the path

$$\mathfrak{c} = F_1 \times I_2 \times I_3 \longrightarrow F_1 \times (0) \times (0) \longleftarrow F_1 \times R_2 \times L_3 \longrightarrow (0) \times K_2 \times K_3 = \mathfrak{w}$$

exists and so there is no thing to prove. Thus we can suppose that $\mathcal{C}^-(K_3) = \varnothing$ and so $K_3 = T_3 \times Q_4 \times \cdots \times Q_n$, where $Q_i \neq (0)$, for every $4 \leq i \leq n$. Setting $J_3 = T_3 \times (0) \times T_5 \times \cdots \times T_n$ and $L_3 = T_3 \times (0) \times \cdots \times (0)$, Remark 2.2 implies that the path

$$\mathfrak{c} = F_1 \times I_2 \times I_3 \longleftarrow F_1 \times R_2 \times J_3 \longrightarrow (0) \times (0) \times L_3 \longleftarrow (0) \times K_2 \times K_3 = \mathfrak{w}$$

exists and so $d(\mathfrak{c}, \mathfrak{w}) \leq 3$. Therefore, in this subcase, $e(\mathfrak{c}) = 3$. So, we are done. □

Let R be an Artinian ring and I and J be two non-trivial ideals of R. Then it is clear that $I \in \mathcal{C}^+(J)$ if and only if $I^c \in \mathcal{C}^-(J^c)$. Moreover, $I \longrightarrow J$ is an arc of $\overrightarrow{\Gamma_{reg}}(R)$ if and only if $J^c \longrightarrow I^c$ of $\overrightarrow{\Gamma_{reg}}(R)$. Thus $d(I, J) = d(I^c, J^c)$ and hence $e(I) = e(I^c)$. Moreover, we have $I \in \mathfrak{C}$ if and only if $I^c \in \mathfrak{W}$. Thus using this facts and applying the similar proof to that of Theorem 3.7, one can prove the following theorem.

Theorem 3.8. *Let R be an Artinian non-reduced ring and $\Gamma_{reg}(R)$ be a connected graph. If $n_F(R) \leq 2$ and $\mathfrak{w} = (0) \times K_2 \times K_3 \in \mathfrak{W}$ is a vertex of $\Gamma_{reg}(R)$, then the following statements hold:*

(i) *If $K_2 = (0)$, then $e(\mathfrak{w}) = 3$.*

(ii) *If K_2 is a non-trivial ideal of R_3 and $K_3 = R_3$, then*

$$e(\mathfrak{w}) = \begin{cases} 3; & n_F(R) = 2 \text{ and } R_3 \text{ is not a field} \\ 4; & \text{Otherwise.} \end{cases}$$

(iii) *If $K_2 = R_2$, then*

$$e(\mathfrak{w}) = \begin{cases} 3; & \text{Either } n_F(R) = 2 \text{ or } \mathcal{C}^-(K_3) \neq \varnothing \\ 4; & \text{Otherwise.} \end{cases}$$

(iv) *If K_2 is a non-trivial ideal of R_2 and $K_3 = (0)$, then $e(\mathfrak{w}) = 3$.*

(v) *Let K_2 and K_3 be non-trivial ideals of R_2 and R_3, respectively. Then*

(a) *If* $n_F(R) = 1$*, then* $e(\mathfrak{w}) = 3$ *if and only if* $\mathcal{C}^-(K_3) \neq \varnothing$*.*

(b) *If* $n_F(R) = 2$*, then* $R_3 \cong T_3 \times T_4 \times \cdots \times T_n$*, where* T_3 *is a field and every* T_i*,* $i \neq 4$*, is an Artinian local ring which is not field. Moreover,* $e(\mathfrak{w}) \neq 3$ *if and only if* $K_3 = T_3 \times Q_4 \times \cdots \times Q_n$*, where every* Q_i *is a non-trivial ideal of* T_i*.*

Finally, we prove the main theorem of this paper.

Theorem 3.9. *If* R *is an Artinian ring and* $\Gamma_{reg}(R)$ *is connected, then* $r(\Gamma_{reg}(R)) = 3$*.*

Proof. Let $\Gamma_{reg}(R)$ be a connected graph. Then [7, Theorem 2.3] implies that $|\mathrm{Max}(R)| \geq 3$ and R contains a field as its direct summand. Since R has at least three maximal ideals, [3, Theorem 8.7] implies that $R \cong F_1 \times R_2 \times R_3$, where F_1 is a field and R_2 and R_3 are Artinian rings. First suppose that R is reduced. Then $R \cong F_1 \times \cdots \times F_n$, where $n \geq 3$ and every F_i is a field. For every ideal $I = I_1 \times \cdots \times I_n$ of R, define

$$\Delta_I = \{k \mid 1 \leq k \leq n \text{ and } I_k = F_k\}$$

and

$$\Omega = \{\Delta_I \mid I \text{ is a non} - \text{trivial ideal of } R\}.$$

Clearly, $\Delta_I = \Delta_J$ if and only if $I = J$. Thus there is a one to one correspondence between Ω and the set of proper and non-empty subsets of $\{1, \ldots, n\}$. We claim that $d(I, J) \leq 3$, for every two distinct vertices I and J of $\Gamma_{reg}(R)$. To see this, we consider the following two cases:

Case 1. $\Delta_I \cap \Delta_J = \varnothing$. Since $n \geq 3$, by pigeon-hole principle and with no loss of generality, we can assume that $|\Delta_I| \geq |\Delta_J|$, $|\Delta_I| \geq 2$ and $1 \in \Delta_I$. Let I_1 and I_2 be two vertices of $\overrightarrow{\Gamma_{reg}}(R)$ such that $\Delta_{I_1} = \{1\}$ and $\Delta_{I_2} = A_J \cup \{1\}$. Then there is the path $I \longrightarrow I_1 \longleftarrow I_2 \longrightarrow J$ in $\overrightarrow{\Gamma_{reg}}(R)$ and hence $d(I, J) \leq 3$.

Case 2. $\Delta_I \cap \Delta_J \neq \varnothing$. If either $\Delta_I \subset \Delta_J$ or $\Delta_J \subset \Delta_I$, then I and J are adjacent. So, we can assume that neither $\Delta_I \nsubseteq \Delta_J$ nor $\Delta_J \nsubseteq \Delta_I$. Choose $i \in \Delta_I \cap \Delta_J$. Then it is clear that $I_i = (0) \times \cdots \times F_i \times \cdots \times (0)$ is adjacent to both I and J and so $d(I, J) = 2$.

So, the claim is proved. Thus from Lemma 3.2, we deduce that $e(I) = 3$. Therefore, in any case, $r(\Gamma_{reg}(R)) = 3$. Now, suppose that R is non-reduced. Then the assertion follows from Theorems 3.3, 3.7, 3.8 and Corollary 3.5. □

Using Theorems 3.3, 3.7 and 3.8, we have the following corollary in which the center of $\Gamma_{reg}(R)$ is determined.

Corollary 3.10. *Let* R *be an Artinian non-reduced ring and* $\Gamma_{reg}(R)$ *be a connected graph. Then the following statements hold:*

(i) *If* $n_F(R) = 1$ *and* $X = \{F_1 \times I_2 \times I_3 | \text{ either } I_2 = R_2 \text{ or } I_3 \nsubseteq \mathrm{Nil}(R_3)\}$*, then the center of* $\Gamma_{reg}(R)$ *equals* $X \cup X^c$*.*

(ii) *If* $n_F(R) = 2$ *and* $|\mathrm{Max}(R)| = 3$*, then every vertex of* $\Gamma_{reg}(R)$ *is centeral.*

(iii) *If* $n_F(R) = 2$ *and* $|\mathrm{Max}(R)| \geq 4$*, then* $R \cong F_1 \times T_2 \times F_3 \times T_4 \times \cdots \times T_n$*, where* F_1 *and* F_3 *are fields and every* T_i *is an Artinian local ring which is not field. Moreover, the vertex* I *is central if and only if neither* $I = F_1 \times I_2 \times (0) \times I_4 \times \cdots \times I_n$ *nor* $I = (0) \times I_2 \times F_3 \times I_4 \times \cdots \times I_n$*, for every non-trivial ideals* I_i *of* T_i*.*

(iv) *If* $n_F(R) \geq 3$*, then every vertex of* $\Gamma_{reg}(R)$ *is central.*

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Farzad Shaveisi

Department of Mathematics, Faculty of Science, Razi University , P.O.Box 67149-67346, Kermanshah, Iran

`Email: f.shaveisi@razi.ac.ir`