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A NOTE ON FALL COLORINGS OF KNESER GRAPHS

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Abstract. A fall coloring of a graph G is a proper coloring of G with k colors such that each vertex sees all k colors on its closed neighborhood. In this paper, we characterize all fall colorings of Kneser graphs of type $KG(n, 2)$.

1. **Introduction**

In this paper, simple graphs with nonempty and finite vertex sets are considered. Also, for each positive integer k, the symbol $[k]$ means $\{i|\ i \in \mathbb{N},\ 1 \leq i \leq k\}$.

Let $G = (V(G), E(G))$ be a graph. A *coloring* of G is a function $f : V(G) \to C$ such that for each c in C, the set $f^{-1}(c)$ is independent; in this case, we consider each c in C as a *color* and call $f^{-1}(c)$ a *color class* of f. The *chromatic number* of G, denoted by $\chi(G)$, is the minimum cardinality of a set C that a coloring $f : V(G) \to C$ exists.

Let G be a graph and $f : V(G) \to C$ be a coloring of G. The vertex v of G is called a *b-dominating* vertex with respect to f if $f(N[v]) = C$, i.e., the vertex v sees all colors on its closed neighborhood. The coloring f is said to be a *fall coloring* of G if all vertices of G are b-dominating [1]. We mean by Fall(G) the set of all natural numbers k for which G admits a fall coloring $f : V(G) \to C$ with $|C| = k$. We may have Fall$(G) = \emptyset$; for example, Fall$(C_5) = \emptyset$, where C_5 is the cycle with five vertices. A graph G is called *f-continuous* if either Fall(G) is empty or Fall$(G) \neq \emptyset$ and Fall$(G) =$

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$\{k \in \mathbb{N}|\ \min(\mathrm{Fall}(G)) \leq k \leq \max(\mathrm{Fall}(G))\}$. Not all graphs are f-continuous; for example, the three dimensional cube Q_3 satisfies $\mathrm{Fall}(Q_3) = \{2, 4\}$.

Let G be a graph. A *b-coloring* of G is a coloring $f : V(G) \to C$ such that each color class contains at least one b-dominating vertex [2]. The set of all natural numbers k that G has a b-coloring $f : V(G) \to C$ with $|C| = k$, is denoted by $\mathrm{B}(G)$. Always $\chi(G) \in \mathrm{B}(G)$. The maximum of the set $\mathrm{B}(G)$, say $\chi_b(G)$, is called the *b-chromatic number* of G. A *b-continuous* graph stands for a graph G with $\mathrm{B}(G) = \{k \in \mathbb{N}|\ \chi(G) \leq k \leq \chi_b(G)\}$. The three dimensional cube Q_3 is not b-continuous; since $\mathrm{B}(Q_3) = \{2, 4\}$.

2. Fall colorings of Kneser graphs

Suppose that $n \geq m$. Hereafter, $\binom{[n]}{m}$ denotes the set of all m-subsets of $[n]$. The *Kneser graph* $KG(n, m)$ has the vertex set $\binom{[n]}{m}$, in which $A \sim B$ iff $A \cap B = \emptyset$. For instance, $KG(5, 2)$ is the celebrated Petersen graph.

In [3], Javadi and Omoomi determined the b-chromatic number of all Kneser graphs of type $KG(n, 2)$. Also, they proved that $KG(n, 2)$ is b-continuous whenever $n \geq 17$. In this paper, we study fall colorings of Kneser graphs of type $KG(n, 2)$. We show that if $n \geq 2$, then $|\mathrm{Fall}(KG(n, 2))| \leq 1$; and as a corollary, these graphs are f-continuous. The procedure uses some facts mentioned in [3]; nevertheless, it is self-contained.

Theorem 2.1. *For each natural number $n \geq 2$, we have*

$$\mathrm{Fall}(KG(n,2)) = \begin{cases} \{1\} & \text{if} \quad n = 2 \text{ or } 3 \\ \{2\} & \text{if} \quad n = 4 \\ \{\frac{n(n-1)}{6}\} & \text{if} \quad n \geq 5,\ n = 1 \text{ or } 3 \pmod 6 \\ \{\frac{(n-1)(n-2)}{6} + 1\} & \text{if} \quad n \geq 5,\ n = 2 \text{ or } 4 \pmod 6 \\ \emptyset & \text{if} \quad n \geq 5,\ n = 0 \text{ or } 5 \pmod 6 \end{cases}$$

Proof. Obviously, $\mathrm{Fall}(KG(2, 2)) = \mathrm{Fall}(KG(3, 2)) = \{1\}$. Also, since the edge set of $KG(4, 2)$ is a matching, therefore, $\mathrm{Fall}(KG(4, 2)) = \{2\}$. Now, let us suppose that $n \geq 5$. We note that by considering the complete graph K_n with vertex set $[n]$, the Kneser graph $KG(n, 2)$ is exactly the complement graph of the line graph of K_n. Therefore, one can think of $KG(n, 2)$ as the graph whose vertex set is $E(K_n)$ in which two elements of $E(K_n)$ are adjacent in $KG(n, 2)$ iff they have not any common vertices in K_n. Now, suppose that f is a fall coloring of $KG(n, 2)$ and $\mathcal{S}$ is an arbitrary color class of f. We have the following two cases :

Case I : The case that $\bigcap_{A\in\mathcal{S}} A = \emptyset$. In this case, let us regard two arbitrary distinct elements of $\mathcal{S}$, say $X := \{a, b\}$ and $Y := \{a, c\}$. Since $a \notin \bigcap_{A\in\mathcal{S}} A$, there exists a vertex Z of $\mathcal{S}$ for which $a \notin Z$. Every such a vertex Z must be equal to $\{b, c\}$. We conclude that in this case where $\bigcap_{A\in\mathcal{S}} A = \emptyset$, there exist three pairwise distinct elements $a, b, c \in [n]$ such that $\mathcal{S} = \{\{a, b\}, \{b, c\}, \{c, a\}\}$. So, $\mathcal{S}$ is the set of edges of a triangle in K_n. We call such color classes *triangular*.

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Case II : The case that $\bigcap_{A\in\mathcal{S}} A \neq \emptyset$. Let $i \in \bigcap_{A\in\mathcal{S}} A$. Each vertex B of $KG(n,2)$ which contains i is an element of $\mathcal{S}$; otherwise, $B \notin \mathcal{S}$ and B has not any neighbors in $\mathcal{S}$, contradicting the fact that f is a fall coloring. Also, since $n \geq 5$, for each $j \in [n] \setminus \{i\}$, there exists a vertex C of $\mathcal{S}$ with $j \notin C$; and therefore, $\bigcap_{A\in\mathcal{S}} A = \{i\}$ and $\mathcal{S} = \{A|\ A \in \binom{[n]}{2},\ i \in A\}$. Accordingly, $\mathcal{S}$ is the set of all edges in K_n that are incident with i. We call such color classes *starlike*. It is obvious that the number of starlike color classes of f is at most one.

We conclude that the set of color classes of a fall coloring of $KG(n,2)$ is either a partition of $E(K_n)$ into one starlike and some triangles or a partition of $E(K_n)$ into triangles. Conversely, every such a partition of $E(K_n)$ is the set of color classes of a fall coloring of $KG(n,2)$. On the other hand, it is well-known [4] that the edge set of a complete graph can be partitioned into triangles iff the number of its vertices is congruent to 1 or 3 modulo 6. Therefore, if $\mathrm{Fall}(KG(n,2)) \neq \emptyset$, then $n = 1$ or 2 or 3 or 4 (mod 6). So, if $n = 0$ or 5 (mod 6), then $\mathrm{Fall}(KG(n,2)) = \emptyset$. Also, for $n = 1$ or 3 (mod 6), we have $\mathrm{Fall}(KG(n,2)) = \{\frac{n(n-1)}{6}\}$. Finally, if $n = 2$ or 4 (mod 6), we can make a partition of $E(K_n)$ into one starlike and some triangles; and therefore, $\mathrm{Fall}(KG(n,2)) = \{\frac{(n-1)(n-2)}{6} + 1\}$. □

The following corollary is an immediate consequence of the theorem.

Corollary 2.2. *For $n \geq 2$, the Kneser graph $KG(n,2)$ is f-continuous.*

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