VISUAL CRYPTOGRAPHY SCHEME ON GRAPHS WITH $m^*(G) = 4$

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Abstract. Let $G = (V, E)$ be a connected graph and $\Gamma(G)$ be the strong access structure where obtained from graph $G$. A visual cryptography scheme (VCS) for a set $P$ of participants is a method to encode a secret image such that any pixel of this image change to $m$ subpixels and only qualified sets can recover the secret image by stacking their shares. The value of $m$ is called the pixel expansion and the minimum value of the pixel expansion of a VCS for $\Gamma(G)$ is denoted by $m^*(G)$. In this paper we obtain a characterization of all connected graphs $G$ with $m^*(G) = 4$ and $\omega(G) = 5$ which $\omega(G)$ is the clique number of graph $G$.

1. Introduction

A secret sharing scheme is a method to share a secret among a set of participants such that only qualified subsets can reconstruct the secret from their shares, in addition non-qualified subsets can not obtain any information about the secret.

Visual cryptography scheme, (VCS), is a kind of secret sharing scheme, was introduced by Naor and Shamir [9]. They investigate the case of $((k, n) - VCS)$ where $2 \leq k \leq n$, in which the secret image is visible if and only if $k$ or more participants stack their shares, whereas any set of less than $k$ participants have no information on the secret image. In a VCS, decoder is human visual system and participants in a qualified set can see secret image without knowledge of cryptographic. Ateniese et al. [3, 4] extended this scheme to general access structures. In this model, $P$ is set of participants and $\Gamma = (Q, F)$ is access structure such that $Q \subseteq 2^P$ is the collection of qualified sets and $F \subseteq 2^P$ is the
collection of forbidden sets. We assume that image secret is collection of black and white pixels. Now in a VCS any pixel of this image is replaced by $m$ subpixels and give to each shares. The number of $m$ is called the pixel expansion and for a given general access structure $\Gamma$, the minimum value of $m$ is denoted by $m^*(\Gamma)$, and is called the optimal pixel expansion.

If the vertex set $V$ in a graph $G = (V, E)$ be the set of participants and any element of $\Gamma$ be the subset of $V$ which contains at least one edge, then this access structure is denoted by $\Gamma(G)$ and the optimal expansion is denoted by $m^*(G)$. Atenies et al. in [3, 4] studied the construction of VCSs that obtained from graphs and proved that $m^*(\Gamma) = 2$ if and only if $\Gamma = \Gamma(G)$ where $G$ is a complete bipartite graph. They have also proved that $m^*(K_n)$ is the smallest $m$ which $n \leq \left(\frac{m}{2}\right)$. So $m^*(K_2) = 2$ and $m^*(K_3) = 3$. In addition they proved that $m^*(H) \leq m^*(G)$ where $H$ is induced subgraph $G$. Since $m^*(K_6) = 4$ and $m^*(K_7) = 5$ thus if $m^*(G) = 4$ and $H$ be the induced subgraph $G$ then the biggest induced complete subgraph $G$ is $K_6$.

Arumugam et al. in [1, 2] obtained a characterization of all connected graphs $G$ for which $m^*(G) = 2$ and 3. In this paper, we study the graphs with $m^*(G) = 4$. We give a characterization of all connected graphs $G$ which $m^*(G) = 4$ and $\omega(G) = 5$ that $\omega(G)$ is the clique number of graph $G$.

2. Preliminaries

Let $P = \{1, 2, \ldots, n\}$ be a set of participants and let $2^P$ denote the set of of all subsets of $P$. If $Q \subseteq 2^P$ and $F \subseteq 2^P$ such that $Q \cap F = \emptyset$, then the pair $\Gamma = (Q, F)$ is called an access structure on $P$. We refer elements of $Q$ as qualified sets and to elements of $F$ as forbidden sets. We say $\Gamma$ is strong access structure whenever $Q$ is monotone increasing and $F$ is monotone decreasing and $Q \cup F = 2^P$. Throughout this paper we consider only strong access structures. Define $\Gamma_0$ to consist of all the minimal qualify sets : $\Gamma_0 = \{A \in Q : A' \notin Q \text{ for all } A' \subseteq A\}$.

Let $S$ be an $n \times m$ boolean matrix. If $X \subseteq P = \{1, 2, \ldots, n\}$ then $S[X]$ denotes the $|X| \times m$ matrix obtained from $S$ by considering its restriction to rows correpsonding to the elements in $X$, further $S_X$ denotes the vector obtained by applying the boolean OR operation to the rows of $S[X]$ and $w(S_X)$ is Hamming weight of $S_X$.

**Definition 2.1.** [3] Let $\Gamma = (Q, F)$ be a strong access access structure on a set of $n$ participants. Two $n \times m$ boolean matrices $S^0$ and $S^1$ construct a VCS if there exist a positive real number $\alpha$ and the set $\{t_X | X \in Q\}$ satisfying the following conditions:

1. Any qualified set $X = \{i_1, i_2, \ldots, i_q\} \in Q$ can recover the shared image by stacking their transparencies. Formally $w(S_X^0) \leq t_X - \alpha m$, whereas $w(S_X^1) \geq t_X$.
2. Any forbidden set $X = \{i_1, i_2, \ldots, i_q\} \in F$ has no information on the shared image. Formally the two $q \times m$ matrices $S^0[X]$ and $S^1[X]$ are equal up to a column permutation.

The first property is attributed to the contrast of the image and the second property is related to security. We assume that the message consist of a collection of black and white pixels. Let $\pi$ be a random permutation of $\{1, 2, \ldots, m\}$. Now a VCS is used to encrypt an image as follows. If a pixel
in the secret image is white (resp. black), then $\pi$ is applied to the columns of $S^0$ (resp. $S^1$) and row $i$ of the permuted matrix is the share of $i$th participant. Therefore each share is a collection of $m$ black and white subpixels. The value of $m$ is called the pixel expansion and the value of $\alpha$ is called relative contrast that measure clarity of reconstructed image.

One problem in a VCS is to minimize the pixel expansion and maximize the relative contrast. Several results on these two concepts can be found in [10, 11]. The minimum value of the pixel expansion $m$ of a VCS for $\Gamma = (Q, F)$ is denoted by $m^*(\Gamma)$.

**Definition 2.2.** Let $\Gamma = (Q, F)$ be an access structure on a set $P$ of participants. Then $\Gamma' = (Q', F')$ is the induced access structure on $P' \subseteq P$ that $Q' = Q \cap 2^{P'}$ and $F' = F \cap 2^{P'}$.

Let $G = (V, E)$ be a graph, then we can define a VCS on $G$ such that a subset $X$ of $V$ is qualified if and only if the induced subgraph $G[X]$ contains at least one edge of $G$. The access structure based on graph $G$ is denoted by $\Gamma(G)$ and $m^*(G)$ is the minimum value of pixel expansion $m$ a VCS that $\Gamma(G)$ is the access structures.

**Theorem 2.3.** [3] Let $\Gamma = (Q, F)$ be an access structure on a set $P$ of participants and let $\Gamma' = (Q|P'|, F|P'|)$ be the induced access structure on the subset of participants $P'$. Then $m^*(\Gamma') \leq m(\Gamma)$.

**Remark 2.4.** If $H = (V', E')$ be an induced subgraph of $G = (V, E)$, then $\Gamma(H)$ is an induced access structure of $\Gamma(G)$ and by Theoram 2.3, $m^*(H) \leq m^*(G)$.

Ateniese et al. [3, 4] studied the construction of VCSs on general access structures and graph access structures. They showed in [3] that how can obtain basis matrices $S^0$ and $S^1$ of a VCS on a complete graph.

**Theorem 2.5.** [3] Let $\Gamma = (Q, F)$ be an access structure on a set $P$ of participants. Let $X, Y \subseteq P$ be two nonempty subsets of participants such that $X \cap Y = \emptyset$, $X \in F$ and $X \cup Y \in Q$. Then in any $(\Gamma, m)$-VCS for this access structure, we have $w(S^1_{X \cup Y}) - w(S^1_X) \geq \alpha m$ where $S^0$ and $S^1$ are basis matrices, $m$ is the pixel expansion and $\alpha$ is the relative contrast.

**Remark 2.6.** [2] By Theorem 2.5, if $Y = \{y\}$, then $S^1[X \cup \{y\}]$ has at least one column with 1 in the row corresponding to $y$ and with zero in all other entries. such a column in $S^1[X \cup \{y\}]$ is called an unavoidable pattern.

For complete graph $K_n$ and complete bipartite graph, we have the following theorems.

**Theorem 2.7.** [3] Let $G = K_n$ be complete graph. Then the value $m^*(K_n)$ is the smallest integer $m$ such that $n \leq \left(\frac{m}{m+1}\right)$.

**Theorem 2.8.** [3] Let $\Gamma$ be a strong access structure on a set of participants $P$. Then $m^*(\Gamma) = 2$ if and only if $\Gamma = \Gamma(G)$ where $G$ is a complete bipartite graph with $V(G) = P$.

An *clique*, $C$, in a graph $G = (V, E)$ is a subset of the vertices such that every two distinct vertices are adjacent. This is equivalent to the condition that the induced subgraph $G[C]$ is complete. A maximum clique of a graph $G$ is a clique such that there is no clique with more vertices. The *clique*
number $\omega(G)$ of a graph $G$ is the number of vertices in a maximum clique in $G$. An independent set, $I$, in a graph is a subset of vertices such that no two vertices in $I$ are adjacent. A maximal independent set is an independent set containing the largest possible number of vertices in graph. The following theorem gives a relation between $m^*(G)$ and number of maximal independent sets in $G$ that proved by Dehkordi and Cheraghi in [7].

**Theorem 2.9.** [7] Let $G$ be a graph with the number of maximal independent sets $l$, then $m^*(G) \geq t$ where $t$ is the smallest integer such that $l \leq \left( \frac{t}{1/2} \right)$.

3. Main results

Let $G = (V, E)$ be a connected graph with $m^*(G) = 4$. Then by Remark 2.4 for any induced subgraph $H$ of $G$ having no isolated vertices, we have $m^*(H) \leq 4$. We have from Theorem 2.9 that the number of maximal independent sets in $H$ is at most 6. Further by Theorem 2.7, we have $\omega(G) \leq 6$. In [6] we characterized all of graphs which $m^*(G) = 4$ and $\omega(G) = 6$. In this paper, we consider case of $\omega(G) = 5$. If $\omega(G) = 5$, then $K_5$ is induced subgraph of graph $G$. We first prove the following lemma.

**Proposition 3.1.** Let $G = K_5$ be a complete graph and $V = \{v_1, v_2, \ldots, v_5\}$ is vertices set. Then one of the pairs of base matrices a VCS for $\Gamma(G)$ is

$$S^1[V] \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}, S^0[V] \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}. $$

**Proof.** By Theorem 2.7, $m^*(K_5) = 4$. From Theorem 6.6 and corollary 6.7 in [3], a $(\Gamma(K_5), 4)$-VCS implies the existence of a sperner family of size 5 over a ground set of size 4. Let ground set is $P = \{a_1, a_2, a_3, a_4\}$, now only sperner family of size 5 over $P$ is $B_1 = \{a_1, a_2\}, B_2 = \{a_2, a_3\}, B_3 = \{a_3, a_4\}, B_4 = \{a_1, a_4\}, B_5 = \{a_1, a_3\}$. From Theorem 7.2 in [3], we obtain basis matrices for a VCS with strong access structure $\Gamma(K_5)$ from following definitions.

$$S^1(i, j) = \begin{cases} 1 & a_j \in B_i \\ 0 & a_j \notin B_i \end{cases}, S^0(i, j) = \begin{cases} 1 & 1 \leq j \leq |B_i| \\ 0 & |B_i| + 1 \leq j \leq 4 \end{cases}. $$

Hence $S^1[V] \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ and $S^0[V] \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$. \hfill \square

**Lemma 3.2.** Let $G$ be a connected graph with $m^*(G) = 4$. If $\omega(G) = 5$ where $\omega(G)$ is the clique number of graph $G$, then $G$ is $(K_5 \cup K_1)$-free.

**Proof.** Assume that $G$ is not $(K_5 \cup K_1)$-free, thus $G$ contains $K_5 \cup K_1$ as an induced subgraph. So if $Z = V(K_5) = \{v_1, \ldots, v_5\}$ and $V(K_1) = \{x\}$, then the vertex $x$ is not connected to any of the
vertices of the $Z$. Given that $G[Z] = K_5$ and by using Proposition 3.1, we have $S^1[Z] \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$. Without loss of generality assume that the rows of $S^1[Z]$ corresponds to $v_1, v_2, \ldots, v_5$ respectively. Using Remark 2.6 with $X = \{x, v_2\}$ and $Y = \{v_1\}$, which $X \in F$ and $X \cup Y \in Q$, then $S^1[X \cup Y]$ has at least one column with 1 in the row corresponding to $v_1$ and with zero in all other entries. Therefore, the row corresponding to $x$ in $S^1[Z \cup \{x\}]$ must be $[0 \ ? \ ? \ ?]$ where $?$ represents the presence of either 0 or 1. So the first entry is zero and the following table shows that other entries of the row corresponding to $x$ are also zero.

<table>
<thead>
<tr>
<th>$X$</th>
<th>$Y$</th>
<th>row of $x$ in $S^1[Z \cup {x}]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x, v_2$</td>
<td>$v_1$</td>
<td>$[0 \ ? \ ? \ ?]$</td>
</tr>
<tr>
<td>$x, v_3$</td>
<td>$v_2$</td>
<td>$[0 \ 0 \ ? \ ?]$</td>
</tr>
<tr>
<td>$x, v_4$</td>
<td>$v_3$</td>
<td>$[0 \ 0 \ 0 \ ?]$</td>
</tr>
<tr>
<td>$x, v_5$</td>
<td>$v_4$</td>
<td>$[0 \ 0 \ 0 \ 0]$</td>
</tr>
</tbody>
</table>

Thus $S^1[Z \cup \{x\}] \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$. This gives a contradiction since no row of $S^1$ can have weight zero. Hence $G$ is $(K_5 \cup K_1)$-free. □

Since graph $G$ is $(K_5 \cup K_1)$-free, then for any vertex of $x \in G$, we have $N(x) \cap Z \neq \emptyset$, where $N(x)$ is the open neighborhood of $x$ consisting of all vertices which are adjacent to $x$. Also, since $\omega(G) = 5$, it follows that $1 \leq |N(x) \cap Z| \leq 4$. For any nonempty proper subset $X \subseteq \{1, 2, \ldots, 5\}$, we define the set $V_X$ as follows:

$$V_X := \{x \in V \setminus Z, N(x) \cap Z = \{v_i : i \in X\}\}$$

We now determine the properties of above sets in following lemmas.

**Lemma 3.3.** Let $G$ be a connected graph with $m^*(G) = 4$ and $\omega(G) = 5$. Then with above definition, we have

(i) If $|X| = 1$, then $V_X = \emptyset$,

(ii) If $|X| = 2$, then $V_X = \emptyset$ except probably $V_{14}$ and $V_{23}$,

(iii) If $|X| = 3$, then $V_X = \emptyset$ except probably $V_{125}$ and $V_{345}$,

(iv) If $|X| = 4$, then $V_X$ can be available.

**Proof.** Let $G[Z] = K_5$ and $Z = \{v_1, v_2, \ldots, v_5\}$. By Lemma 3.1, $S^1[Z \cup \{x\}] \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$. Without loss of generality we assume that rows of $S^1[Z]$ corresponds to $v_1, v_2, \ldots, v_5$ respectively.

(i). Suppose $V_1 \neq \emptyset$ and let $x \in V_1$. Using Remark 2.6, we have the following table:

http://dx.doi.org/10.22108/toc.2019.113671.1599
Therefore row of \( x \) in \( S^1[Z \cup \{x\}] \) is \([0 \ 0 \ 0 \ 0]\). This gives a contradiction, hence \( V_1 = \emptyset \). A similar proof shows that other \( V_i \)'s are empty.

(ii). According to \( S^1[Z] \), \( v_2 \) and \( v_3 \) are zero in first column and other entries in this column are nonzero, so \( V_{23} \neq \emptyset \) and if \( x \in V_{23} \) then we have

\[
S^1[Z \cup \{x\}] \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix},
\]

\[
S^0[Z \cup \{x\}] \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}
\] or

\[
\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}
\].

Similarly if \( y \in V_{14} \) then we have

\[
S^1[Z \cup \{y\}] \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix},
\]

\[
S^0[Z \cup \{y\}] \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}
\] or

\[
\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}
\].

The other \( V_{ij} \)'s are empty. For example let \( x \in V_{12} \), then by Remark 2.6, we have the following table:

<table>
<thead>
<tr>
<th>( X )</th>
<th>( Y )</th>
<th>row of ( x ) in ( S^1[Z \cup {x}] )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x, v_2 )</td>
<td>( v_1 )</td>
<td>([0 \ ? \ ? \ ?])</td>
</tr>
<tr>
<td>( x, v_3 )</td>
<td>( v_2 )</td>
<td>([0 \ 0 \ ? \ ?])</td>
</tr>
<tr>
<td>( x, v_4 )</td>
<td>( v_3 )</td>
<td>([0 \ 0 \ 0 \ ?])</td>
</tr>
<tr>
<td>( x, v_5 )</td>
<td>( v_4 )</td>
<td>([0 \ 0 \ 0 \ 0])</td>
</tr>
</tbody>
</table>

Therefore row of \( x \) in \( S^1[Z \cup \{x\}] \) is \([0 \ 0 \ 0 \ 0]\). This gives a contradiction, hence \( V_{12} = \emptyset \). A similar proof shows that other \( V_{ij} \)'s are empty.

(iii). In \( S^1[Z] \), \( v_1, v_2 \) and \( v_5 \) are zero in last column and other entries in this column are nonzero, so \( V_{125} \neq \emptyset \). If \( x \in V_{125} \), then we have

\[
S^1[Z \cup \{x\}] \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix},
\]

\[
S^0[Z \cup \{x\}] \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}
\] or

\[
\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}
\].

Similarly if \( y \in V_{345} \), then we have

\[
S^1[Z \cup \{y\}] \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix},
\]

\[
S^0[Z \cup \{y\}] \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}
\] or

\[
\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}
\].

By the (ii), graph \( G \) can contains \( V_{14} \) and \( V_{23} \). From each one, we can make three \( V_{ijk} \), where \( 1 \leq i, j, k \leq 5 \) and \( i, j, k \) are different as follows:

http://dx.doi.org/10.22108/toc.2019.113671.1599
All of \(V_{ijk}\)'s in above table are \(\emptyset\). Let \(V_{123}\) that is obtained from \(V_{23}\) is not empty and let \(x \in V_{123}\).

Then by (ii) row of \(x\) in \(S^1[Z \cup \{x\}]\) is \([1 \ 0 \ 0 \ 0]\). However \(\{x, v_1\} \in F\), Thus in \(S^1[Z \cup \{x\}]\) we must have the unavoidable patterns of \([1 \ 0]\) and \([0 \ 1]\) while the first pattern doesn’t exist. Hence \(V_{123} = \emptyset\). Similarly all of \(V_{ijk}\)'s in above table are empty. Now, it is sufficient that we show \(V_{135}\) and \(V_{245}\) are empty.

Let \(V_{135}\) and \(V_{245}\) are not empty and \(x \in V_{245}\); \(y \in V_{135}\). Using Remark 2.6, we have the following tables:

\[
\begin{array}{c|c|c|c|c|c|c}
 X & Y & \text{row of } x \text{ in } S^1[Z \cup \{x\}] \\
\hline
 x, v_3 & v_4 & \begin{bmatrix} 0 & ? & ? & ? \end{bmatrix} \\
 x, v_3 & v_2 & \begin{bmatrix} 0 & 0 & ? & ? \end{bmatrix} \\
 x, v_1 & v_2 & \begin{bmatrix} 0 & 0 & 0 & ? \end{bmatrix} \\
 x, v_1 & v_4 & \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix} \\
\end{array}
\]

\[
\begin{array}{c|c|c|c|c|c|c|c|c}
 X & Y & \text{row of } y \text{ in } S^1[Z \cup \{y\}] \\
\hline
 y, v_2 & v_1 & \begin{bmatrix} 0 & ? & ? & ? \end{bmatrix} \\
 y, v_4 & v_1 & \begin{bmatrix} 0 & 0 & ? & ? \end{bmatrix} \\
 y, v_4 & v_5 & \begin{bmatrix} 0 & 0 & 0 & ? \end{bmatrix} \\
 y, v_2 & v_3 & \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix} \\
\end{array}
\]

Therefore rows of \(x\) and \(y\) in \(S^1[Z \cup \{x, y\}]\) are \([0 \ 0 \ 0 \ 0]\). This gives a contradiction, hence \(V_{245}\) and \(V_{135}\) are empty.

(iv). Let \(x \in V_{1234}\). By Remark 2.6, we have the following table:

\[
\begin{array}{c|c|c|c|c|c|c|c|c}
 X & Y & \text{row of } x \text{ in } S^1[Z \cup \{x\}] \\
\hline
 x, v_5 & v_1 & \begin{bmatrix} ? & 0 & ? & ? \end{bmatrix} \\
 x, v_5 & v_3 & \begin{bmatrix} ? & 0 & ? & 0 \end{bmatrix} \\
 v_2 & x & \begin{bmatrix} 1 & 0 & ? & 0 \end{bmatrix} \\
 v_4 & x & \begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix} \\
\end{array}
\]

Hence \(S^1[Z \cup \{x\}] \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}\).

Also, since \(\{x, v_5\}\) is forbidden set and \(w(S^1_{\{x\}}) = 2\), then \(S^0[Z \cup \{x\}] \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}\). Let \(y \in V_{1235}\). By Remark 2.6, we have the following table:
<table>
<thead>
<tr>
<th>$X$</th>
<th>$Y$</th>
<th>row of $y$ in $S^1[Z \cup {y}]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y, v_4$</td>
<td>$v_1$</td>
<td>[? 0 ? ?]</td>
</tr>
<tr>
<td>$y, v_4$</td>
<td>$v_3$</td>
<td>[? 0 0 ?]</td>
</tr>
<tr>
<td>$v_3$</td>
<td>$y$</td>
<td>[1 0 0 ?]</td>
</tr>
<tr>
<td>$v_5$</td>
<td>$y$</td>
<td>[1 0 0 1]</td>
</tr>
</tbody>
</table>

Hence $S^1[Z \cup \{y\}] \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$.

Also, since $\{y, v_4\}$ is forbidden set and $w(S^1_{\{y\}}) = 2$, then $S^0[Z \cup \{y\}] \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$. Let $z \in V_{1245}$. By Remark 2.6, we have the following table:

<table>
<thead>
<tr>
<th>$X$</th>
<th>$Y$</th>
<th>row of $z$ in $S^1[Z \cup {z}]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$z, v_3$</td>
<td>$v_4$</td>
<td>[0 ? ? ?]</td>
</tr>
<tr>
<td>$z, v_3$</td>
<td>$v_2$</td>
<td>[0 0 ? ?]</td>
</tr>
<tr>
<td>$v_2$</td>
<td>$z$</td>
<td>[0 0 ? 1]</td>
</tr>
<tr>
<td>$v_4$</td>
<td>$z$</td>
<td>[0 0 1 1]</td>
</tr>
</tbody>
</table>

Hence $S^1[Z \cup \{z\}] \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$.

Also, since $\{z, v_3\}$ is forbidden set and $w(S^1_{\{z\}}) = 2$, then $S^0[Z \cup \{z\}] \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$. Let $s \in V_{1345}$. By Remark 2.6, we have the following table:

<table>
<thead>
<tr>
<th>$X$</th>
<th>$Y$</th>
<th>row of $s$ in $S^1[Z \cup {s}]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s, v_2$</td>
<td>$v_1$</td>
<td>[0 ? ? ?]</td>
</tr>
<tr>
<td>$s, v_2$</td>
<td>$v_3$</td>
<td>[0 ? ? 0]</td>
</tr>
<tr>
<td>$v_3$</td>
<td>$s$</td>
<td>[0 1 ? 0]</td>
</tr>
<tr>
<td>$v_1$</td>
<td>$s$</td>
<td>[0 1 1 0]</td>
</tr>
</tbody>
</table>

Hence $S^1[Z \cup \{s\}] \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$.

Also, since $\{s, v_2\}$ is forbidden set and $w(S^1_{\{s\}}) = 2$, then $S^0[Z \cup \{s\}] \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$. Let $t \in V_{2345}$. By Remark 2.6, we have the following table:

http://dx.doi.org/10.22108/toc.2019.113671.1599
\[ \begin{array}{c|c|c}
X & Y & \text{row of } t \text{ in } S^1[Z \cup \{t\}] \\
\hline
t, v_1 & v_2 & [? \ ? \ 0 \ ?] \\
t, v_1 & v_3 & [? \ ? \ 0 \ 0] \\
v_2 & t & [1 \ ? \ 0 \ 0] \\
v_4 & t & [1 \ 1 \ 0 \ 0] \\
\end{array} \]

Hence \( S^1[Z \cup \{t\}] \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \).

Also, since \( \{t, v_1\} \) is forbidden set and \( w(S^1_{\{t\}}) = 2 \), then \( S^0[Z \cup \{t\}] \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \). \( \square \)

**Lemma 3.4.** Let \( G \) be a connected graph with \( m^*(G) = 4 \) and \( \omega(G) = 5 \). Also, suppose that \( Z = \{v_1, v_2, \ldots, v_5\} \) is a clique of \( G \). Then \( V_X \) is an independent set for any \( X \subseteq \{1, 2, \ldots, 5\} \) with \( 2 \leq |X| \leq 4 \).

**Proof.** Let \( |X| = 2 \). Then by Lemma 3.3, we consider the cases \( X = \{1, 4\} \) and \( X = \{2, 3\} \). Let \( x, y \in V_{14} \). If \( \{x, y\} \in Q \), then we have 8 maximal independent sets as follows: \( \{x, v_2\}, \{x, v_3\}, \{x, v_5\}, \{y, v_2\}, \{y, v_3\}, \{y, v_5\}, \{v_1\}, \{v_4\} \). By Theorem 2.9, \( G \) has at most 6 maximal independent sets, thus this gives a contradiction. Hence \( V_{14} \) is an independent set. Similarly, the set of \( V_{23} \) is independent. Now let \( x, y \in V_{125} \). If \( \{x, y\} \in Q \), then we have 7 maximal independent sets as follows: \( \{x, v_3\}, \{x, v_4\}, \{y, v_3\}, \{y, v_4\}, \{v_1\}, \{v_2\}, \{v_5\} \) and \( \{v_5\} \). By Theorem 2.9, this gives a contradiction, hence \( V_{125} \) is an independent set. Similarly, the set of \( V_{345} \) is independent.

Now we show that \( V_{1234} = \emptyset \). If \( V_{1234} \neq \emptyset \), let \( x, y \in V_{1234} \). Then by Lemma 3.3, we have

\[ S^1[Z \cup \{x, y\}] \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}, \quad S^0[Z \cup \{x, y\}] \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}. \]

So \( w(S^1_{\{x,y\}}) = w(S^0_{\{x,y\}}) = 2 \), thus \( \{x, y\} \in F \). Hence \( V_{1234} \) is a independent set and this complete proof. \( \square \)

**Lemma 3.5.** Let \( G \) be a connected graph with \( m^*(G) = 4 \) and \( \omega(G) = 5 \). Then \( V_{14} \cup V_{25} \) is an independent set.

**Proof.** From lemma 3.3, we have

\[ S^1[Z \cup \{x, y\}] \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}, \quad S^0[Z \cup \{x, y\}] \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}. \]

http://dx.doi.org/10.22108/toc.2019.113671.1599
If $S^0[Z \cup \{x,y\}] \sim \left[\begin{array}{c}
1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0
\end{array}\right]$ then $w(S^1_{\{x,y\}}) > w(S^0_{\{x,y\}})$, thus $\{x,y\} \in Q$. However $w(S^1_{\{x,y,v_3\}}) = w(S^0_{\{x,y,v_3\}})$, which is contradiction since $\Gamma(G)$ is strong access structure. Hence $S^0[Z \cup \{x,y\}] \sim \left[\begin{array}{c}
1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 \\
1 & 0 & 1 & 0
\end{array}\right]$. In this case $w(S^1_{\{x,y\}}) = w(S^0_{\{x,y\}})$ and this show that $\{x,y\} \in F$. Hence $x$ and $y$ are nonadjacent. Thus $V_{14} \cup V_{23}$ is an independent set.

\[\square\]

**Remark 3.6.** Suppose that $V_{125}, V_{345} \neq \emptyset$. Let $x \in V_{125}$, $y \in V_{345}$. From Lemma 3.3, we have

\[S^1[Z \cup \{x,y\}] \sim \left[\begin{array}{c}
1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 \\
1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1
\end{array}\right], \quad S^0[Z \cup \{x,y\}] \sim \left[\begin{array}{c}
1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0
\end{array}\right] \quad \text{or} \quad \left[\begin{array}{c}
1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{array}\right].\]

In first form of $S^0[Z \cup \{x,y\}]$, we have $w(S^1_{\{x,y\}}) > w(S^0_{\{x,y\}})$, hence $\{x,y\}$ is qualified set and this show that $x$ and $y$ are adjacent. In second form of $S^0[Z \cup \{x,y\}]$, we have $w(S^1_{\{x,y\}}) = w(S^0_{\{x,y\}})$, hence $\{x,y\}$ is forbidden set, thus $x$ and $y$ are nonadjacent.

**Lemma 3.7.** Let $G$ be a connected graph with $m^*(G) = 4$ and $\omega(G) = 5$. Also, $X$, $Y$ and $W$ be nonempty proper subsets of $\{1, 2, \ldots, 5\}$. If $|X| = 2$, $|Y| = 3$ and $|W| = 4$, then with previous definitions,

(i) The sets of $V_X \cup V_Y$ are independent sets,

(ii) The sets of $V_X \cup V_W$ are independent sets,

(iii) The induced subgraphs $G[V_Y \cup V_W]$ are complete bipartite graphs.

**Proof.** (i). Let $V_{14}, V_{125} \neq \emptyset$. If $x \in V_{14}$ and $y \in V_{125}$, then by Lemma 3.3, we have

\[S^1[Z \cup \{x,y\}] \sim \left[\begin{array}{c}
1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1
\end{array}\right], \quad S^0[Z \cup \{x,y\}] \sim \left[\begin{array}{c}
1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 \\
1 & 0 & 1 & 0
\end{array}\right] \quad \text{or} \quad \left[\begin{array}{c}
1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{array}\right].\]

Hence $w(S^1_{\{x,y\}}) > w(S^0_{\{x,y\}})$, thus $\{x,y\} \in Q$. However $w(S^1_{\{x,y,v_3\}}) = w(S^0_{\{x,y,v_3\}})$, which is contradiction since $\Gamma(G)$ is strong access structure. So $\{x,y\} \in F$ and hence $V_{14} \cup V_{125}$ is independent set.

A similar proof shows that $V_{14} \cup V_{345}, V_{23} \cup V_{125}$ and $V_{23} \cup V_{345}$ are independent sets.

(ii). Let $V_{14}, V_{1234} \neq \emptyset$. If $x \in V_{14}$ and $y \in V_{1234}$, then by Lemma 3.3, we have

\[S^1[Z \cup \{x,y\}] \sim \left[\begin{array}{c}
1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1
\end{array}\right], \quad S^0[Z \cup \{x,y\}] \sim \left[\begin{array}{c}
1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 \\
1 & 0 & 1 & 0
\end{array}\right] \quad \text{or} \quad \left[\begin{array}{c}
1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0
\end{array}\right].\]

Hence $w(S^1_{\{x,y\}}) = w(S^0_{\{x,y\}})$, thus $\{x,y\} \in F$. A similar proof shows that $V_X \cup V_W$ is an independent set.
(iii). Let $V_{125}, V_{1234} \neq \emptyset$. If $x \in V_{125}$ and $y \in V_{1234}$, then by Lemma 3.3, we have

$$S^1[Z \cup \{x, y\}] \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad S^0[Z \cup \{x, y\}] \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$ 

Hence $w(S^1_{\{x,y\}}) > w(S^0_{\{x,y\}})$, thus $\{x, y\} \in Q$. A similar proof shows that the induced subgraph $G[V_Y \cup V_W]$ is a complete bipartite graph. \hfill \Box

**Lemma 3.8.** Let $G$ be a connected graph with $\omega(G) = 5$. If $V_{ij}$’s are not empty, then $V_{ijk}$’s are empty.

**Proof.** Let $V_{ij}$’s are not empty and $x \in V_{23}$ and $y \in V_{14}$. If $V_{125} \neq \emptyset$, then by Lemma 3.3 we have

$$S^1[Z \cup \{x, y, z\}] \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad S^0[Z \cup \{x, y, z\}] \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ * & * & * & * \end{bmatrix}.$$ 

Hence $w(S^1_{\{x,y,z\}}) > w(S^0_{\{x,y,z\}})$, therefore $\{x, y\} \in Q$ and this is contradiction with Lemma 3.5. \hfill \Box

We now proceed to characterize connected graphs $G$ such that $\omega(G) = 5$.

Let $F$ be the family of graphs that obtained from complete graph $K_5$ with $V(K_5) = \{v_1, v_2, \ldots, v_5\}$ by adding nine independent sets $V_{14}, V_{23}, V_{125}, V_{345}, V_{1234}, V_{1235}, V_{1245}, V_{1345}$ and $V_{2345}$ where $|V_{14}| = n_1$, $|V_{23}| = n_2$, $|V_{125}| = n_3$, $|V_{345}| = n_4$, $|V_{1234}| = n_5$, $|V_{1235}| = n_6$, $|V_{1245}| = n_7$, $|V_{1345}| = n_8$, $|V_{2345}| = n_9$ and these sets satisfy in Lemma 3.4 to Lemma 3.8. According to the definition of $V_{1234}$, if $x \in V_{1234}$, then $\{x, v_5\}$ is independent, so we can replace set of $v_5$ with a set of independent vertices, with the name $V'_5$, instead of set $V_{1234}$. Similarly, the vertices $v_1, v_2, v_3$ and $v_4$ can replace by independent sets $V'_1, V'_2, V'_3$ and $V'_4$. A few graphs in the family $F$ are given in Figure 1.

![Graphs in family F](http://dx.doi.org/10.22108/toc.2019.113671.1599)
Theorem 3.9. Let $G$ be a connected graph with $\omega(G) = 5$. Then $m^*(G) = 4$ if and only if for specified values $n_1, n_2, \ldots, n_9$, $G$ is isomorphic to a graph $H$ in $\mathcal{F}$.

Proof. Let $G = (V, E)$ be a connected graph containing the complete graph $V(K_5) = \{v_1, v_2, \ldots, v_5\}$ and $m^*(G) = 4$. Let $Z = \{v_1, v_2, \ldots, v_5\}$. It follows from Lemma 3.2 that every vertex $u \in V - Z$ is adjacent to at least one vertex in $Z$. Further since $G$ is $K_6$ free, $u$ is adjacent to at most four vertices in $Z$. Therefore by Lemma 3.3 to Lemma 3.8, we conclude that $G \in \mathcal{F}$. To prove the converse, consider $H \in \mathcal{F}$. By Lemma 3.8, if $|V_{14}| = n_1$ and $|V_{23}| = n_2$, then $V_{125}$ and $V_{345}$ are empty sets. Given that $|V_{1234}| = n_5, |V_{1235}| = n_6, |V_{1245}| = n_7, |V_{1345}| = n_8$ and $|V_{2345}| = n_9$, then the basis matrices for VCS of the access structure $\Gamma(H)$ are:

$$
S^1 = \begin{bmatrix}
1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 \\
1 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 \\
0_{n_1} & 0_{n_1} & 1_{n_1} & 0_{n_1} \\
1_{n_2} & 0_{n_2} & 0_{n_2} & 1_{n_2} \\
1_{n_3} & 0_{n_3} & 1_{n_3} & 0_{n_3} \\
1_{n_4} & 0_{n_4} & 0_{n_4} & 1_{n_4} \\
0_{n_5} & 0_{n_5} & 1_{n_5} & 1_{n_5} \\
0_{n_6} & 1_{n_6} & 0_{n_6} & 0_{n_6} \\
0_{n_7} & 0_{n_7} & 1_{n_7} & 1_{n_7} \\
0_{n_8} & 1_{n_8} & 1_{n_8} & 0_{n_8} \\
1_{n_9} & 1_{n_9} & 0_{n_9} & 0_{n_9}
\end{bmatrix}
$$

and

$$
S^0 = \begin{bmatrix}
1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 \\
1_{n_1} & 0_{n_1} & 0_{n_1} & 1_{n_1} \\
0_{n_2} & 1_{n_2} & 1_{n_2} & 0_{n_2} \\
0_{n_3} & 1_{n_3} & 1_{n_3} & 0_{n_3} \\
0_{n_4} & 1_{n_4} & 1_{n_4} & 0_{n_4} \\
0_{n_5} & 1_{n_5} & 1_{n_5} & 0_{n_5} \\
0_{n_6} & 1_{n_6} & 1_{n_6} & 0_{n_6} \\
0_{n_7} & 1_{n_7} & 1_{n_7} & 0_{n_7} \\
0_{n_8} & 1_{n_8} & 1_{n_8} & 0_{n_8} \\
0_{n_9} & 1_{n_9} & 1_{n_9} & 0_{n_9}
\end{bmatrix}
$$

where $1_n$ ($0_n$) denotes the $n \times 1$ column matrix with all entries one (zero). It is simple work that $S^0$ and $S^1$ are basis matrices for a VCS of the access structure $\Gamma(H)$. Hence $m^*(H) \leq 4$. However $H$ contains $K_5$, thus $m^*(H) = 4$. Now if $G \in \mathcal{F}$, then $G$ is an induced subgraph of $H$ and since $G$ contains $K_5$ as a subgraph, we have $m^*(G) = 4$.

Further if $V_{125}$ and $V_{345}$ are not empty and $|V_{125}| = n_3$ and $|V_{345}| = n_4$, then by Lemma 3.8 don't exist $V_{14}$ and $V_{23}$ simultaneously. Let $V_{14} \neq \emptyset$, in this case the basis matrices for VCS of the access structure $\Gamma(H)$ are:

http://dx.doi.org/10.22108/toc.2019.113671.1599
\[ S^1 \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & n_1 & 1 & n_1 \\ 0 & n_3 & 0 & n_3 \\ 0 & n_4 & 1 & n_4 \\ 1 & n_5 & 0 & n_5 \\ 1 & n_6 & 0 & n_6 \\ 0 & n_7 & 0 & n_7 \\ 0 & n_8 & 1 & n_8 \\ 1 & n_9 & 0 & n_9 \end{bmatrix} \quad \text{and} \quad S^0 \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & n_1 & 0 & n_1 \\ 1 & n_3 & 0 & n_3 \\ 1 & n_4 & 0 & n_4 \\ 1 & n_5 & 0 & n_5 \\ 1 & n_6 & 0 & n_6 \\ 1 & n_7 & 0 & n_7 \\ 1 & n_8 & 0 & n_8 \\ 1 & n_9 & 0 & n_9 \end{bmatrix}. \]

In this case, similar to discussion of above, we have \( m^*(G) = 4 \). \( \square \)

4. Conclusion

Ateniese et al. [3] have proved that \( m^*(\Gamma) = 2 \) if and only if \( \Gamma = \Gamma(G) \) where \( G \) is a complete bipartite graph. Also Arumugam et al. [1] have obtained a characterization of all connected graphs \( G \) where \( m^*(G) = 3 \). If \( m^*(G) = 4 \) then \( \omega(G) \leq 6 \). We have obtained previously in [6] a characterization of all connected graphs \( G \) for which \( \omega(G) = 6 \). In this paper, we obtained a characterization of all connected graphs \( G \) for which \( \omega(G) = 5 \). The next problem is to characterize all graphs where \( m^*(G) = 4 \) and \( \omega(G) \leq 4 \).

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