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ON LABEL GRAPHOIDAL COVERING NUMBER-I

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Abstract. Let $G = (V, E)$ be a graph with p vertices and q edges. An *acyclic graphoidal cover* of G is a collection ψ of paths in G which are internally-disjoint and cover each edge of the graph exactly once. Let $f : V \to \{1, 2, \ldots, p\}$ be a bijective labeling of the vertices of G. Let $\uparrow G_f$ be the directed graph obtained by orienting the edges uv of G from u to v provided $f(u) < f(v)$. If the set ψ_f of all maximal directed paths in $\uparrow G_f$, with directions ignored, is an acyclic graphoidal cover of G, then f is called a *graphoidal labeling* of G and G is called a *label graphoidal graph* and $\eta_l = \min\{|\psi_f| : f$ is a graphoidal labeling of $G\}$ is called the *label graphoidal covering number* of G. In this paper we characterize graphs for which (i) $\eta_l = q - m$, where m is the number of vertices of degree 2 and (ii) $\eta_l = q$. Also, we determine the value of label graphoidal covering number for unicyclic graphs.

1. Introduction

Graph decomposition problems rank among the most prominent areas of research in graph theory and combinatorics and further have numerous applications in various fields such as networking, engineering and DNA analysis.

A decomposition of a graph G is a collection of its subgraphs such that every edge of G lies in exactly one member of the collection. Various types of decompositions have been introduced and studied by imposing conditions on the members of the decomposition. For instance, Harary introduced the notion of path cover [9] which demands each member of a decomposition to be a path. Following Harary, several variations of decomposition have been introduced and extensively studied. Unrestricted path cover [10], simple path cover [4] and cycle decomposition [6] are some variations of decomposition.

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In this direction Acharya and Sampathkumar [1] introduced the concept of graphoidal cover and the notion of label graphoidal graph. Further, Arumugam and Sahul Hamid [3] coined the term label graphoidal covering number for label graphoidal graphs and investigated its properties. In this paper we further extend the study of the label graphoidal covering number by establishing some bounds along with the characterization.

2. Label graphoidal covering number

By a graph $G = (V, E)$, we mean a finite, undirected graph with neither loops nor multiple edges. The order and size of G are denoted by p and q respectively. All graphs in this paper are assumed to be connected and non-trivial. For graph theoretic terminology we refer to Harary [8].

If P is an $u - v$ path in a graph G, the vertices of P other than u and v are called the *internal vertices* of P, while u and v are called respectively the *origin* and *terminal* of the path P. We say that two paths are *internally-disjoint* if an internal vertex of one path is not an internal vertex of the other. For a collection ψ of internally-disjoint paths in G, a vertex is said to be *interior* if it is an internal vertex of a path in ψ; otherwise it is called an *exterior vertex* to ψ.

Definition 2.1. [1] *A graphoidal cover of a graph G is a collection ψ of paths and cycles in G satisfying the following conditions.*

(i) *Every member in ψ has at least two vertices.*
(ii) *Every vertex of G is an internal vertex of at most one member in ψ.*
(iii) *Every edge of G is in exactly one member in ψ.*

If further, no member of ψ is a cycle in G, then ψ is called an acyclic graphoidal cover of G. That is, an acyclic graphoidal cover is a collection of internally-disjoint paths such that every edge of G lies in exactly one path in the collection. The minimum cardinality of a graphoidal cover of G is called the graphoidal covering number of G and is denoted by $\eta(G)$. Similarly the acyclic graphoidal covering number $\eta_a(G)$ is defined.

The notion of acyclic graphoidal cover was introduced by Arumugam and Suresh Sussela [5]. For more details about these topics see [2].

The study of graph labelings is one of the fastest growing areas within graph theory which has been extensively studied. A *graph labeling* is an assignment of integers to the vertices or edges, or both, subject to certain conditions. Graph labelings were first studied in the late 1960s. In the intervening years dozens of graph labelings have been studied in over 600 papers. A detailed survey of graph labeling is given in [7]. Acharya and Sampathkumar [1] introduced a new graph labeling, namely graphoidal labeling, using the notion of graphoidal cover. As we know an *orientation* of a graph is a digraph obtained from G by assigning a direction to each edge of G.

Definition 2.2. [1] *Let $G = (V, E)$ be a graph with p vertices and let $f : V \to \{1, 2, \ldots, p\}$ be a bijective labeling of the vertices of G. Orient the edges uv of G from u to v provided $f(u) < f(v)$.*

Such an orientation of G is called a low-to-high orientation of G with respect to the given labeling f. By $\uparrow G_f$, we mean that G is a graph together with the labeling f with respect to which the edges of G are oriented low-to-high. Let $\pi^*(\uparrow G_f)$ be the set of all maximal directed paths in $\uparrow G_f$. Let $\pi(\uparrow G_f)$ be the set of all members in $\pi^*(\uparrow G_f)$ with directions ignored. We say that f is a graphoidal labeling of G if $\pi(\uparrow G_f)$ is a graphoidal cover of G and if G admits such a labeling f of its vertices, then G is called a label graphoidal graph.

If G is a label graphoidal graph with a graphoidal labeling f, then we use ψ_f to denote the graphoidal cover $\pi(\uparrow G_f)$ and we call ψ_f, the *label graphoidal cover of G induced by f.*

A complete characterization of label graphoidal graphs is presented in [1] (see Theorem 2.4) with the help of an interesting theorem which we state below. Here by *sink* in $\uparrow G_f$ we mean a vertex with out-degree zero and we say a vertex with in-degree zero a *source* in $\uparrow G_f$.

Theorem 2.3. [1] *Suppose G has a graphoidal labeling f. Then any vertex v of G with $\deg v > 2$ is either a sink or a source in $\uparrow G_f$.*

Theorem 2.4. [1] *A graph G is label graphoidal if and only if every odd cycle in G has a vertex of degree 2.*

Obviously, a label graphoidal graph G may admit more than one graphoidal labelings and hence we get a number of acyclic graphoidal covers of G corresponding to each graphoidal labeling of G. Certainly, for different graphoidal labelings f the cardinality of the corresponding acyclic graphoidal covers ψ_f may vary. So it is quite natural to ask that what is the minimum cardinality of an acyclic graphoidal cover ψ_f of G amongest all such possible covers?. Motivated by this, Arumugam and Sahul Hamid [3] introduced the concept of label graphoidal covering number of a label graphoidal graph and investigated its properties.

Definition 2.5. [3] *Let G be a label graphoidal graph. The label graphoidal covering number η_l of G is defined to be $\eta_l = \min\{|\psi_f| : f$ is a graphoidal labeling of G$\}$.*

Theorem 2.6. [3] *Let G be a label graphoidal graph. For a graphoidal labeling f of G, let s_f denote the number of vertices of degree 2 which are exterior to ψ_f and let $s = \min_f s_f$. Then $\eta_l = q - m + s$, where m is the number of vertices of degree 2.*

Corollary 2.7. [3] *For any label graphoidal graph G, $\eta_l \geq q - m$. Further, equality holds if and only if there exists a graphoidal labeling f of G such that every vertex of degree 2 is interior to ψ_f.*

Theorem 2.8. [3] *If G is a label graphoidal graph with $\eta_l(G) = q$, then G is bipartite.*

A number of bounds for the label graphoidal covering number of a graph have been obtained in [3] and also the following problems are left open.

(i) Characterize label graphoidal graphs for which $\eta_l = q - m$, where m is the number of vertices of degree 2.

(ii) Characterize label graphoidal graphs for which $\eta_l = q$.

(iii) Characterize label graphoidal graphs for which $\eta_l = \eta_a$.

In this paper we settle the first two problems. Also, we determine the value of label graphoidal covering number for unicyclic graphs.

3. Graphs with $\eta_l = q - m$

Arumugam and Sahul Hamid [3] observed that for any label graphoidal graph $G, \eta_l \geq q - m$, where m is the number of vertices of degree 2 in G and posed the problem of characterizing the graphs attaining this bound. We now proceed to characterize those graphs.

Theorem 3.1. *Let G be a label graphoidal graph which is not a cycle. Let m denote the number of vertices of degree 2 in G. Then $\eta_l = q - m$ if and only if every cycle in G has an even number of vertices of degree at least three in G.*

Proof. Suppose $\eta_l = q - m$, where m is the number of vertices of degree 2 in G. Then it follows from Corollary 2.7 that there is a graphoidal labeling f of G such that every vertex of degree 2 is interior to ψ_f. Now, let C be a cycle in G and let v_i be a vertex on it of degree greater than 2. Then by Theorem 2.3 the vertex v_i is either a sink or a source in $\uparrow G_f$. Assume without loss of generality that it is a source. Suppose v_j is the next vertex to v_i on C (in clockwise order) of degree greater than 2. Then all the vertices of C between v_i and v_j (in the clockwise direction) are of degree 2 so that they are interior to ψ_f which implies that the in-degree of the vertex v_j is positive. Hence it follows from Theorem 2.3 that v_j must be a sink in $\uparrow G_f$. Thus the vertices of degree greater than 2 on C are alternatively source and source in $\uparrow G_f$ and consequently C has an even number of vertices of degree greater than two.

Conversely, suppose every cycle in G has an even number of vertices of degree greater than 2 in G. To prove $\eta_l = q - m$, by virtue of Corollary 2.7, it suffices to show that G admits a graphoidal labeling f such that every vertex of degree 2 is interior to ψ_f. We prove this by induction on m. Suppose $m = 0$. Then it follows from Theorem 2.3 that, for any graphoidal labeling f of G, every vertex is either a sink or a source in $\uparrow G_f$. Certainly, a vertex in G which is either a sink or a source in $\uparrow G_f$ is exterior to ψ_f so that $s = m$, where s is the number defined in Theorem 2.6. Hence by Theorem 2.6, we have $\eta_l = q = q - m$. Assume that $m \geq 1$ and that the result is true for all label graphoidal graphs with fewer than m vertices of degree 2. Let u be a vertex in G with deg u =2 and let x and y be its neighbours.

Case 1. x and y are not adjacent.

Let G' be the graph obtained from G by replacing the path (x, u, y) by the edge xy. We claim that G' is label graphoidal. For this, we prove that every odd cycle in G' has a vertex of degree 2 in G' and then Theorem 2.4 completes the claim. This is vacuously true when G' has no odd cycle and so

assume that G' has some odd cycle and let C' be an arbitrary odd cycle in G'. Suppose C' does not contain the edge xy. Then C' is an odd cycle in G too, so that C' contains a vertex z of degree 2 in G being G label graphoidal. Since the edge xy is not on C' it follows that $z \notin \{x, y\}$ so that the degree of z in G' is also 2. Thus z is a vertex on C' of degree 2 in G'. Suppose the edge xy lies on C'. Then the cycle C obtained from C' by replacing the edge xy by the path (x, u, y) is of even length in G in which u is of degree 2 in G. By the assumption that every cycle in G has an even number of vertices of degree greater than 2, the cycle C must contain a vertex z' of degree 2 in G other than u. Also, the degree of z' in G' is also 2 (even if z' is either x or y). Thus z' is a vertex on C' of degree 2 in G'. Hence G' is label graphoidal.

Further, every cycle in G' has an even number of vertices of degree greater than 2 in G' and the number of vertices of degree 2 in G' is $m - 1$. Hence, by the induction hypothesis, G' admits a graphoidal labeling f' such that every vertex of degree 2 in G' is interior to $\psi_{f'}$. Assume without loss of generality that $f'(x) < f'(y)$. Let P be the maximal directed path in $\uparrow G'_{f'}$ containing the arc (x, y) and let w be the vertex at which P ends. We now define a labeling $f : V(G) \to \{1, 2, \ldots, p = |V(G)|\}$ as follows.

Assign the labels $f'(y)$ and p to the vertices u and w respectively. Further, when $w \neq y$, for each vertex in the (y, w) - section of P other than w, assign the label (under f') of its very next vertex in this section. Now, one can observe that f is a graphoidal labeling of G such that every vertex of degree 2 in G is interior to ψ_f so that $\eta_l(G) = q - m$.

Case 2. x and y are adjacent.

Since G is not a cycle either deg $x \geq 3$ or deg $y \geq 3$. So, if deg $x = 2$, for instance, then deg $y \geq 3$ so that (x, u, y, x) is a cycle in G with y being the only vertex of degree greater than 2 in G, contradicting the assumption that every cycle in G contains an even number of vertices of degree greater than 2. Hence deg $x \geq 3$ and deg $y \geq 3$ in G.

Subcase 2.1. deg $x \geq 4$ or deg $y \geq 4$.

Assume without loss of generality that deg $y \geq 4$. Now, let $G' = G - uy$. It is not difficult to see that G' is label graphoidal and every cycle in G' contains an even number of vertices of degree greater than 2. Further, the number of vertices of degree 2 in G' is $m-1$. Hence, by the induction hypothesis, we can find a graphoidal labeling f' of G' such that every vertex of degree 2 in G' is interior to $\psi_{f'}$. Now, we define a labeling f of G as follows.

(i) Suppose $f'(x) < f'(y)$. If $f'(u) < f'(y)$, let $f = f'$. Otherwise, let f be the labeling obtained from f' by interchanging the labels of u and y and keeping all other labels unchanged.
(ii) Suppose $f'(x) > f'(y)$. If $f'(u) > f'(y)$, let$f = f'$. Otherwise, let f be the labeling obtained from f' by interchanging the labels of u and y and keeping all other labels unchanged.

Then f is a graphoidal labeling of G such that every vertex in G of degree 2 is interior to ψ_f so that $\eta_l(G) = q - m$.

Subcase 2.2. deg $x = 3$ and deg $y = 3$.

Let G' be the graph obtained from G by deleting the edge ux and adding a new vertex w and joining it with the vertex x by an edge. Then G' is a label graphoidal graph on $p+1$ vertices satisfying the requirements to apply the induction hypothesis so that it admits a graphoidal labeling f' such that every vertex of degree 2 in G' is interior to $\psi_{f'}$. We now define a graphoidal labeling $f : V(G) \to \{1, 2, \ldots, p\}$ of G as follows.

(i) Suppose $f'(w) = p+1$. If $f'(x) < f'(u)$, let f be the restriction of f' on G (that is , $f(v) = f'(v)$, for all $v \in V(G)$). If $f'(x) > f'(u)$, let f be the labeling obtained from the restriction of f' on G by interchanging $f'(x)$ and $f'(u)$.

(ii) Suppose $f'(w) < p+1$. Let v be the vertex with $f'(v) = p+1$. Let z be the neighbour of v having the maximum label among the neighbours of v. Let P be the maximal directed path in $\uparrow G_{f'}$ ending with v containing the arc (z, v), say $P = (z_1, \ldots, z_k = z, z_{k+1} = v)$. Let j, where $1 \leq j \leq k+1$, be the least positive integer such that $f'(w) < f'(z_j)$. Now, define $f : V(G) \to \{1, 2, \ldots, p\}$ by $f(z_j) = f'(w)$, $f(z_{j+i}) = f'(z_{j+i-1})$, for all $i = 1, 2, \ldots, k+1-j$ and $f(x) = f'(x)$ for all other vertices x of G.

It is not difficult to see that direction of an edge e in G other than xu under the labeling f is same as the direction under f' treating e as an edge of G'. Also, the vertex u is interior to ψ_f. That is, f is a graphoidal labeling of G such that every vertex in G of degree 2 is interior to ψ_f so that $\eta_l(G) = q - m$. □

As a consequence of the above theorem we get the following which has already been proved in [3].

Corollary 3.2. *If T is tree with m vertices of degree 2, then $\eta_l(T) = q - m$.*

Also the above theorem is very useful in determining the value of η_l for unicyclic graphs which we present below.

Proposition 3.3. *Let G be a unicyclic graph which is label graphoidal. Let C be the cycle of G and let r be the number of vertices on C of degree greater than 2. Let m be the number of vertices of degree 2 in G. Then*

$$\eta_l(G) = \begin{cases} 2 & \text{if } r = 0 \\ q - m & \text{if } r > 0 \text{ and is even} \\ q - m + 1 & \text{if } r \text{ is odd.} \end{cases}$$

Proof. If $r = 0$, then $G = C$ so that $\eta_l(G) = 2$. Assume $r \geq 1$. If r is even, it follows from Theorem 3.1 that $\eta_l(G) = q - m$.

Suppose r is odd. Now, if C is an odd cycle, by Theorem 2.4, the cycle C contains a vertex of degree 2 (in fact, in this case there will be at least two vertices of degree 2 on C) and if C is an even cycle, then also C has a vertex of degree 2 because r is odd. Let x be a vertex of degree 2 on C. Let H be the graph obtained from G by introducing a vertex y and joining it to x by an edge. Then H is a label graphoidal graph in which C has an even number of vertices of degree greater than 2 in H and hence it follows from Theorem 3.1 that $\eta_l(H) = q(H) - m'$, where m' is the number of vertices of degree 2 in H. Let f be a graphoidal labeling of H with $|\psi_f| = \eta_l(H)$. Then x is exterior to ψ_f and the vertices of degree 2 in H are interior to ψ_f.

We now define a graphoidal labeling g of G such that x is the only vertex of degree 2 in G exterior to ψ_g using the graphoidal labeling f of H.

If $f(y) = p + 1$, where $p = |V(G)|$, then define $g : V(G) \to \{1, 2, \ldots, p\}$ by $g(v) = f(v)$ for all $v \in V(G)$. If $f(y) = t \neq p + 1$, let $v_1, v_2, \ldots, v_{p+1-t}$ be the vertices in G such that $f(v_i) = t + i$. Now, define $g : V(G) \to \{1, 2, \ldots, p\}$ by

$$g(v_i) = t + i - 1, \text{ for } i = 1, 2, \ldots, p + 1 - t$$
$$g(v) = f(v), \text{ for } v \in V(G) \setminus \{v_1, \ldots, v_{p+1-t}\}.$$

Now, it is easy to verify that in either case the orientation of G with respect to the labeling f is the same as the orientation of G with respect to the labeling g. Hence g is a graphoidal labeling of G such that x is the only vertex in G of degree two which is exterior to ψ_g so that $s \leq 1$ and consequently it follows from Theorem 2.6 that $\eta_l(G) \leq q - m + 1$. Further, since r is odd, Theorem 3.1 and Corollary 2.7 together infer that $\eta_l(G) \geq q - m + 1$. Thus $\eta_l(G) = q - m + 1$. □

4. Graphs with $\eta_l = q$

It has been proved in [3] that $\eta_l(G) \leq q$ and also that if this equality holds then G is bipartite. We now proceed to settle the question of when the equality holds.

Theorem 4.1. *Let G be a label graphoidal graph. Then $\eta_l(G) = q$ if and only if G is bipartite and every vertex of degree 2 lies on a cycle in which all other vertices are of degree greater than 2.*

Proof. Suppose $\eta_l(G) = q$. It follows from Theorem 2.8 that G is bipartite. Let x be a vertex of degree 2 in G. Let y and z be the vertices adjacent to x. Since G is connected and $\eta_l(G) = q$ it follows that $\deg y \geq 2$ or $\deg z \geq 2$. Assume $\deg z \geq 2$. We have to prove that x lies on a cycle in which all other vertices are of degree greater than 2. Suppose not. That is, either x does not lie on any cycle or every cycle containing x has a vertex of degree 2 other than x.

Let H be the graph obtained from G by adding the edge yz. We now claim that H is a label graphoidal graph by proving that every odd cycle in H contains a vertex of degree 2 in H. Let C be an odd cycle in H. Since G is bipartite, it follows that C contains the edge yz. Now, if x

does not lie on any cycle in G, then $C = (x, y, z, x)$ which implies that C has a vertex of degree 2, namely x. If every cycle in G containing the vertex x has a vertex of degree 2 other than x, then the cycle C' in G obtained from C by replacing the edge yz by the path (y, x, z) has a vertex $u \neq x$ of degree 2 in G and consequently u is a vertex of degree 2 on C in H. Thus H is a label graphoidal graph.

Further, H is not bipartite so that $\eta_l(H) < q(H) = q(G) + 1$. Let f be a graphoidal labeling of H such that $|\psi_f| = \eta_l(H)$. Now, define $g : V(G) \to \{1, \ldots, p\}$ as follows:

(i) If x is neither a sink nor a source under f, let $g(v) = f(v)$, for all $v \in V(G)$.

(ii) If x is either a sink or a source under f, let $g(x) = f(y), g(y) = f(x)$ and $g(v) = f(v)$, for all $v \in V(G) \setminus \{x, y\}$.

Then g is a graphoidal labeling of G such that $|\psi_g| = |\psi_f| - 1 < q(G)$, which is a contradiction. Thus x lies on a cycle in which all other vertices are of degree greater than 2.

Conversely, suppose G is bipartite and every vertex of degree 2 lies on a cycle in which all other vertices are of degree greater than 2. Let f be a graphoidal labeling of G. We have to prove that every vertex of degree 2 is exterior to ψ_f. Let x be a vertex of degree 2 in G. By assumption x lies on a cycle C in which all other vertices are of degree greater than 2 and hence they are alternatively source and sink in $\uparrow G_f$. Now, since G is bipartite the cycle C is even and hence the two neighbours of x are source or both are sink and hence x is exterior to ψ_f. Thus every vertex of degree 2 is exterior to ψ_f for any graphoidal labeling f of G so that $\eta_l(G) = q$. □

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