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ON THE DEFENSIVE ALLIANCES IN GRAPH

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Abstract. Let $G = (V, E)$ be a graph. We say that $S \subseteq V$ is a defensive alliance if for every $u \in S$, the number of neighbors u has in S plus one (counting u) is at least as large as the number of neighbors it has outside S. Then, for every vertex u in a defensive alliance S, any attack on a single vertex by the neighbors of u in $V - S$ can be thwarted by the neighbors of u in S and u itself. In this paper, we study alliances that are containing a given vertex u and study their mathematical properties.

1. Introduction

The concept of defensive alliances in graphs, together with a variety of other kinds of alliances, were first studied by Hedetniemi, Hedetniemi and Kristiansen in [6]. In the cited paper, the authors initiate the study of mathematical properties of alliances and give some sharp bounds on the defensive alliance number. Then some authors study the other kind of alliances such as global defensive alliances [3], defensive k-alliances [11], global defensive k-alliances [7], offensive alliances [1, 9], global offensive alliances [8, 13], and some authors study the algorithmic complexity of alliances in graphs [4, 5]. Also, for defensive alliances in line graphs and spectral study of alliances in graphs, we refer the reader to [10, 12]. The authors in [6] list some interest questions which are important in application. In this paper, we study the existence of a defensive alliance in graph $G = (V, E)$ which is containing a given vertex $u \in V$ (listed as an interest question in [6]) and study mathematical properties of the

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alliance number which is containig a given vertex, say, u. For the rest of this section, we provide some notations, definitions and theorems from the literature which will be needed in the sequel.

Let $G = (V, E)$ be a graph of order n and size m. According to notations of [14], for any vertex $u \in V$, the closed neighborhood of u is denoted by $N_G[u]$, the subscript G will be omitted when it is understood, degree of a vertex u is denoted by $deg_G(u)$ or simply $deg(u)$. The complement of a set $E \subseteq V$ is denoted by $\overline{E}$ and is defined as $\overline{E} = V - E$. The subgraph induced by S, that is, the subgraph $(S, E \cap (E \times E))$ is denoted by $\langle S \rangle$. Given $S, T \subseteq V$, we write $[S, T]$ for the set of edges having one endpoint in S and the other in T. An edge cut is an edge set of the form $[S, \overline{S}]$, where S is a non-empty proper subset of V.

The authors in [6] introduced several types of alliances, including defensive alliances and strong defensive alliances that we mention here.

A non-empty set of vertices $S \subseteq V$ is called a defensive alliances if for every $u \in S$, $\left|N[u] \cap S\right| \geqslant \left|N[u] - S\right|$. In other word, every vertex in S is defended from possible attack by vertices $\overline{S}$. A defensive alliance is called strong if for every vertex $u \in S$, $\left|N[u] \cap S\right| > \left|N[u] - S\right|$. Any two vertices u, v in an alliance S are called allies with respect to S, we also say that u and v are allied. An alliance S is called critical if no proper subset of S is an alliance. The alliance number $a(G)$ is the minimum cardinality of any critical alliance in G, and the strong alliance number $\hat{a}(G)$ is the minimum cardinality of any critical strong alliance in G.

In the following definition, we introduce the (strong) defensive alliance number of a graph such that this alliance is containing a given vertex.

Definition 1.1. *Let $u \in V$ be an arbitrary vertex. The alliance number $a_u(G)$ is the minimum cardinality of any critical defensive alliance in G which is containing u and the strong alliance number $\hat{a}_u(G)$ is the minimum cardinality of any critical strong alliance in G which is containing u. It is evident that $a(G) = \min_{u \in V} a_u(G)$ and $\hat{a}(G) = \min_{u \in V} \hat{a}_u(G)$.*

In order to prove our main results, we need the following theorem [2]. This result guarantees the existence of a defensive alliance of cardinality $\lceil \frac{n}{2} \rceil$ or $\lfloor \frac{n}{2} \rfloor$ in a graph of order n.

Theorem 1.2. *Let $G = (V, E)$ be a graph of order n. Then G has a defensive alliance of cardinality $\lceil \frac{n}{2} \rceil$ or $\lfloor \frac{n}{2} \rfloor$.*

Note that the above theorem implies that if $G = (V, E)$ is a graph of even order n, then G has a defensive alliance S such that $|S| = \frac{n}{2}$

2. Properties of $a_u(G)$ and $\hat{a}_u(G)$

In this section we study mathematical properties of the alliance number $a_u(G)$ and strong alliance number $\hat{a}_u(G)$. First observe that $a_u(G) = 1$ if and only if $\deg(u) \leqslant 1$ and $\hat{a}_u(G) = 1$ if and only if

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$\deg(u) = 0$. The authors in [6] show that any critical (strong) alliance S in a graph G must induce a connected subgraph of G. Then, our first observation is immediate.

Observation 2.1. *Let $G = (V, E)$ be a graph and $u \in V$. For any critical alliance S such that $u \in S$, the induced graph $\langle S \rangle$ is a connected graph.*

In the following result, we characterize vertices for which $a_u(G) = 2$ and vertices for which $\hat{a}_u(G) = 2$. This result is immediate consequence of Observation 2.1.

Proposition 2.1. *For any graph G,*

(i) $a_u(G) = 2$ *if and only if* $2 \leqslant deg(u) \leqslant 3$ *and* u *has a neighbour of degree at most three.*

(ii) $\hat{a}_u(G) = 2$ *if and only if* $deg(u) = 2$ *and* u *has a neighbour of degree at most two.*

Example 2.1. *For any cycle C_n and path P_n,*

i) $a_u(C_n) = \hat{a}_u(C_n) = 2$, *for all* $u \in V(C_n)$.

ii) $a_u(P_n) = \begin{cases} 1 & deg(u) = 1, \\ 2 & deg(u) = 2. \end{cases}$

iii) $\hat{a}_u(P_n) = 2$, *for all* $u \in V(P_n)$.

Let K_n be the complete graph of order n.

Example 2.2. *For any $u \in V(K_n)$,*

i) $a_u(K_n) = \left\lceil \frac{n}{2} \right\rceil$,

ii) $\hat{a}_u(K_n) = \left\lfloor \frac{n}{2} \right\rfloor + 1$.

Proof. It is clear that every subset $S \subseteq V(K_n)$ such that $|S| = \left\lceil \frac{n}{2} \right\rceil$ is a defensive alliance and every subset $S \subseteq V(K_n)$ such that $|S| = \left\lfloor \frac{n}{2} \right\rfloor + 1$ is a strong defensive alliance. It follows that $a_u(K_n) \leqslant \left\lceil \frac{n}{2} \right\rceil$ and $\hat{a}_u(K_n) \leqslant \left\lfloor \frac{n}{2} \right\rfloor + 1$ for all $u \in V(K_n)$.On the other hand we know that $a(K_n) = \left\lceil \frac{n}{2} \right\rceil$ and $\hat{a}(K_n) = \left\lfloor \frac{n}{2} \right\rfloor + 1$ (see [6]), so this graph has no (strong) defensive alliance of cardinalty less than $\left\lceil \frac{n}{2} \right\rceil$ $(\left\lfloor \frac{n}{2} \right\rfloor + 1)$. This completes the proof. □

The wheel graph W_n is the graph whose vertex set is $V = \{v_1, v_2, \dots, v_2, w\}$ where the induced subgraph on $V' = \{v_1, v_2, \dots, v_n\}$ is a cycle C_n of length n and w is a central vertex which is adjacent to every vertex of the cycle. The proof of the following result is similar to thr proof of the above result.

Example 2.3. *For the complete bipartite graph $K_{m,n}$ and the wheel W_n,*

i) $a_u(K_{m,n}) = \begin{cases} 1 & m \geqslant n = 1 \text{ and } deg(u) = 1, \\ \left\lfloor \frac{m}{2} \right\rfloor + 1 & m \geqslant n = 1 \text{ and } deg(u) = m, \\ \left\lfloor \frac{m}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor & m \geqslant n > 1. \end{cases}$

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ii) $\hat{a}_u(K_{m,n}) = \left\lceil \frac{n}{2} \right\rceil + \left\lceil \frac{m}{2} \right\rceil$.

iii) $a_u(W_n) = \begin{cases} 2 & deg(u) = 3, \\ \left\lfloor \frac{n}{2} \right\rfloor + 1 & deg(u) = n. \end{cases}$

iv) $\hat{a}_u(W_n) = \left\lceil \frac{n}{2} \right\rceil + 1$.

3. Upper and lower bounds

In this section, we give some upper and lower bounds on the alliance and strong alliance number $a_u(G)$ and $\hat{a}_u(G)$. First, note that both $a_u(G)$ and $\hat{a}_u(G)$ can equal one for some graphs G and for some $u \in V$. Also, it is clear that for any $u \in V$, the entire vertex set V is always a defensive alliance and strong defensive alliance which is containing u. Then, for any graph G of order n and for any $u \in V$,

$$1 \leqslant a_u(G) \leqslant \hat{a}_u(G) \leqslant n. \tag{3.1}$$

In the following result, we improve the above lower bound slightly.

Theorem 3.1. *Let $G = (V, E)$ be a graph. Then for all $u \in V$,*

i) $\left\lfloor \frac{\deg(u)}{2} \right\rfloor + 1 \leqslant a_u(G)$,

ii) $\left\lceil \frac{\deg(u)}{2} \right\rceil + 1 \leqslant \hat{a}_u(G)$,

and these bounds are sharp.

Proof. It is obvious that these bounds are sharp. Suppose that $S \subseteq V$ is a defensice alliance which is containing u, then $|N[u] \cap S| \geqslant |N[u] - S|$. This means that S must have at least $\left\lfloor \frac{\deg(u)}{2} \right\rfloor$ of neighbors u and u itself. Thus $\left\lfloor \frac{\deg(u)}{2} \right\rfloor + 1 \leqslant |S|$. Similarly, one can show that if $S \subseteq V$ is a strong defensive alliance which is containing u, then S must have at least $\left\lceil \frac{\deg(u)}{2} \right\rceil$ of neighbors u and u itself. Thus $\left\lceil \frac{\deg(u)}{2} \right\rceil + 1 \leqslant |S|$. □

Now, consider graph $G = (V, E)$ of order n and $u \in V$ such that $\deg(u) = 2$. Suppose that u is adjacent to v_1, v_2 and v_1 is adjacent to all vertices except v_2 and v_2 is adjacent to all vertices except v_1. Moreover, assume that $\langle V - \{u, v_1, v_2\} \rangle \cong K_{n-3}$. It is not hard to see that $a_u(G) = \left\lfloor \frac{n}{2} \right\rfloor + 1$. This example show that $a_u(G)$ can be at least as large as $\left\lfloor \frac{n}{2} \right\rfloor + 1$. At present we know of no example where $a_u(G) > \left\lfloor \frac{n}{2} \right\rfloor + 1$ for a graph G of order n. This leads us to conjecture:

Conjecture 3.1. *For any graph G of order n and for any $u \in V$,*

$$a_u(G) \leqslant \left\lfloor \frac{n}{2} \right\rfloor + 1.$$

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In the following results, we show that the above conjecture is valid when $G = (V, E)$ is a graph of order n and $u \in V$ such that $n - 5 \leqslant \deg(u) \leqslant n - 1$. The ideas of the proof in all obtained results in this section are similar to [2].

Theorem 3.2. *Let $G = (V, E)$ be a graph of order n. Let $u \in V$ such that $deg(u) = n - 1$. Then there exists a defensive alliance S such that $u \in S$ and $|S| = \lceil \frac{n}{2} \rceil$. Furthermore, $a_u(G) = \lceil \frac{n}{2} \rceil$.*

Proof. First, note that by Theorem 3.1, $\left\lceil \dfrac{n}{2} \right\rceil = \left\lfloor \dfrac{n-1}{2} \right\rfloor + 1 \leqslant a_u(G)$. Then it is sufficient to prove that $a_u(G) \leqslant \left\lceil \dfrac{n}{2} \right\rceil$. To do this, let $\mathcal{P}$ be the set of edge cuts of form $[E, \overline{E}]$ in graph G such that $|E| = \lceil \frac{n}{2} \rceil$. Assume that $[A, \overline{A}] \in \mathcal{P}$ such that

$$\left|[A, \overline{A}]\right| = \min\left\{ \left|[E, \overline{E}]\right| : [E, \overline{E}] \in \mathcal{P} \right\}. \tag{3.2}$$

In fact, according to paper [2], $[A, \overline{A}]$ is a balanced bi-partition of V and either A or $\overline{A}$ is a defensive alliance. First, let n be an even number. If A is a defensive alliance and $u \in A$, then the proof is complete. If A is a defensive alliance and $u \in \overline{A}$, then choose $a \in A$ and define $B = (A - \{a\}) \cup \{u\}$. It can be easily checked that $|N[u] \cap B| = \frac{n}{2} = |N[u] - B|$, and for all $v \in B$ such that $a \notin N[v]$, $\left|N[v] \cap B\right| = \left|N[v] \cap (A - \{a\})\right| + 1 = \left|N[v] \cap A\right| + 1 \geqslant \left|N[v] - A\right| + 1 = \left|N[v] - (A - \{a\})\right| + 1 = \left|N[v] - B\right|$, and for all $v \in B$ such that $a \in N[v]$, $\left|N[v] \cap B\right| = \left|N[v] \cap (A - \{a\})\right| + 1 = \left|N[v] \cap A\right| \geqslant \left|N[v] - A\right| = \left|N[v] - (A - \{a\})\right| - 1 = \left|N[v] - B\right|$. Thus B is a defensive alliance which is containing u and $|B| = \frac{n}{2}$. In the case that $\overline{A}$ is a defensive alliance the proof is similar. Now, let n be an odd number. If A is a defensive alliance and $u \in A$, then the proof is complete. If A is a defensive alliance and $u \in \overline{A}$, then by similar argument, as it mentioned in previous case, one can show that for all $a \in A$ the set $B = (A - \{a\}) \cup \{u\}$ is a defensive alliance which is containing u and $|B| = \lceil \frac{n}{2} \rceil$. Now, If $\overline{A}$ is a defensive alliance, then $u \in A$, since otherwise, if $u \in \overline{A}$, then $|N[u] \cap \overline{A}| = \lfloor \frac{n}{2} \rfloor = \frac{n-1}{2} < \lceil \frac{n}{2} \rceil = \frac{n+1}{2} = |N[u] - \overline{A}|$ which is a contradiction. If $\overline{A}$ is a defensive alliance, then one can easily check that $B = A \cup \{u\}$ is a defensive alliance which is containing u and $|B| = \lceil \frac{n}{2} \rceil$. This completes the proof. □

Theorem 3.3. *Let $G = (V, E)$ be a graph of even order n and $u \in V$ such that $deg(u) = n - 2$. Then there exists a defensive alliance S such that $u \in S$ and $|S| = \frac{n}{2}$. Furthermore, $a_u(G) = \frac{n}{2}$.*

Proof. Let $v \in V$ such that v is not adjacent to u. Consider the induced graph $G' = \langle V - \{u, v\} \rangle$. It is clear that G' is a graph of oder $n - 2$. By Theorem 1.2, G' has a defensive alliance S' such that $|S'| = \frac{n-2}{2}$. Now, define the set $S = S' \cup \{u\}$. Obviously, $|S| = \frac{n}{2}$ and $u \in S$. On the other hand $|N_G[u] \cap S| = \frac{n}{2} = |N_G[u] - S|$, and for all $w \in S$, $|N_G[w] \cap S| = |N_{G'}[w] \cap S'| + 1 \geqslant |N_{G'}[w] - S'| + 1 \geqslant |N_G[w] - S|$. It follows that S is a defensive alliance, so $a_u(G) \leqslant \frac{n}{2}$. On the other hand, by Theorem 3.1, we have $\frac{n}{2} = \lfloor \frac{n-2}{2} \rfloor + 1 \leqslant a_u(G)$. □

Theorem 3.4. *Let $G = (V, E)$ be a graph of odd order n and $u \in V$ such that $deg(u) = n - 2$. Then there exists a defensive alliance S such that $|S| = \frac{n+1}{2}$ and $u \in S$. Furthermore, $a_u(G) = \frac{n+1}{2}$ or $a_u(G) = \frac{n-1}{2}$.*

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Proof. By Theorem 3.1 , n-1$\frac{}{2=\left\lfloor\frac{n-2}{2}\right\rfloor+1\leqslant a_u(G)}$. Then it is sufficient to find a defensive alliance of cardinality $\frac{n+1}{2}$ containing u. Define

$$\mathcal{P}:=\Big\{[E,\overline{E}]:E\subseteq V,u\in E,|E|=\frac{n+1}{2}\Big\}.$$

Assume that $[A,\overline{A}]\in\mathcal{P}$ such that $\Big|[A,\overline{A}]\Big|=\min\Big\{\Big|[E,\overline{E}]\Big|:[E,\overline{E}]\in\mathcal{P}\Big\}$. Note that $|N[u]\cap A|\geqslant|N[u]-A|$. Since otherwise, define $B=\overline{A}\cup\{u\}$. It is easy to see that $[B,\overline{B}]\in\mathcal{P}$ and $\Big|[B,\overline{B}]\Big|<\Big|[A,\overline{A}]\Big|$, which is a contradiction. By the same argument, as in the proof of Theorem 3.2, one can show that A or $\overline{A}$ is a defensive alliance. If A is a defensive alliance, the proof is complete. If $\overline{A}$ is a defensive alliance, define $B=\overline{A}\cup\{u\}$. It is easy to see that $|N[u]\cap B|\geqslant\frac{n-1}{2}\geqslant|N[u]-B|$, and for all $v\in B$, $v\neq u$, $|N[v]\cap B|\geqslant|N[v]\cap\overline{A}|\geqslant|N[v]-\overline{A}|\geqslant|N[v]-B|$. So B is a defensive alliance, $u\in B$ and $|B|=\frac{n+1}{2}$. □

Theorem 3.5. *Let $G=(V,E)$ be a graph of even order n and $u\in V$ such that $deg(u)=n-3$. Then there exists a defensive alliance S such that $|S|=\frac{n}{2}$ and $u\in S$. Furthermore, $a_u(G)=\frac{n}{2}$ or $a_u(G)=\frac{n-2}{2}$.*

Proof. By Theorem 3.1, $\frac{n-2}{2}=\left\lfloor\frac{n-3}{2}\right\rfloor+1\leqslant a_u(G)$. Then it is sufficient to find a defensive alliance of cardinality $\frac{n}{2}$ containing u. Suppose that $u_1,u_2\in V$ such that are not adjacent to u and $C:=\{u,u_1,u_2\}$. Define $\mathcal{P}:=\Big\{[E,\overline{E}]:E\subseteq V,|E\cap C|=2,|E|=\frac{n}{2}\Big\}$. Assume that $[A,\overline{A}]\in\mathcal{P}$ such that $\Big|[A,\overline{A}]\Big|=\min\Big\{\Big|[E,\overline{E}]\Big|:[E,\overline{E}]\in\mathcal{P}\Big\}$. We divide the proof into two steps.

1) $u\in A$.

By symmetry, it is sufficient to consider the case $u,u_1\in A$. First, note that $|N[u]\cap A|=\frac{n-2}{2}=|N[u]-A|$ and $|N[u_2]\cap\overline{A}|\geqslant|N[u_2]-\overline{A}|$. Since otherwise, if $|N[u_2]\cap\overline{A}|<|N[u_2]-\overline{A}|$, then define $B=(A-\{u\})\cup\{u_2\}$. We have $[B,\overline{B}]\in\mathcal{P}$ and $\big|[B,\overline{B}]\big|<\big|[A,\overline{A}]\big|$ which is a contradiction. Now, assume that neither A nor $\overline{A}$ is a defensive alliance. Thus, there exist undefended vertices $a\in A$ and $b\in\overline{A}$ such that $|N[a]\cap A|<|N[a]-A|$ and $|N[b]\cap\overline{A}|<|N[b]-\overline{A}|$ (note that $a\neq u$ and $b\neq u_2$). But then $[B,\overline{B}]\in\mathcal{P}$ (or $[\overline{B},B]\in\mathcal{P}$) and $\big|[B,\overline{B}]\big|<\big|[A,\overline{A}]\big|$, where $B=(A-\{a\})\cup\{b\}$, which is a contradiction. If A is a defensive alliance, we are done. If $\overline{A}$ is a defensive alliance, define $B=(\overline{A}-\{u_2\})\cup\{u\}$. Hence $|N[u]\cap B|=\frac{n}{2}>\frac{n-4}{2}=|N[u]-B|$, and for all $v\in B-\{u\}$, $\big|N[v]\cap B\big|=\big|N[v]\cap(\overline{A}-\{u_2\})\big|+1\geqslant\big|N[v]\cap\overline{A}\big|\geqslant\big|N[v]-\overline{A}\big|\geqslant\big|N[v]-(\overline{A}-\{u_2\})\big|-1=\big|N[v]-B\big|$. It follows that B is a defensive alliance of cardinality $\frac{n}{2}$ such that $u\in B$.

2) $u\in\overline{A}$.

Note that $|N[u]\cap\overline{A}|=\frac{n}{2}>\frac{n}{2}-2=|N[u]-\overline{A}|$. Here, we suppose that there exists no $[F,\overline{F}]\in\mathcal{P}$ such that $u,u_1\in F$ or $u,u_2\in F$ and $\big|[F,\overline{F}]\big|=\big|[A,\overline{A}]\big|$. Since otherwise, by step (1), the proof is complete. If $\overline{A}$ is a defensive alliance, we are done. If not, there exists $b\in\overline{A}$ ($b\neq u$) such that $|N[b]\cap\overline{A}|<|N[b]-\overline{A}|$. Now, we show that A is a strong defensive alliance.

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We argue by contradiction. If A is not a strong defensive alliance, there exists $a \in A$ such that $|N[a] \cap A| \leqslant |N[a] - A|$. Now define $B = (A - \{a\}) \cup \{b\}$. We have $[B, \overline{B}] \in \mathcal{P}$ (or $[\overline{B}, B] \in \mathcal{P}$) and $\big|[B, \overline{B}]\big| < \big|[A, \overline{A}]\big|$, which is impossible. If $|N[u_2] \cap A| = |N[u_2] - A| + 1$, define $F = (\overline{A} - \{b\}) \cup \{u_2\}$. It is easy to see that $[F, \overline{F}] \in \mathcal{P}$, $u, u_2 \in F$ and $\big|[F, \overline{F}]\big| = \big|[A, \overline{A}]\big|$, contradicting our hypothesis. Now, consider the set $S = (A - \{u_1\}) \cup \{u\}$. We have $|N[u] \cap S| = \frac{n}{2} - 1 = |N[u] - S|$, and $\big|N[u_2] \cap S\big| = \big|N[u_2] \cap (A - \{u_1\})\big| \geqslant \big|N[u_2] \cap A\big| - 1 \geqslant \big|N[u_2] - A\big| + 1 \geqslant \big|N[u_2] - (A - \{u_1\})\big| = \big|N[u_2] - S\big|$, and for all $v \in S - \{u, u_2\}$, $\big|N[v] \cap S\big| = \big|N[v] \cap (A - \{u_1\})\big| + 1 \geqslant \big|N[v] \cap A\big| \geqslant \big|N[v] - A\big| + 1 \geqslant \big|N[v] - (A - \{u_1\})\big| \geqslant \big|N[v] - S\big|$. So S is a defensive alliance containing u and $|S| = \frac{n}{2}$.

□

Theorem 3.6. *Let $G = (V, E)$ be a graph of odd order n and $u \in V$ such that $deg(u) = n - 3$. Then there exists a defensive alliance S such that $|S| = \frac{n+1}{2}$ and $u \in S$. Furthermore, $a_u(G) = \frac{n-1}{2}$ or $a_u(G) = \frac{n+1}{2}$.*

Proof. By Theorem 3.1, $\frac{n-1}{2} = \left\lfloor \frac{n-3}{2} \right\rfloor + 1 \leqslant a_u(G)$. Then to complete the proof, it is sufficient to find a defensive alliance of cardinality $\frac{n+1}{2}$ which is containing u. To do this, suppose that $u_1, u_2 \in V$ such that u_1, u_2 are not adjacent to u and $C := \{u, u_1, u_2\}$. Define $\mathcal{P} := \Big\{[E, \overline{E}] : E \subseteq V, |E \cap C| = 2, |E| = \frac{n+1}{2}\Big\}$. Let $[A, \overline{A}] \in \mathcal{P}$ such that $\Big|[A, \overline{A}]\Big| = \min\Big\{\Big|[E, \overline{E}]\Big| : [E, \overline{E}] \in \mathcal{P}\Big\}$. We divide the proof into two steps.

1) $u \in A$.

It is sufficient to consider the case $u, u_1 \in A$. Note that $|N[u] \cap A| = \frac{n-1}{2} > \frac{n-3}{2} = |N[u] - A|$, $|N[u_2] \cap \overline{A}| \geqslant |N[u_2] - \overline{A}|$ and $|N[u_1] \cap A| \geqslant |N[u_1] - A|$. Since otherwise, if $|N[u_2] \cap \overline{A}| < |N[u_2] - \overline{A}|$, define $B = (A - \{u\}) \cup \{u_2\}$ and if $|N[u_1] \cap A| < |N[u_1] - A|$, define $B = \overline{A} \cup \{u_1\}$. We have $[B, \overline{B}] \in \mathcal{P}$ and $\big|[B, \overline{B}]\big| < \big|[A, \overline{A}]\big|$ which is a contradiction. Now, assume that neither A nor $\overline{A}$ is a defensive alliance. Thus, there exist vertices $a \in A$ and $b \in \overline{A}$ such that $|N[a] \cap A| < |N[a] - A|$ and $|N[b] \cap \overline{A}| < |N[b] - \overline{A}|$ (note that $a \neq u, u_1, b \neq u_2$). But then $[B, \overline{B}] \in \mathcal{P}$ and $\big|[B, \overline{B}]\big| < \big|[A, \overline{A}]\big|$, where $B = (A - \{a\}) \cup \{b\}$, which is impossible. If A is a defensive alliance, we are done. If $\overline{A}$ is a defensive alliance, define $S = \overline{A} \cup \{u\}$. Hence $|N[u] \cap S| = \frac{n-1}{2} > \frac{n-3}{2} = |N[u] - S|$, and for all $v \in S - \{u\}$, $\big|N[v] \cap S\big| \geqslant \big|N[v] \cap \overline{A}\big| \geqslant \big|N[v] - \overline{A}\big| \geqslant \big|N[v] - S\big|$.

2) $u \in \overline{A}$.

Note that $|N[u] \cap \overline{A}| = \frac{n-1}{2} > \frac{n-3}{2} = |N[u] - \overline{A}|$. Here, we suppose that there exists no $[F, \overline{F}] \in \mathcal{P}$ such that $u, u_1 \in F$ or $u, u_2 \in F$ and $\big|[F, \overline{F}]\big| = \big|[A, \overline{A}]\big|$. Since otherwise, by step (1), we are done. If A is a strong defensive alliance, then $|N[u_2] \cap A| \geqslant |N[u_2] - A| + 2$. Since otherwise, If $|N[u_2] \cap A| = |N[u_2] - A| + 1$, define $F = \overline{A} \cup \{u_2\}$. We have $[F, \overline{F}] \in \mathcal{P}$, $u, u_2 \in F$ and $\big|[F, \overline{F}]\big| = \big|[A, \overline{A}]\big|$, contradicting our hypothesis. Now, consider the set

$S = (A - \{u_1\}) \cup \{u\}$. We have $|N[u] \cap S| = \frac{n-1}{2} > \frac{n-3}{2} = |N[u] - S|$, and $\big|N[u_2] \cap S\big| = \big|N[u_2] \cap (A - \{u_1\})\big| \geqslant \big|N[u_2] \cap A\big| - 1 \geqslant \big|N[u_2] - A\big| + 1 \geqslant \big|N[u_2] - (A - \{u_1\})\big| = \big|N[u_2] - S\big|$, and for all $v \in S - \{u, u_2\}$, $\big|N[v] \cap S\big| = \big|N[v] \cap (A - \{u_1\})\big| + 1 \geqslant \big|N[v] \cap A\big| \geqslant \big|N[v] - A\big| + 1 \geqslant \big|N[v] - (A - \{u_1\})\big| \geqslant \big|N[v] - S\big|$. Therefore S is a defensive alliance containing u and $|S| = \frac{n+1}{2}$.

Now, suppose that A is not a strong defensive alliance. If $\big|N[u_1] \cap A\big| \leqslant \big|N[u_1] - A\big|$ define $B = \overline{A} \cup \{u_1\}$ and if $\big|N[u_2] \cap A\big| \leqslant \big|N[u_2] - A\big|$ define $B = \overline{A} \cup \{u_2\}$. Clearly $[B, \overline{B}] \in \mathcal{P}$ and $\big|[B, \overline{B}]\big| < \big|[A, \overline{A}]\big|$, which is impossible. Since A is not a strong defensive alliance, there exists $a \in A$ ($a \neq u_1, u_2$) such that $|N[a] \cap A| \leqslant |N[a] - A|$. Now, if $\overline{A}$ is not a defensive alliance, then there exists $b \in \overline{A}$ such that $|N[b] \cap \overline{A}| < |N[b] - \overline{A}|$. Define $B = (A - \{a\}) \cup \{b\}$. We have $[B, \overline{B}] \in \mathcal{P}$ and $\big|[B, \overline{B}]\big| < \big|[A, \overline{A}]\big|$, which is impossible. Hence A is a defensive alliance. Clearly, $S = \overline{A} \cup \{a\}$ is a defensive alliance such that $|S| = \frac{n+1}{2}$ and $u \in S$.

□

Theorems 3.2, 3.3, 3.4, 3.5 and 3.6 imply that.

Corollary 3.7. *Let $G = (V, E)$ be a graph of order n and $u \in V$ such that $deg(u) = n - 1$ or $deg(u) = n - 2$ or $deg(u) = n - 3$. Then there exists a defensive alliance S such that $|S| = \left\lceil \frac{n}{2} \right\rceil$ and $u \in S$.*

Theorem 3.8. *Let $G = (V, E)$ be a graph of even order n and $u \in V$ such that $deg(u) = n - 4$. Then there exists a defensive alliance S such that $|S| = \frac{n+2}{2}$ or $|S| = \frac{n}{2}$ and $u \in S$. Furthermore, $a_u(G) = \frac{n+2}{2}$ or $a_u(G) = \frac{n}{2}$ or $a_u(G) = \frac{n-2}{2}$.*

Proof. By Theorem 3.1, $\frac{n-2}{2} = \left\lfloor \frac{n-4}{2} \right\rfloor + 1 \leqslant a_u(G)$. Then it is sufficient to find a defensive alliance of cardinality $\frac{n+2}{2}$ or $\frac{n}{2}$ containing u. Suppose that $u_1, u_2, u_3 \in V$ such that are not adjacent to u and $C := \{u, u_1, u_2, u_3\}$. Define

$$\mathcal{P} := \Big\{[E, \overline{E}] : E \subseteq V, |E| = \frac{n}{2}, |E \cap C| = 2 \text{ or } |E \cap C| = 3 \text{ and } u \in E\Big\}.$$

Let $[A, \overline{A}] \in \mathcal{P}$ such that $\Big|[A, \overline{A}]\Big| = \min\Big\{\Big|[E, \overline{E}]\Big| : [E, \overline{E}] \in \mathcal{P}\Big\}$. We divide the proof into two steps.

1) $|A \cap C| = 2$.

It is sufficient to consider the case $u, u_1 \in A$. Note that $|N[u] \cap A| = \frac{n-2}{2} > \frac{n-4}{2} = |N[u] - A|$, and $|N[u_2] \cap \overline{A}] \geqslant |N[u_2] - \overline{A}|$. Since otherwise, defining $B = (A - \{u\}) \cup \{u_2\}$ we have $[B, \overline{B}] \in \mathcal{P}$ and $\big|[B, \overline{B}]\big| < \big|[A, \overline{A}]\big|$ which is impossible. Similarly, one can also show that $|N[u_3] \cap \overline{A}] \geqslant |N[u_3] - \overline{A}|$. If $\overline{A}$ is not a defensive alliance, there exists $v \in \overline{A}$ ($v \neq u_2, u_3$) such that $|N[v] \cap \overline{A}| < |N[v] - \overline{A}|$. Defining $B = (\overline{A} - \{v\}) \cup \{u\}$, we have $[B, \overline{B}] \in \mathcal{P}$ and $\big|[B, \overline{B}]\big| < \big|[A, \overline{A}]\big|$, which is impossible. Now, define $S = \overline{A} \cup \{u\}$. Clearly, $u \in S$, $|S| = \frac{n+2}{2}$ and S is a defensive alliance.

2) $|A \cap C| = 3$ and $u \in A$.

It is sufficient to consider the case $u, u_1, u_2 \in A$. Note that $|N[u] \cap A| = \frac{n-4}{2} < \frac{n-2}{2} = |N[u] - A|$. If $\overline{A}$ is a defensive alliance, define $S = \overline{A} \cup \{u\}$. Obviously $|S| = \frac{n+2}{2}$ and S is a defensive alliance containing u. If not, there exists $v \in \overline{A}$ such that $|N[v] \cap \overline{A}| < |N[v] - \overline{A}|$. If $v \neq u_3$, define $B = (A - \{u\}) \cup \{v\}$. Clearly $[B, \overline{B}] \in \mathcal{P}$ and $\big|[B, \overline{B}]\big| < \big|[A, \overline{A}]\big|$, which is impossible. So $v = u_3$, and for all $b \in \overline{A} - \{u_3\}$, $|N[b] \cap \overline{A}| \geqslant |N[b] - \overline{A}|$. Now define $S = (\overline{A} - \{u_3\}) \cup \{u\}$. we have $|N[u] \cap S| = \frac{n}{2} > \frac{n-6}{2} = |N[u] - S|$, and for all $b \in S - \{u\}$, $\big|N[b] \cap S\big| = \big|N[b] \cap (\overline{A} - \{u_3\})\big| + 1 \geqslant \big|N[b] \cap \overline{A}\big| \geqslant \big|N[b] - \overline{A}\big| \geqslant \big|N[b] - (\overline{A} - \{u_3\})\big| - 1 = \big|N[b] - S\big|$. □

Theorem 3.9. *Let $G = (V, E)$ be a graph of odd order n and $u \in V$ such that $deg(u) = n - 4$. Then there exists a defensive alliance S such that $|S| = \frac{n+1}{2}$ and $u \in S$. Furthermore, $a_u(G) = \frac{n+1}{2}$ or $a_u(G) = \frac{n-1}{2}$ or $a_u(G) = \frac{n-3}{2}$.*

Proof. Theorem 3.1 implies that $\frac{n-3}{2} = \lfloor \frac{n-4}{2} \rfloor + 1 \leqslant a_u(G)$. Then it is sufficient to find a defensive alliance of cardinality $\frac{n+1}{2}$ containing u. Suppose that $u_1, u_2, u_3 \in V$ such that are not adjacent to u and $C := \{u_1, u_2, u_3\}$. Define

$$\mathcal{P} := \Big\{[E, \widetilde{V} - E] : E \subseteq \widetilde{V}, |E| = \frac{n-1}{2}, |E \cap C| = 2\Big\},$$

where $\widetilde{V} := V - \{u\}$. Let $[A, \widetilde{V} - A] \in \mathcal{P}$ such that $\Big|[A, \widetilde{V} - A]\Big| = \min\Big\{\Big|[E, \widetilde{V} - E]\Big| : [E, \widetilde{V} - E] \in \mathcal{P}\Big\}$. It is sufficient to consider the case $u_1, u_2 \in A$. If $\widetilde{V} - A$ is a defensive alliance, define $S = (\widetilde{V} - A) \cup \{u\}$. Clearly, S is a defensive alliance, $|S| = \frac{n+1}{2}$ and $u \in S$. Similarly, if A is a defensive alliance, then $S = A \cup \{u\}$ is a defensive alliance of cardinality $\frac{n+1}{2}$ containing u. Now, suppose that both A and $\widetilde{V} - A$ are not defensive alliances. Then there exist vertices $a \in A$ and $b \in \widetilde{V} - A$ such that $|N[a] \cap A| < |N[a] - A|$ and $|N[b] \cap (\widetilde{V} - A)| < |N[b] - (\widetilde{V} - A)|$. If $a = u_1$ or $a = u_2$, put $B = (A - \{a\}) \cup \{b\}$ and if $b \neq u_3$, put $B = (A - \{a\}) \cup \{b\}$. We have, $[B, \widetilde{V} - B] \in \mathcal{P}$ (or $[\widetilde{V} - B, B] \in \mathcal{P}$) and $\big|[B, \widetilde{V} - B]\big| < \big|[A, \widetilde{V} - A]\big|$ which is impossible. So $a \neq u_1$, $a \neq u_2$, $b = u_3$ and for all $v \in (\widetilde{V} - A) - \{u_3\}$, $|N[v] \cap (\widetilde{V} - A)| \geqslant |N[v] - (\widetilde{V} - A)|$. Note that if there exists $v \in (\widetilde{V} - A) - \{u_3\}$ such that $|N[v] \cap (\widetilde{V} - A)| = |N[v] - (\widetilde{V} - A)|$, then $\big|[B, \widetilde{V} - B]\big| < \big|[A, \widetilde{V} - A]\big|$, where $B = (A - \{a\}) \cup \{b\}$ and this is a contradiction. Now, define $S = \big((\widetilde{V} - A) - \{u_3\}\big) \cup \{u, a\}$. It is easy to see that $|S| = \frac{n+1}{2}$, $u \in S$, $|N[a] \cap (\widetilde{V} - A)| \geqslant |N[a] - (\widetilde{V} - A)|$, $|N[u] \cap (\widetilde{V} - A)| = \frac{n+1}{2} > \frac{n-5}{2} = |N[u] - (\widetilde{V} - A)|$ and for all $v \in S - \{a, u\}$, $\big|N[v] \cap S\big| \geqslant \big|N[v] \cap \big((\widetilde{V} - A) - \{u_3\}\big)\big| + 1 \geqslant \big|N[v] \cap (\widetilde{V} - A)\big| > \big|N[v] - (\widetilde{V} - A)\big| \geqslant \big|N[v] - S\big| - 1$. □

Theorem 3.10. *Let $G = (V, E)$ be a graph of even order n and $u \in V$ such that $deg(u) = n - 5$. Then there exists a defensive alliance S such that $|S| = \frac{n+2}{2}$ or $|S| = \frac{n}{2}$ and $u \in S$. Furthermore, $\frac{n-4}{2} \leqslant a_u(G) \leqslant \frac{n+2}{2}$.*

Proof. By Theorem 3.1, $\frac{n-4}{2} = \lfloor \frac{n-5}{2} \rfloor + 1 \leqslant a_u(G)$. Then it is sufficient to find a defensive alliance of cardinality $\frac{n+2}{2}$ or $\frac{n}{2}$ containing u. Suppose that $u_1, u_2, u_3, u_4 \in V$ such that are not adjacent to u

and $C := \{u_1, u_2, u_3, u_4\}$. Define

$$\mathcal{P} := \Big\{[E, \widetilde{V} - E] : E \subseteq \widetilde{V}, |E| = \frac{n}{2}, |E \cap C| = 2 \text{ or } 3\Big\},$$

where $\widetilde{V} := V - \{u\}$. Let $[A, \widetilde{V} - A] \in \mathcal{P}$ such that $\Big|[A, \widetilde{V} - A]\Big| = \min\Big\{\Big|[E, \widetilde{V} - E]\Big| : [E, \widetilde{V} - E] \in \mathcal{P}\Big\}$. We divide the proof into two steps.

1) $|A \cap C| = 2$.

It is sufficient to consider the case $u_1, u_2 \in A$. If A is a defensive alliance, $S = A \cup \{u\}$ is a defensive alliance of cardinality $\frac{n+2}{2}$ containig u. If not, there exists vertex $a \in A$ such that $\big|N[a] \cap A\big| < \big|N[a] - A\big|$. But this means that $[B, \widetilde{V} - B] \in \mathcal{P}$, where $B = (\widetilde{V} - A) \cup \{a\}$, and $\big|[B, \widetilde{V} - B]\big| < \big|[A, \widetilde{V} - A]\big|$ which is impossible.

2) $|A \cap C| = 3$.

It is sufficient to consider the case $u_1, u_2, u_3 \in A$. If A is a defensive alliance, define $S = A \cup \{u\}$. Clearly, S is a defensive alliance, $|S| = \frac{n+2}{2}$ and $u \in S$. If not, there exists vertex $a \in A$ such that $\big|N[a] \cap A\big| < \big|N[a] - A\big|$. If $\widetilde{V} - A$ is a defensive alliance, $S = (\widetilde{V} - A) \cup \{u, a\}$ is a defensive alliance of cardinality $\frac{n+2}{2}$ containing u. If not, there exists vertex $b \in \widetilde{V} - A$ such that $|N[b] \cap (\widetilde{V} - A)| < |N[b] - (\widetilde{V} - A)|$. If $a = u_1, u_2$ or u_3, put $B = (A - \{a\}) \cup \{b\}$ and if $b \neq u_4$, put $B = (A - \{a\}) \cup \{b\}$. We have $[B, \widetilde{V} - B] \in \mathcal{P}$ (or $[\widetilde{V} - B, B] \in \mathcal{P}$) and $\big|[B, \widetilde{V} - B]\big| < \big|[A, \widetilde{V} - A]\big|$ which is a contradiction. So $a \neq u_1$, $a \neq u_2$, $a \neq u_3$, $b = u_4$ and for all $v \in (\widetilde{V} - A) - \{u_4\}$, $|N[v] \cap (\widetilde{V} - A)| \geqslant |N[v] - (\widetilde{V} - A)|$. Note that if there exists $v \in (\widetilde{V} - A) - \{u_4\}$ such that $|N[v] \cap (\widetilde{V} - A)| = |N[v] - (\widetilde{V} - A)|$, then $\big|[B, \widetilde{V} - B]\big| < \big|[A, \widetilde{V} - A]\big|$, where $B = (A - \{a\}) \cup \{b\}$ and this is impossible. Now, define $S = \big((\widetilde{V} - A) - \{u_4\}\big) \cup \{u, a\}$. It is not hard to see that $|S| = \frac{n}{2}$, $u \in S$, $|N[a] \cap (\widetilde{V} - A)| \geqslant |N[a] - (\widetilde{V} - A)|$, $|N[u] \cap (\widetilde{V} - A)| = \frac{n}{2} > \frac{n}{2} - 4 = |N[u] - (\widetilde{V} - A)|$ and for all $v \in S - \{a, u\}$, $\big|N[v] \cap S\big| \geqslant \big|N[v] \cap \big((\widetilde{V} - A) - \{u_4\}\big)\big| + 1 \geqslant \big|N[v] \cap (\widetilde{V} - A)\big| > \big|N[v] - (\widetilde{V} - A)\big| \geqslant \big|N[v] - S\big| - 1$.

□

Theorem 3.11. *Let $G = (V, E)$ be a graph of odd order n and $u \in V$ such that $deg(u) = n - 5$. Then there exists a defensive alliance S such that $|S| = \frac{n+1}{2}$ or $|S| = \frac{n-1}{2}$ and $u \in S$. Furthermore, $a_u(G) = \frac{n+1}{2}$ or $a_u(G) = \frac{n-1}{2}$ or $a_u(G) = \frac{n-3}{2}$.*

Proof. Theorem 3.1 implies that $\frac{n-3}{2} = \big\lfloor \frac{n-5}{2} \big\rfloor + 1 \leqslant a_u(G)$. Then, it is sufficient to find a defensive alliance of cardinality $\frac{n+1}{2}$ or $\frac{n-1}{2}$ containing u. Suppose that $u_1, u_2, u_3, u_4 \in V$ such that are not adjacent to u and $C := \{u_1, u_2, u_3, u_4\}$. Define

$$\mathcal{P} := \left\{[E, \widetilde{V} - E] : E \subseteq \widetilde{V},\ \frac{n-1}{2} \leqslant |E| \leqslant \frac{n+1}{2},\ 1 \leqslant |E \cap C| \leqslant 3\right\},$$

where $\widetilde{V} := V - \{u\}$. Assume that $[A, \widetilde{V} - A] \in \mathcal{P}$ such that $\Big|[A, \widetilde{V} - A]\Big| = \min\Big\{\Big|[E, \widetilde{V} - E]\Big| : [E, \widetilde{V} - E] \in \mathcal{P}\Big\}$. We divide the proof into three steps.

i) $|A \cap C| = 1$.

It is sufficient to consider the case $u_1 \in A$. First, let $|A| = \frac{n-1}{2}$. If A is a strong defensive alliance, $S = A \cup \{u\}$ is a defensive alliance of cardinality $\frac{n+1}{2}$ containig u. If not, there exists $a \in A$ such that $\big|N[a] \cap A\big| \leqslant \big|N[a] - A\big|$. If $a \neq u_1$, defining $B = (\widetilde{V} - A) \cup \{a\}$ we have $[B, \widetilde{V} - B] \in \mathcal{P}$ and $\big|[B, \widetilde{V} - B]\big| < \big|[A, \widetilde{V} - A]\big|$ which is impossible, therefore $a = u_1$ and for all $v \in A - \{a\}$, $\big|N[v] \cap A\big| > \big|N[v] - A\big|$. Now, define $S = (A - \{u_1\}) \cup \{u\}$. It is not hard to see that $|S| = \frac{n-1}{2}$, $u \in S$, $|N[u] \cap S| = \frac{n-1}{2} > \frac{n+1}{2} - 4 = |N[u] - S|$ and for all $v \in S - \{u\}$, $\big|N[v] \cap S\big| \geqslant \big|N[v] \cap (A - \{u_1\})\big| + 1 \geqslant \big|N[v] \cap A\big| > \big|N[v] - A\big| \geqslant \big|N[v] - S\big| - 1$, so $\big|N[v] \cap S\big| \geqslant |N[v] - S\big|$. Now, let $|A| = \frac{n+1}{2}$. If A is a strong defensive alliance, $S = (A - \{u_1\}) \cup \{u\}$ is a defensive alliance of cardinality $\frac{n+1}{2}$ containig u. If not, there exists $a \in A$ such that $\big|N[a] \cap A\big| \leqslant \big|N[a] - A\big|$. If $a \neq u_1$, define $B = (\widetilde{V} - A) \cup \{a\}$. We have, $[B, \widetilde{V} - B] \in \mathcal{P}$ and $\big|[B, \widetilde{V} - B]\big| < \big|[A, \widetilde{V} - A]\big|$, therefore $a = u_1$ and for all $v \in A - \{a\}$, $\big|N[v] \cap A\big| > \big|N[v] - A\big|$. Now, define $S = (A - \{u_1\}) \cup \{u\}$. It is easy to see that $|S| = \frac{n+1}{2}$, $u \in S$, $|N[u] \cap S| = \frac{n+1}{2} > \frac{n-1}{2} - 4 = |N[u] - S|$ and for all $v \in S - \{u\}$, $\big|N[v] \cap S\big| \geqslant \big|N[v] \cap (A - \{u_1\})\big| + 1 \geqslant \big|N[v] \cap A\big| > \big|N[v] - A\big| \geqslant |N[v] - S\big| - 1$.

ii) $|A \cap C| = 2$.

It is sufficient to consider the case $u_1, u_2 \in A$. First, let $|A| = \frac{n-1}{2}$. If A is a defensive alliance, define $S = A \cup \{u\}$. Clearly, S is a defensive alliance, $|S| = \frac{n+1}{2}$ and $u \in S$. If not, there exists vertex $a \in A$ such that $\big|N[a] \cap A\big| < \big|N[a] - A\big|$. But then $[B, \widetilde{V} - B] \in \mathcal{P}$, where $B = (\widetilde{V} - A) \cup \{a\}$, and $\big|[B, \widetilde{V} - B]\big| < \big|[A, \widetilde{V} - A]\big|$. Now, let $|A| = \frac{n+1}{2}$. If A is a strong defensive alliance and $\big|N[u_2] \cap A\big| \geqslant \big|N[u_2] - A\big| + 2$, define $S = (A - \{u_1\}) \cup \{u\}$. It is easy to see that $|S| = \frac{n+1}{2}$, $u \in S$ and S is a defensive alliance. If A is a strong defensive alliance and $\big|N[u_2] \cap A\big| = \big|N[u_2] - A\big| + 1$, define $B = A - \{u_2\}$. Obviously, $[B, \widetilde{V} - B] \in \mathcal{P}$, $\big|[B, \widetilde{V} - B]\big| = \big|[A, \widetilde{V} - A]\big|$, $|B| = \frac{n-1}{2}$ and $|B \cap C| = 1$. Thus, by step (i), we are done. If not, there exists $a \in A$ such that $\big|N[a] \cap A\big| \leqslant \big|N[a] - A\big|$. But then $[B, \widetilde{V} - B] \in \mathcal{P}$, where $B = A - \{a\}$, and $\big|[B, \widetilde{V} - B]\big| < \big|[A, \widetilde{V} - A]\big|$ which is impossible.

iii) $|A \cap C| = 3$.

It is sufficient to consider the case $u_1, u_2, u_3 \in A$. First, let $|A| = \frac{n-1}{2}$. If $\widetilde{V} - A$ is a strong defensive alliance, $S = (\widetilde{V} - A) \cup \{u\}$ is a defensive alliance of cardinality $\frac{n+1}{2}$ containig u. If not, there exists $b \in \widetilde{V} - A$ such that $\big|N[b] \cap (\widetilde{V} - A)\big| \leqslant \big|N[b] - (\widetilde{V} - A)\big|$. If $b \neq u_4$, define $B = A \cup \{b\}$. We have $[B, \widetilde{V} - B] \in \mathcal{P}$ and $\big|[B, \widetilde{V} - B]\big| < \big|[A, \widetilde{V} - A]\big|$, therefore $b = u_4$ and for all $v \in (\widetilde{V} - A) - \{b\}$, $\big|N[v] \cap (\widetilde{V} - A)\big| > \big|N[v] - (\widetilde{V} - A)\big|$. Now, define $S = \big((\widetilde{V} - A) - \{u_4\}\big) \cup \{u\}$. We have $|S| = \frac{n-1}{2}$, $u \in S$, $|N[u] \cap S| = \frac{n-1}{2} > \frac{n+1}{2} - 4 = |N[u] - S|$ and for all $v \in S - \{u\}$, $\big|N[v] \cap S\big| \geqslant \big|N[v] \cap \big((\widetilde{V} - A) - \{u_4\}\big)\big| + 1 \geqslant \big|N[v] \cap (\widetilde{V} - A)\big| > \big|N[v] - (\widetilde{V} - A)\big| \geqslant |N[v] - S\big| - 1$, so $\big|N[v] \cap S\big| \geqslant |N[v] - S\big|$. Now, let $|A| = \frac{n+1}{2}$. If A is a strong defensive alliance and $\big|N[u_2] \cap A\big| \geqslant \big|N[u_2] - A\big| + 2$ and $\big|N[u_3] \cap A\big| \geqslant \big|N[u_3] - A\big| + 2$, define $S = (A - \{u_1\}) \cup \{u\}$. It is easy to see that $|S| = \frac{n+1}{2}$, $u \in S$ and S is a defensive alliance. If A is a strong defensive alliance and $\big|N[u_2]\cap A\big| = \big|N[u_2]-A\big|+1$, define $B = A-\{u_2\}$, and if $\big|N[u_3]\cap A\big| = \big|N[u_3]-A\big|+1$, define $B = A-\{u_3\}$. Obviously, $[B, \widetilde{V}-B] \in \mathcal{P}$, $\big|[B, \widetilde{V}-B]\big| = \big|[A, \widetilde{V}-A]\big|$, $|B| = \frac{n-1}{2}$ and $|B\cap C| = 2$. Thus, by step (ii), we are done. If not, there exists $a \in A$ such that $\big|N[a] \cap A\big| \leqslant \big|N[a] - A\big|$. But then $[B, \widetilde{V} - B] \in \mathcal{P}$, where $B = A - \{a\}$, and $\big|[B, \widetilde{V} - B]\big| < \big|[A, \widetilde{V} - A]\big|$ which is impossible.

□

To conclude this paper, we mention an upper bound for alliance number $a_u(G)$ which improves upper bound (3.1).

Theorem 3.12. *Let $G = (V, E)$ be a graph of order n and $u \in V$. Then*

$$a_u(G) \leqslant n - \left\lceil \frac{deg(u)}{2} \right\rceil. \tag{3.3}$$

In addition, if $deg(u) = n - 1$, then this bound is sharp.

Proof. Note that if $\deg(u) = n-1$, then, by Theorem 3.2, $a_u(G) = \left\lceil \frac{n}{2} \right\rceil = n - \left\lceil \frac{\deg(u)}{2} \right\rceil$. Thus, in this case upper bound (3.3) is sharp and the proof is complete. Now, consider the case that $\deg(u) \leqslant n-2$. Suppose that $u_1, u_2, \ldots, u_{\deg(u)} \in V$ such that are adjacent to u and $C := \big\{u, u_1, u_2, \ldots, u_{\deg(u)}\big\}$. Define

$$\mathcal{P} := \left\{ [E, \overline{E}] : E \subseteq V,\ \left\lfloor \frac{n}{2} \right\rfloor \leqslant |E| \leqslant n - \left\lceil \frac{\deg(u)}{2} \right\rceil,\ |E \cap C| = \left\lceil \frac{\deg(u) + 1}{2} \right\rceil, u \in E \right\}.$$

Let $[A, \overline{A}] \in \mathcal{P}$ such that $\Big|[A, \overline{A}]\Big| = \min\Big\{\Big|[E, \overline{E}]\Big| : [E, \overline{E}] \in \mathcal{P}\Big\}$. We divide the proof into three steps.

i) $|A| = n - \left\lceil \frac{\deg(u)}{2} \right\rceil$.

Note that in this case, $\overline{A} \subseteq C$ and $|\overline{A}| = \left\lceil \frac{\deg(u)}{2} \right\rceil$. If $\overline{A}$ is a defensive alliance, $\overline{A} \cup \{u\}$ is a defensive alliance of cardinality $\left\lceil \frac{\deg(u)}{2} \right\rceil + 1 \leqslant n - \left\lceil \frac{\deg(u)}{2} \right\rceil$ containing u. If not, then there exists $b \in \overline{A}$ such that $\big|N[b] \cap \overline{A}\big| < \big|N[b] - \overline{A}\big|$ (note that b is adjacent to u). If A is a defensive alliance, the proof is complete. If not, there exists $a \in A$ such that $\big|N[a] \cap A\big| < \big|N[a] - A\big|$ (note that $a \neq u$). If a is adjacent to u, define $B = (A - \{a\}) \cup \{b\}$ and if a is not adjacent to u, define $B = A - \{a\}$. We have $[B, \overline{B}] \in \mathcal{P}$ and $\big|[B, \overline{B}]\big| < \big|[A, \overline{A}]\big|$ which is impossible.

ii) $\left\lfloor \frac{n}{2} \right\rfloor < |A| < n - \left\lceil \frac{\deg(u)}{2} \right\rceil$.

Let $|A| = m$. If $\overline{A}$ is a defensive alliance, $\overline{A} \cup \{u\}$ is a defensive alliance of cardinality $m + 1 \leqslant n - \left\lceil \frac{\deg(u)}{2} \right\rceil$ containing u. If not, there exists $b \in \overline{A}$ such that $\big|N[b] \cap \overline{A}\big| < \big|N[b] - \overline{A}\big|$. If A is a defensive alliance, the proof is complete, so assume that A is not a defensive alliance. There exists $a \in A$ such that $\big|N[a] \cap A\big| < \big|N[a] - A\big|$ (note that $a \neq u$). If both a and b are adjacent to u or both a and b are not adjacent to u, define $B = (A - \{a\}) \cup \{b\}$, if a is not adjacent to u and b is adjacent to u, define $B = A - \{a\}$ and if a is adjacent to u and b is not adjacent to u, define $B = A \cup \{b\}$. We have $[B, \overline{B}] \in \mathcal{P}$ and $\big|[B, \overline{B}]\big| < \big|[A, \overline{A}]\big|$ which is a contradiction.

iii) $|A| = \left\lfloor \dfrac{n}{2} \right\rfloor$.

If $\overline{A}$ is a defensive alliance, $\overline{A} \cup \{u\}$ is a defensive alliance of cardinality $\left\lfloor \dfrac{n}{2} \right\rfloor + 1 \leqslant n - \lceil \frac{\deg(u)}{2} \rceil$ containing u. If not, then there exists $b \in \overline{A}$ such that $\big|N[b] \cap \overline{A}\big| < \big|N[b] - \overline{A}\big|$. If A is a defensive alliance, the proof is complete. If not, there exists $a \in A$ such that $\big|N[a] \cap A\big| < \big|N[a] - A\big|$ $(a \neq u)$. If both a and b are adjacent to u or both a and b are not adjacent to u, define $B = (A - \{a\}) \cup \{b\}$, if a is adjacent to u and b is not adjacent to u, define $B = A \cup \{b\}$ and if a is not adjacent to u and b is adjacent to u, in the case that $\deg(u)$ is an odd number, define $B = (\overline{A} - \{b\}) \cup \{u\}$ and in the case that $\deg(u)$ is an even number, define $B = \overline{A} \cup \{a, u\}$. It is easy to see that $[B, \overline{B}] \in \mathcal{P}$ and $\big|[B, \overline{B}]\big| < \big|[A, \overline{A}]\big|$ which is a contradiction.

□

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