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ON THE ZERO FORCING NUMBER OF GENERALIZED SIERPIŃSKI GRAPHS

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Abstract. In this article we study the Zero forcing number of Generalized Sierpiński graphs $S(G, t)$. More precisely, we obtain a general lower bound on the Zero forcing number of $S(G, t)$ and we show that this bound is tight. In particular, we consider the cases in which the base graph G is a star, path, a cycle or a complete graph.

1. Introduction

Let $G = (V, E)$ be a non-empty graph of order n, vertex set V and edge set E. For a given graph G and $S \subset V(G)$, we denote by $\langle S \rangle$ the subgraph induced by S. For a vertex $v \in V(G)$, the set $N_G(v) = \{u : uv \in E(G)\}$ is the open neighborhood of v, and the degree of a vertex $v \in V(G)$ is $deg_G(v) = |N_G(v)|$.

The letters of a word u of length t are denoted by $u_1u_2 \dots u_t$ and the concatenation of two words u and v is denoted by uv. Let V^t be the set of words of size t on alphabet V. In [12], Klavžar and Milutinović introduced the graph $S(K_n, t)$ whose vertex set is V^t, where u is adjacent to v if and only if there exists $1 \leq i \leq t$ such that:

$$\text{(i) } u_j = v_j, \text{ if } j < i; \text{ (ii) } u_i \neq v_i; \text{ (iii) } u_j = v_i \text{ and } v_j = u_i \text{ if } j > i.$$

When $n = 3$, those graphs are isomorphic to the Tower of Hanoi graphs. In [13], those graphs have been called Sierpiński graphs. This construction was generalized in [10] for any graph $G = (V, E)$, by defining the *generalized Sierpiński graph*, $S(G, t)$, as the graph with vertex set V^t and edge set defined

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as follows. The vertices u and v are adjacent in $S(G,t)$ if and only if there exists $i \in \{1,\dots,t\}$ such that:

$$(i)\, u_j = v_j, \text{ if } j < i; \qquad (ii)\, u_i \neq v_i \text{ and } \{u_i, v_i\} \in E; \qquad (iii)\, u_j = v_i \text{ and } v_j = u_i \text{ if } j > i.$$

It shows that if $\{u,v\} \in E(S(G,t))$, then there is $\{a,b\} \in E(G)$ and a word w such that $u = wabb\cdots b$ and $v = wba\cdots a$. In general, $S(G,t)$ can be constructed recursively from G with the following process: $S(G,1) = G$ and, for $t \geq 2$, we have n copies of $S(G,t-1)$ by adding the letter a at the beginning of each label of the vertices belonging to the copy of $S(G,t-1)$ corresponding to a. Then there is an edge between vertices $ab\cdots b$ and vertex $ba\cdots a$, if $\{a,b\}$ is an edge of G. (See Figures 1 and 2). Moreover, $deg_{S(G,t)}(ba\cdots a) = d_G(a)+1$ and $deg_{S(G,t)}(ab\cdots b) = d_G(b)+1$. Vertices of the form $a\cdots a$ are called *extreme vertices* which has the same degree of a. Also, for $t \geq 2$, $S(G,t)$ has exactly n extreme vertices. In addition, for any $w \in V^{t-i}$, $t \geq 2$ and $i \leq t-1$, the subgraph $\langle V_w \rangle$ of $S(G,t)$ induced by $V_w = \{wx : x \in V^i\}$ is isomorphic to $S(G,i)$.

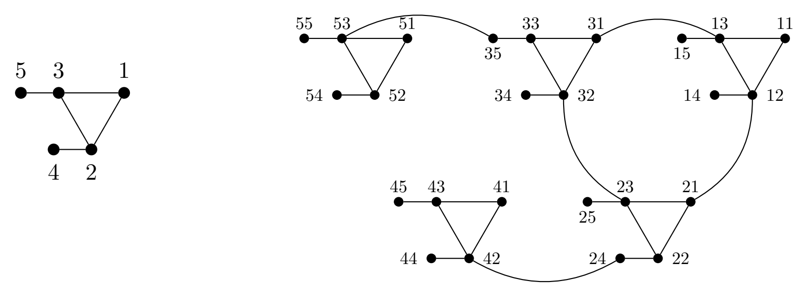

FIGURE 1. A graph G and the Sierpiński graph $S(G,2)$.

AIM group in [1] introduced the notion of zero forcing number, of a simple graph to bound the minimum rank of graphs. Let each vertex of a graph G be given one of two colors "black" and "white". Let Z denote the (initial) set of black vertices of G. If the white vertex u_2 is the only white neighbor of a black vertex u_1, then u_1 changes the color of u_2 to black (color -change rule) and we say "u_1 forces u_2" which we denote by $u_1 \to u_2$. A sequence, $u_1 \to u_2 \to \cdots \to u_i \to u_{i+1} \to \cdots \to u_t$, obtained through repetitious applications of the color-change rule is called a forcing chain. The set Z is said to be a zero forcing set of G if all vertices of G will be turned black after nitely many applications of the color-change rule. The zero forcing number, $Z(G)$, of G is the minimum cardinality among all zero forcing sets. In [1] it is shown that for any graph G, $M(G) \leq Z(G)$. A path covering of a graph is a family of induced disjoint paths in the graph that cover (or include) all vertices of the graph. The minimum number of such paths that cover the vertices of a graph G is the path cover number of G and is denoted by $P(G)$. Since the forcing chains form a set of covering paths we have $P(G) \leq Z(G)$.

These motivated us to consider the zero forcing number of Generalized Sierpiński graphs. For this aim

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we obtain the lower bound for $Z(S(G,t))$ for any graph G and we discuss the tightness of this bound. Also, the zero forcing numbers of Generalized Sierpiński graph of path, cycle, star and complete graph are determined.

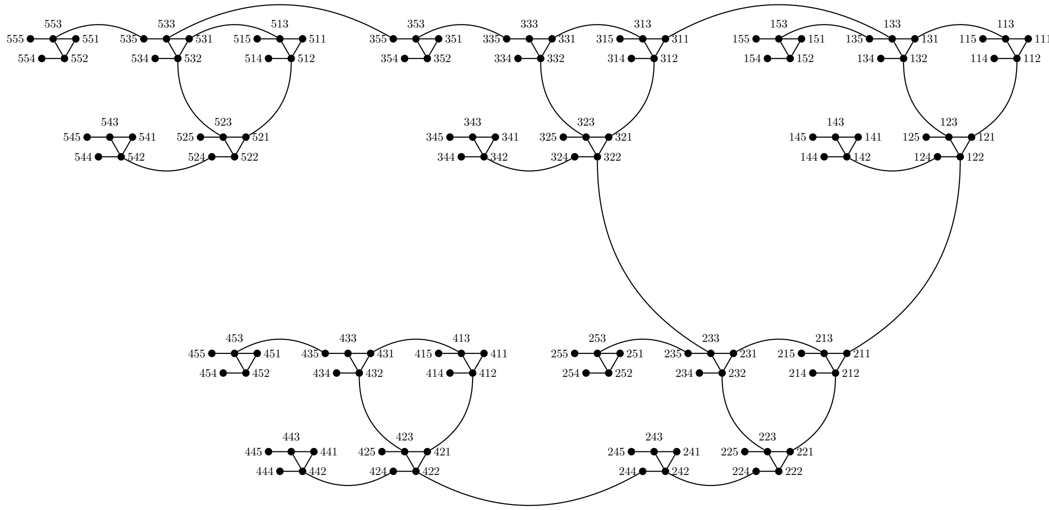

Figure 2. The Sierpiński graph $S(G,3)$ for the graph G of Figure 1.

2. Preliminaries

First we give some facts that we need in later sections.

Theorem 2.1. [1][5][17] *Let G be a connected graph of order $n \geq 2$. Then*

i. $Z(G) = 1$ *if and only if* $G \simeq P_n$.

ii. $Z(G) = n - 1$ *if and only if* $G \simeq K_n$.

iii. *If G is a tree, then* $Z(G) = P(G)$.

iv. *For any integer* $n \geq 2$, $Z(K_{1,n}) = n - 1$.

Theorem 2.2. [4] *Let G be any graph. Then*

i. *For* $v \in V(G)$, $Z(G) - 1 \leq Z(G \setminus \{v\}) \leq Z(G) + 1$.

ii. *For* $e \in E(G)$, $Z(G) - 1 \leq Z(G \setminus \{e\}) \leq Z(G) + 1$.

Theorem 2.3. [16] *For any tree T and any positive integer t, $S(T,t)$ is a tree.*

3. Main Results

In this section we obtain a lower bound for $Z(S(G,t))$ and then we show that this bound is tight.

Theorem 3.1. *Let G be a graph of order n and size m. Then for any integer $t \geq 2$,*

$$Z(S(G,t)) \geq n^{t-1}Z(G) - m\frac{n^{t-1}-1}{n-1}.$$

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Proof. Let $e_{ij}^t = \{ij\dots j, ji\dots i\} \in E(S(G,t))$ where $\{i,j\} \in E(G)$. Notice that in this notation $e_{ij}^t = e_{ji}^t$. Thus

$$S(G,t) \setminus \{e_{ij}^t : \{i,j\} \in E(G)\} \simeq nS(G,t-1).$$

Now, we use Theorem 2.2 to achieve the result. Let $\{i,j\}$ be an edge in G. Then $Z(S(G,t)) \geq Z\left(S(G,t) \setminus \{e_{ij}^t\}\right) - 1$. By using this structure for all $e_{ij}^t \in E(S(G,t))$, we have

$$Z(S(G,t)) \geq nZ(S(G,t-1)) - |\{e_{ij}^t : \{i,j\} \in E(G)\}| = nZ(S(G,t-1)) - m.$$

Again

$$Z(S(G,t-1)) \geq nZ(S(G,t-1)) - |\{e_{ij}^{t-1} : \{i,j\} \in E(G)\}| = nZ(S(G,t-2)) - m.$$

Therefore, $Z(S(G,t)) \geq n^2 Z(G,t-2) - nm - m$. With similar argument we have

$$Z(S(G,t)) \geq n^{t-1}Z(G) - m\frac{n^{t-1}-1}{n-1}.$$

□

In the next Theorem, the zero forcing number of Generalized Sierpiński graph of K_n is obtained and we will see that the lower bound in Theorem 3.1 is tight.

Theorem 3.2. *For any positive integers n and t,*

$$Z(S(K_n,t)) = \frac{n^t - 2n^{t-1} + n}{2}$$

Proof. To obtain the upper bound we define the following sets.

For $t = 2$, $\quad Z_2 = \{ij : 1 \leq i \leq n-1 \text{ and } i \leq j \leq n-1\}$,

For $t \geq 3$, $\quad Z_t = \{iz : 1 \leq i \leq n \text{ and } z \in Z_{t-1}\} \setminus \{ij\cdots j : 2 \leq i \leq n \text{ and } 1 \leq j < i\}$.

By induction on $t \geq 2$ we show that Z_t is a forcing set of $S(K_n,t)$. For $t = 2$, use the following instructions from $i = 1$ to $i = n-1$ to make all vertices black.

$$ii \to in$$
$$ij \to ji \text{ for } i+1 \leq j \leq n,$$

and at the end, $n1 \to nn$. Hence, Z_2 is a forcing set of $S(K_n,2)$. Now suppose that for any $t = k$, Z_t is a forcing set of $S(K_n,t)$ and we show that Z_{k+1} is a forcing set of $S(K_n,k+1)$. Since Z_k is a forcing set, all vertices in V_1 will be forced by Z_{k+1}. Use the following structures from $i = 1$ to $i = n-1$.

$$in\cdots n \to ni\cdots i$$
$$ij\cdots j \to ji \text{ for } i+1 \leq j \leq n,$$

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and at the end, $n\cdots n1 \to n\cdots n$. So Z_{k+1} is a forcing set of $S(K_n, k+1)$. Therefore, for any $t \geq 2$

$$\begin{aligned}
Z(S(K_n,t)) \leq |Z_t| &= n|Z_{t-1}| - \frac{n(n-1)}{2} \\
&= n^2|Z_{t-2}| - \frac{n^2(n-1)}{2} - \frac{n(n-1)}{2} \\
&\vdots \\
&= n^{t-2}|Z_2| - \frac{n^{t-2}(n-1)}{2} - \cdots - \frac{n^2(n-1)}{2} - \frac{n(n-1)}{2} \\
&= \frac{n(n-1)}{2}\left(n^{t-2} - \frac{n^{t-2}-1}{n-1}\right) \\
&= \frac{n^t - 2n^{t-1} + n}{2}.
\end{aligned}$$

Theorems 3.1 and 2.1 complete the proof. □

Zero Forcing Number On $S(C_n, t)$

Here we will give the similar result for $Z(S(C_n, t))$.

Theorem 3.3. *For any integers $n \geq 4$ and $t \geq 2$, $Z(S(C_n,t)) = \frac{n^t - 2n^{t-2} + n}{n-1}$.*

Proof. Let $t = 2$. Let $Z_2 = \{ii : 1 \leq i \leq n\} \cup \{12, 13\}$. Follow this structure to force all vertices. Notice that the addition is taken modulo n.

$$11 \to 1n \to n1,\ 13 \to 14 \to \cdots \to 1(n-1)$$

for $1 \leq i \leq n-1$

$$i(i+1) \to (i+1)i \to (i+1)(i+n-1) \to (i+1)(i+n-2) \to \cdots \to (i+1)(i+2)$$

Hence, $Z(S(C_n, 2)) \leq |Z_2| = n+2$. Now we show that $Z(S(C_n, 2)) = n+2$. Let Z be a forcing set of minimum cardinality. Since for each $1 \leq i \leq n$, V_i has two vertices of degree three , $|V_i \cap Z| \geq 1$. We can assume that the starting forcing chain starts in V_1.

Case I. $\{12, 1n\} \cap Z \neq \emptyset$. Let $12 \in Z$. Since $deg(12) = 3$, $|V_1 \cap Z| = 3$ or $|V_1 \cap Z| = 2$ and $21 \in Z$. If $|V_1 \cap Z| = 3$, then $|Z| \geq n+2$. Otherwise, since $deg(21) = 3$, $\{22, 2n\} \cap Z \neq \emptyset$. Thus $|V_2 \cap Z| \geq 2$.

Case II. $\{12, 1n\} \cap Z = \emptyset$. If $11 \in Z$, then $|V_1 \cap Z| = 3$. Otherwise, $|V_1 \cap Z| = 2$ and $\{21, n1\} \cap Z \neq \emptyset$. Let $21 \in Z$. Since $deg(21) = 3$, $22 \in Z$ or $2n \in Z$. So $|V_2 \cap Z| \geq 2$.

In both cases, $|Z| \geq n+2$, so that $Z(S(C_n, 2)) = n+2$. Therefore, $Z(S(C_n, 2)) = n+2$. For $t \geq 3$, let

$$Z_t = \{iz : z \in Z_{t-1} \text{ and } 1 \leq i \leq n\} \setminus (\{i(i-1)\cdots(i-1) : 2 \leq i \leq n\} \cup \{n1\cdots 1\}.$$

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By induction, we see that Z_t is a forcing set of $S(C_n,t)$. First suppose that $t=3$. Since $V_1 \cap Z_3 = \{1z : z \in Z_2\}$, so all vertices in V_1 will be blacked by Z_3. Now for omitted vertices of $\{iz : z \in Z_2, 1 \le i \le n\}$, use the following forcing chain:

$$122 \to 211$$
$$233 \to 322$$
$$\vdots$$
$$(n-1)nn \to n(n-1)(n-1)$$
$$1nn \to n11$$

Hence, Z_3 is a forcing set of $S(C_n,3)$. Suppose that for any $t \ge k$, Z_t is a forcing set of $S(C_n,t)$. Now we show that this is true for $t=k+1$. Since, Z_k is a forcing set, all vertices in V_1 will be black. Also $1n\cdots n \to n1\cdots 1$. For $2 \le i \le n-1$, $(i-1)i\cdots i \to i(i-1)\cdots(i-1)$ and since, $\langle V_i\rangle \simeq S(C_n,k)$, all vertices in V_i will be forced by $Z_{k+1} \cap V_i$.
Hence, Z_t is a forcing set for $S(C_n,t)$ for each $t \ge 3$ and so

$$Z(S(C_n,t)) \le |Z_t| = n|Z_{t-1}| - n = \cdots = n^{t-2}|Z_2| - \sum_{i=1}^{t-2} n^i = \frac{n^t - 2n^{t-2} + n}{n-1}.$$

To obtain the lower bound, we use the similar argument as in proof of Theorem 3.1.

$$\begin{aligned} Z(S(C_n,t)) &\ge n^{t-2}|Z(S(C_n,2))| - \frac{n(n^{t-2}-1)}{n-1} \\ &= n^{t-2}(n+2) - \frac{n(n^{t-2}-1)}{n-1} \\ &= \frac{n^t - 2n^{t-2} + n}{n-1}. \end{aligned}$$

This completes the proof. □

Zero Forcing Number On $S(K_{1,n},t)$

Let $V(K_{1,n}) = \{0,1,\ldots,n\}$ where $deg(0) = n$. Let $S(K_{1,n},0) = K_1$. For any positive integer t, we use the notation

$$S(K_{1,n},t) \sim S(K_{1,n},t-1)$$

when vertex $00\cdots00$ of $S(K_{1,n},t)$ is adjacent to an extreme of V_i in $S(K_{1,n},t-1)$ for some $1 \le i \le n$. Now let $G_2 : S(K_{1,n},1) \sim K_1$ and $G_t : S(K_{1,n},t-1) \sim G_{t-1}$. With this notations in mind we will prove the following results.

Lemma 3.4. *For any positive integer n and $t \ge 2$, $S(K_{1,n},t) \setminus \{00\cdots00\} \simeq nG_t$ and also for $t \ge 3$ we have $G_t \setminus \{i0\cdots0\} \simeq (n+1)G_{t-1}$ for some $1 \le i \le n$.*

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Proof. We use induction on t to reach the result. For $t = 2$, we have

$$S(K_{1,n}, 2) \setminus \{00\} \simeq \bigcup_{i=1}^{n} (V_i \sim K_1)$$

where $V_i \simeq S(K_{1,n}, 1)$ for $1 \leq i \leq n$. Thus $V_i \sim K_1 \simeq S(K_{1,n}, 1) \sim K_1$ for any $1 \leq i \leq n$ and so $S(K_{1,n}, 2) \setminus \{00\} \simeq nG_2$. Now, suppose that for any $k \geq t$,

$$S(K_{1,n}, k) \setminus \{0 \cdots 0\} \simeq nG_k$$

and we will prove the result for $t = k+1$. For any $1 \leq i \leq n$ there is the following path in $S(K_{1,n}, k+1)$:

$$0 \cdots 0 - 0 \cdots 0i - 0 \cdots 0i0 - 0 \cdots 0ii - 0 \cdots 0i00 - \cdots - 0i \cdots i0 - 0i \cdots i - i0 \cdots 0.$$

Hence,

$$S(K_{1,n}, k+1) \setminus \{0 \cdots 0\} \simeq \bigcup_{i=1}^{n} \left(V_i \sim V_{0i} \sim V_{00i} \sim \cdots \sim V_{0 \cdots 0i}\right).$$

As we know $V_i \simeq S(K_{1,n}, k)$, $V_{0i} \simeq S(K_{1,n}, k-1)$ and so on. Therefore,

$$\begin{aligned} S(K_{1,n}, k+1) \setminus \{0 \cdots 0\} &\simeq n(S(K_{1,n}, k) \sim S(K_{1,n}, k-1) \sim \cdots S(K_{1,n}, 1) \sim K_1) \\ &\simeq n(S(K_{1,n}, k) \sim G_k) \\ &\simeq nG_{k+1}. \end{aligned}$$

Since $N_{G_t}(\{i0 \cdots 0\}) = \{i0 \cdots 0j : 1 \leq j \leq n\} \cup \{0i \cdots i\}$ for some $1 \leq i \leq n$,

$$G_t \setminus \{i0 \cdots 0\} \simeq nG_{t-1} \cup G_{t-1} \simeq (n+1)G_{t-1}.$$

This completes the proof. □

Theorem 3.5. *For any positive integers n and t,*

$$Z(S(K_{1,n}, t)) = (n-1)(n+1)^{t-1}.$$

Proof. First, we use Lemma 3.4 and Theorem 2.2 to obtain the lower bound.

$$\begin{aligned} Z(S(K_{1,n}, t)) &\geq Z(S(K_{1,n}, t) \setminus \{0 \cdots 0\}) - 1 \\ &= nZ(G_t) - 1 \\ &\geq n(Z(G_t \setminus \{i0 \cdots 0\}) - 1) - 1 \\ &= n(n+1)Z(G_{t-1}) - (n+1) \\ &\geq n(n+1)\left(Z(G_{t-1} \setminus \{0i0 \cdots 0\}) - 1\right) - (n+1) \\ &= n(n+1)^2 Z(G_{t-2}) - (n+1)^2 \\ &\vdots \\ &\geq n(n+1)^{t-2} Z(G_2) - (n+1)^{t-2} \end{aligned}$$

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But $Z(G_2) = Z(S(K_{1,n}, 1) \sim K_1) = Z(K_{1,n} \sim K_1) = Z(K_{1,n+1}) = n$. Hence,

$$Z(S(K_{1,n}, t)) \geq n^2(n+1)^{t-2} - (n+1)^{t-2} = (n+1)^{t-1}(n-1).$$

To obtain he upper bound we define the following sets. For $t = 2$, let

$$Z_2 = \{ij : 1 \leq i, j \leq n\} \setminus \{nn\},$$

for $t = 3$, let

$$Z_3 = \{iz : 0 \leq i \leq n \text{ and } z \in Z_2\} \cup \{inn : 1 \leq i \leq n\} \setminus \{0ii, 0n(n-1) : 1 \leq i \leq n\}$$

and for $t \geq 4$, let

$$Z_t = \{iz : 0 \leq i \leq n \text{ and } z \in Z_{t-1}\} \cup \{i0 \cdots 0n(n-1) : 1 \leq i \leq n\} \setminus \{0i \cdots i : 1 \leq i \leq n\}.$$

By induction on $t \geq 2$ we show that Z_t is a forcing set of $S(K_{1,n}, t)$. Let $t = 2$. For any $1 \leq i, j \leq n-1$, $deg(ij) = 1$ and $ij \in Z_2$. So ij forces $i0$. Since $N(i0) \setminus Z_2 = \{0i\}$, $i0$ forces $0i$ for $1 \leq i \leq n-1$. With following the path

$$i0 \to 00 \to 0n \to n0 \to nn$$

all vertices will be black. Hence, Z_2 is a forcing set of $Z(S(K_{1,n}, 2))$. Let $t = 3$. All vertices in $\{iz : 0 \leq i \leq n \text{ and } z \in Z_2\} \cup \{inn : 1 \leq i \leq n\}$ force $ij0$ for $1 \leq i, j \leq n$ and $ij0$ forces $i0j$ and as following it forces $i00$. Now, $\{0ii : 1 \leq i \leq n\}$ will be forced by $\{i00 : 1 \leq i \leq n\}$. Since $\{0ii : 1 \leq i \leq n\} \cup \{0ij : 1 \leq i \leq n-1 \text{ and } 1 \leq j \leq n\}$ are black, $\{0i0 : 1 \leq i \leq n-1\}$ and then $\{00i : 1 \leq i \leq n-1\}$ will be black. Now, by following the path

$$00i \to 000 \to 00n \to 0n0 \to 0n(n-1)$$

all verices will get black. We suppose that for $t = k-1$, Z_{k-1} is a forcing set and we will prove it for Z_k. Since Z_{k-1} is a forcing set, $\{iz : 0 \leq i \leq n \text{ and } z \in Z_{k-1}\} \cup \{i0 \cdots 0n(n-1) : 1 \leq i \leq n\}$ force all vertices in V_i for $1 \leq i \leq n$. By similar argument as in $t = 3$, we see that the set $\{0i \cdots i : 1 \leq i \leq n\} \cup \{0 \cdots 0n(n-1)\}$ will be forced and so Z_k is a forcing set. Hence,

$$\begin{aligned} Z(S(K_{1,n}), t) &\leq |Z_t| = (n+1)|Z_{t-1}| = (n+1)^2|Z_{t-2}| = \cdots = (n+1)^{t-2}|Z_2| \\ &= (n+1)^{t-2}(n^2-1) = (n+1)^{t-1}(n-1). \end{aligned}$$

This completes the proof. □

Zero Forcing Number On $S(P_n, t)$

Let $V = \{1, 2, \ldots, n\}$ be the vertex set of P_n, and $\langle V_{wu} \rangle$ be a copy of P_n in $S(P_n, t)$ for $w \in V^{t-2}$ and $u \in V$. Also we say $\langle V_{wu} \rangle$ and $\langle V_{w'v} \rangle$ are two consecutive paths when $\{x, y\}$ is an edge in $S(P_n, t)$ for $x \in V_{wu}$ and $y \in V_{w'v}$ where $w, w' \in v^{t-2}$ and $u, v \in V$. Also we use $V_{wu} \sim V_{w'v}$ for induced subgraph on $V_{wu} \cup V_{w'v}$. With these notations in mind we will prove the following results.

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Lemma 3.6. *Let $t \geq 2$, $w, w' \in v^{t-2}$ and $u, v \in V$. If $\langle V_{wu} \rangle$ and $\langle V_{w'v} \rangle$ are two consecutive paths in $S(P_n, t)$, then the path cover number of $V_{wu} \sim V_{w'v}$ is two.*

Proof. Since $\langle V_{wu} \rangle$ and $\langle V_{w'v} \rangle$ are two consecutive paths, there are $x \in V_{wu}$ and $y \in V_{w'v}$ such that $\{x, y\} \in E(S(P_n, t))$. Hence, $\{u, v\} \in E(P_n)$ and so $u = v + 1$ or $v = u + 1$. Also $deg(x) = 3$ or $deg(y) = 3$. Thus $V_{wu} \sim V_{w'v}$ is not a path and by Theorem 2.1, $P(V_{wu} \sim V_{w'v}) \geq 2$. On the other hand, $P(V_{wu} \sim V_{w'v}) \leq P(\langle V_{wu} \rangle) + P(\langle V_{w'v} \rangle) = 2$. This completes the proof. □

Theorem 3.7. *For any positive integers n and t, $Z(S(P_n, t)) = n^{t-1}$.*

Proof. As we know there are n^{t-1} copies of P_n in $S(P_n, t)$ and by Lemma 3.6, the path cover number of each pair of consecutive paths is two. Hence, $P(S(P_n, t)) = n^{t-1}$ and by Theorems 2.1 and 2.3 we have $Z(S(P_n, t)) = n^{t-1}$. □

Question 1. *As we see the zero forcing number of $S(K_n, t)$ is equal to lower bound in Theorem 3.1. For which other family of graphs the zero forcing number is exactly the lower bound given in this work?*

Question 2. *Let G be a universal graph of order n with exactly one vertex of degree $n - 1$. Which is the relation between $Z(S(G, t))$ and $Z(G)$?*

A tree is called Starlike if it has exactly one vertex of degree more than two. It is denoted by $S(\ell_1, \ell_2, \dots \ell_r)$ such that $S(\ell_1, \ell_2, \dots \ell_r) \setminus \{v\} = P_{\ell_1} \cup P_{\ell_2} \cup \cdots \cup P_{\ell_r}$ where v is the vertex of degree more than two. One can see that $Z(S(\ell_1, \ell_2, \dots \ell_r)) = r - 1$.

Question 3. *What is the zero forcing number of Generalized Sierpiński graph of Starlike?*

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