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A GENERALIZATION OF GLOBAL DOMINATING FUNCTION

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Abstract. Let G be a graph. A function $f : V(G) \longrightarrow \{0,1\}$, satisfying the condition that every vertex u with $f(u) = 0$ is adjacent with at least one vertex v such that $f(v) = 1$, is called a dominating function (DF). The weight of f is defined as $wet(f) = \Sigma_{v\in V(G)} f(v)$. The minimum weight of a dominating function of G is denoted by $\gamma(G)$, and is called the domination number of G. A dominating function f is called a global dominating function (GDF) if f is also a DF of $\overline{G}$. The minimum weight of a global dominating function is denoted by $\gamma_g(G)$ and is called global domination number of G. In this paper we introduce a generalization of global dominating function. Suppose G is a graph and $s \geq 2$ and K_n is the complete graph on $V(G)$. A function $f : V(G) \longrightarrow \{0,1\}$ on G is s-dominating function ($s - DF$), if there exists some factorization $\{G_1, \ldots, G_s\}$ of K_n, such that $G_1 = G$ and f is dominating function of each G_i.

1. **Introduction**

In this paper all graphs are finite, simple and undirected. We denote the vertex set and the edge set of a graph G by $V(G)$ and $E(G)$, respectively.
A function $f : V(G) \longrightarrow \{0,1\}$, satisfying the condition that every vertex u with $f(u) = 0$ is adjacent with at least one vertex v such that $f(v) = 1$, is called a *dominating function* (DF). The weight of f is defined as $wet(f) = \Sigma_{v\in V(G)} f(v)$. The minimum weight of a dominating function of G is denoted by $\gamma(G)$, and is called the *domination number* of G. A dominating function f is called a *global*

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dominating function (GDF) if f is also a DF of $\overline{G}$. The minimum weight of a global dominating function is denoted by $\gamma_g(G)$ and is called *global domination number* of G.

Let G be a graph. An *end vertex* is a vertex of degree one. The length of a shortest path between two vertices u, v in G is defined as the distance of u and v, denoted by $d(u, v)$. The diameter of G denoted by $diam(G)$ is defined as $\max\{d(u, v) : u, v \in V(G), u \neq v\}$. The length of a smallest cycle in a graph G (containing cycles) is the *girth* of G, denoted by $g(G)$. Two vertices that are not adjacent in G are said to be *independent*. A set S of vertices is called *independent* if every two vertices of S are independent. The *independence number* $\alpha(G)$ of a graph G, is the maximum cardinality of an independent set of G.

A *factor* of G is spanning subgraph of G. A graph G is said to be *factorable into factors* $G_1, G_2, \ldots, G_t$ if

(i) $E(G_i) \cap E(G_j) = \emptyset$ for, $i \neq j$,

(ii) $E(G) = \cup_{i=1}^{t} E(G_i)$.

In this case we say, $T = \{G_1, G_2, \ldots, G_t\}$ is a *factorization* of G.

2. s-dominating function ($s - DF$)

Definition 2.1. *Suppose that $s \geq 2$. A function $f : V(G) \longrightarrow \{0, 1\}$, is called an s-dominating function ($s - DF$), if there exists some factorization $\{G_1, \ldots, G_s\}$ of K_n, such that $G_1 = G$ and f is a dominating function for each of G_i. If $s = 1$ then s-dominating function is the same as dominating function. We denote the minimum weight of a s-dominating function of G by $\gamma_s(G)$.*

Obviously according to the definition if $s = 2$, an $s - DF$ is the same as GDF.

Theorem 2.2. *Let G be a graph of order n. A function $f : V(G) \longrightarrow \{0, 1\}$ is an $s - DF$ if and only if each vertex v of G with $f(v) = 0$ is adjacent with at least a vertex u with $f(u) = 1$, and is not adjacent with at least $s - 1$ vertices x with $f(x) = 1$.*

Proof. If $s = 1$ then the assertion is obvious. Thus we can suppose that $s \geq 2$.

Suppose that f is an $s - DF$. If G does not have a vertex v with $f(v) = 0$ then the result is obvious. Otherwise since f is a dominating function of G, each vertex v with $f(v) = 0$ is adjacent with at least one vertex x with $f(x) = 1$ and since f is a dominating function for $G_2, \ldots, G_s$, and these subgraphs are edge disjoint, v is not adjacent with at least $s - 1$ vertices y with $f(y) = 1$.

Suppose that $f : V(G) \longrightarrow \{0, 1\}$ is a function such that each vertex u of G with $f(u) = 0$ is adjacent with at least a vertex v with $f(v) = 1$ and is not adjacent with at least $s - 1$ vertices x with $f(x) = 1$. If there is no vertex v with $f(v) = 0$ then we let $G_2 = \overline{G}$ and $G_i = \overline{K_n}$, $3 \leq i \leq s$. In this case obviously f is a dominating function of all G_i and G. So f is an $s - DF$. Now suppose that $v_1, \cdots, v_t$ are all vertices with value 0 in G, and $v_i, 1 \leq i \leq t$ is not adjacent with $w_i^1, w_i^2, \ldots, w_i^{s-1}$ and $f(w_i^j) = 1, 1 \leq j \leq s - 1$. Suppose that for each $2 \leq j \leq s - 1$, $E_j = \{v_i w_i^{j-1} : 1 \leq i \leq t\}$, and

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$E_s = E(\overline{G}) - \cup_{j=2}^{s-1} E_j$. So, if $G_1 = G$ and G_i is the induced graph on E_i in K_n for $i \geq 2$, then we have $E(K_n) = \bigcup_{i=1}^{s} E(G_i)$, and f is dominating function for each of G_i. □

We denote the set of vertices with value 1 in an $s - DF$, by $s - DS$. According to Theorem 2.2 a set $S \subseteq V(G)$ is an $s - DS$ if and only if each $v \in V(G) \setminus S$ is adjacent with at least a vertex $w \in S$, and is not adjacent with at least $s - 1$ vertices in S.

Observation 2.3. *For any graph G with n vertices and $1 \leq s \leq n$ we have*

1. $1 \leq \gamma \leq \gamma_s \leq n$,
2. $\gamma_s \leq \gamma_{s+1}$,
3. $\gamma_s \geq s$ *for* $s \leq n$,
4. $\gamma_s = n$ *for* $s > n$.

We know that $\gamma(P_n) = \gamma(C_n) = \lfloor (n+2)/3 \rfloor$. Also in [5] can be found:

Theorem 2.4. [5] *If S is a γ-set of $P_n (n \geq 6)$ then S is a global dominating set of P_n. Also $\gamma(P_n) = \gamma_g(P_n)$.*

Theorem 2.5. *If s is an integer such that $1 \leq s \leq \lfloor (n-1)/3 \rfloor$, then we have $\gamma_s(P_n) = \gamma_s(C_n) = \lfloor (n+2)/3 \rfloor$.*

Proof. We prove the assertion for P_n . The proof for C_n is similar.

We know that $\gamma(P_n) = \lfloor (n+2)/3 \rfloor$. Let f be a $\gamma-$function on P_n (with weight $\lfloor (n+2)/3 \rfloor$). Thus each vertex u with $f(u) = 0$ is adjacent with a vertex v with $f(v) = 1$ and is not adjacent with at least $\lfloor (n+2)/3 \rfloor - 2$ vertices x with $f(x) = 1$. Since $s \leq \lfloor (n-1)/3 \rfloor$ we have $s - 1 \leq \lfloor (n+2)/3 \rfloor - 2$ and so each vertex v with $f(v) = 0$ is not adjacent with at least $s - 1$ vertices x with $f(x) = 1$. Thus f is an s-dominating function and $\gamma_s \leq \gamma$. The converse is obvious. □

Theorem 2.6. *For any connected graph G we have*

$$\gamma_s(G) \geq min \gamma_s(T_G)$$

where the minimum is over all spanning trees T_G of G.

Proof. If G dose not have any cycle the result is obvious.

Otherwise we consider an $s - DF$ function f of weight γ_s and a cycle C in G. On this cycle either there is a vertex v with $f(v) = 0$ and a vertex w with $f(w) = 1$ such that v and w are adjacent on this cycle or not. If there is no such a vertex we delete an arbitrary edge of C and obtain a connected graph G' such that, each vertex x with $f(x) = 0$, is adjacent with a vertex y with $f(y) = 1$. If there is such a vertex, by deleting the edge vz, on C, in which $z \neq w$ we obtain a connected graph G' such that, each vertex x with $f(x) = 0$, is adjacent with a vertex y with $f(y) = 1$. Therefore we obtain a connected subgraph G' with at least one fewer cycle such that f is an $s - DF$ of G' of weight $\gamma_s(G)$. So by repeating this procedure, the result follows. □

Corollary 2.7. *For any connected graph G we have*

$$\gamma(G) \geq min\gamma(TG)$$

and

$$\gamma_g(G) \geq min\gamma_g(TG)$$

Theorem 2.8. *Let G be a graph without any isolated vertex and suppose that $|V(G)| = n$ and $|E(G)| = m$ and α is independence number of G, then we have*

$$n - \alpha + s - 1 \geq \gamma_s \geq (m - \binom{n}{2})(1/(s-1)) + n$$

Proof. Suppose that B is a set of independent vertices of G with α vertices. Since G has no isolated vertex, $V(G) \setminus B$ is a dominating set of G. First we assume that $\alpha \geq s-1$. For any subset of B with $s-1$ vertices say T we have $(V-B) \cup T$ is $s-DS$ of cardinality $n-\alpha+s-1$ and so $\gamma_s \leq n-\alpha+s-1$.

If $\alpha < s-1$, then $s-1-\alpha > 0$ and again we conclude the result as $\gamma_s \leq n$.

To prove the other inequality suppose $M \subseteq V(G)$ is an $s-DS$.

According to the definition, K_n is factorable into G and $s-1$ graphs such that, each vertex of $V(G) \setminus M$ in each of these $s-1$ subgraphs is adjacent with a vertex of M. Thus we have at least $(n-|M|)(s-1)$ edges in $E(K_n) \setminus E(G)$.

Therefore $m \leq \binom{n}{2} - (n-|M|)(s-1)$. If $|M| = \gamma_s$, then the result follows. □

Similar to [1, Proposition 1] we have:

Theorem 2.9. *If G is a graph with at least s components ($s \geq 2$) then we have $\gamma_s(G) = \gamma_g(G) = \gamma(G)$.*

Proof. Suppose that f is a dominating function of G of weight $\gamma(G)$. Obviously each component of G has at least one vertex with value 1. Since f is a dominating function, each vertex v with $f(v) = 0$ is adjacent with a vertex w with $f(w) = 1$, and since we have at least s components each of them has a vertex with value 1, v is not adjacent with at least $s-1$ vertices x with $f(x) = 1$. Thus f is an $s-DF$ of weight $\gamma(G)$ and $\gamma_s(G) \leq \gamma(G)$. The converse is obvious. □

Similar to [1, Theorem 4] we have:

Theorem 2.10. *If G is a graph with order n such that $n \geq 4$ and $\Delta(G)$ is the maximum degree of the vertices of G. Then $\gamma_t(G) \leq \gamma(G) + \Delta(G)(t-1)$.*

Proof. First we prove that for each positive integer $s \geq 2$, we have $\gamma_s(G) \leq \gamma_{s-1}(G) + \Delta(G)$, that by induction on s, the result follows.

If $\Delta(G) = 0$ the result is obvious. If $\Delta(G) \geq 1$ suppose that f is an $(s-1)-DF$ of minimum weight ($\gamma_{s-1}(G) - function$). If $\gamma_{s-1}(G) = n$ then the result is obvious. Otherwise we can suppose that u

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is a vertex such that $f(u) = 0$. We define B to be the set of vertices x of G adjacent with u and $f(x) = 0$.

Obviously u has a neighbor v, such that $f(v) = 1$. We define $g : V(G) \longrightarrow \{0,1\}$ as follows: $g(u) = 1$, $g(x) = 1$ for any $x \in B$, and $g = f$ for other vertices.

It is not difficult to see that g is an $s - DF$. By the way we defined g by modifying f,

$$w(g) \leq (\Delta(G) - 1) + 1 + w(f) = \Delta(G) + \gamma_{s-1}(G).$$

So $\gamma_s(G) \leq \gamma_{s-1}(G) + \Delta(G)$. The proof is now complete. □

Theorem 2.11. *If G is a triangle-free graph with order n such that $n \geq 4$ then*

$$\gamma_t(G) \leq \gamma(G) + 2(t-1).$$

Proof. First we prove that for each positive integer $s \geq 2$, we have $\gamma_s(G) \leq \gamma_{s-1}(G) + 2$, that by induction the result follows.

Suppose that f is an $(s-1) - DF$ of minimum weight ($\gamma_{s-1}(G) - function$). If $\gamma_{s-1}(G) = n$ then the result is obvious. So we can suppose that there is a vertex v with $f(v) = 0$. If there is a vertex v with $f(v) = 0$, such that for each of its neighbors x we have $f(x) = 1$, we define g as $g(v) = 1$ and $g = f$ for other vertices of $V(G)$. Obviously g is an $s - DF$ and its weight is at most $\gamma_{s-1}(G) + 1$. Otherwise there are two adjacent vertices v and w with $f(v) = f(w) = 0$. In this case we define g by, $g(v) = g(w) = 1$ and $g = f$ for other vertices of $V(G)$. Since the graph is triangle-free, g is an $s - DF$ and its weight is at most $\gamma_{s-1}(G) + 2$. So the result follows. □

Theorem 2.12. *If T is a tree with p end vertices then for any $s \leq n - p$, we have*

$$\gamma_s(T) \leq n - p.$$

Proof. We assign 0 to each end vertex and 1 to the other vertices of T. Then clearly this function is an $s - DF$ of weight $n - p$. So $\gamma_s(T) \leq n - p$. □

Theorem 2.13. *If G is a graph and $uv \in E(G)$, then we have*
(i) $\gamma_{s-1}(G) \leq \gamma_s(G - uv) \leq \gamma_s(G) + 1$,
(ii) $\gamma_{s-1}(G) \leq \gamma_s(G \cup K_1) - 1$.

Proof. For (i) let f be a $\gamma_s(G)-$function on G and $G' = G - uv$. Define g on $V(G) = V(G')$ as follows:
(1) if $f(u) = f(v) = 0$ then $g(u) = g(v) = 0$;
(2) if $f(u) = f(v) = 1$ then $g(u) = g(v) = 1$;
(3) if $f(u) = 0$ and $f(v) = 1$ or $f(u) = 1$ and $f(v) = 0$ then $g(u) = g(v) = 1$.
Also $g = f$ for other vertices of G'.

Since f is an $s - DF$ on G, clearly g is an $s - DF$ on G' of weight at most $\gamma_s(G) + 1$. So $\gamma_s(G - uv) \leq \gamma_s(G) + 1$.

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For the other inequality in (i) let f be a γ_s-function on $G-uv$. Then clearly f is an $(s-1)-DF$ of G of weight $\gamma_s(G-uv)$ and so $\gamma_{s-1}(G)\leq\gamma_s(G-uv)$ as desired.

For (ii) let f be a γ_s-function on $G\cup K_1$. Then the restriction of f on G is an $(s-1)-DF$ and the value of the vertex of K_1 is 1. So $\gamma_{s-1}(G)\leq\gamma_s(G\cup K_1)-1$. □

Corollary 2.14. *If G is a graph and $uv\in E(G)$ such that v is an end vertex, then we have*

$$\gamma_{s-1}(G-v)\leq\gamma_s(G).$$

Proof. According to Theorem 2.13, $\gamma_{s-1}(G-v)\leq\gamma_s(G-v\cup K_1)-1\leq\gamma_s(G)$.

This completes the proof. □

Corollary 2.15. *If G is a graph and $uv\in E(G)$, then we have*
(i) $\gamma(G)\leq\gamma_g(G-uv)\leq\gamma_g(G)+1$;
(ii) $\gamma(G)\leq\gamma_g(G\cup K_1)-1$.
If v is an end vertex then we have
(iii) $\gamma(G-v)\leq\gamma_g(G)$.

We know that if α is the independence number of a graph G and γ is the domination number of G, then we have $\gamma\leq\alpha$. We have a similar result for the global domination number.

Theorem 2.16. *Suppose that G is a graph and $\Delta(G)$ is the maximum degree of G and $\alpha(G)$ is the independence number of G such that $\alpha(G)-\Delta(G)\geq s-1$. Then we have $\gamma_s(G)\leq\alpha(G)$.*

Proof. Suppose that S is an independent set of G with, $\alpha(G)$ vertices. Therefore S is dominating set and since $\alpha(G)-\Delta(G)\geq s-1$ each vertex in $V(G)\setminus S$ is not adjacent with at least $s-1$ vertices of S. So $\gamma_s(G)\leq|S|=\alpha(G)$. □

Theorem 2.17. *Let G be a graph and $d=diam(G)$,*
(i) If $d\geq 3(s-1)$ then $\gamma_s(G)\leq\gamma(G)+s$.
(ii) If $d>3(s-1)$ then $\gamma_s(G)\leq n-2s+1$.

Proof. (i) Suppose that $v_1,\cdots,v_{d+1}$ is a path of G with $d(v_1,v_{d+1})=d$, and suppose that, f is a $\gamma-$function of G. We assign 1 to each of $v_1,v_4,\ldots,v_{3t+1},\ldots,v_{3(s-1)+1}$ and assign to the other vertices of G the same value of f. Clearly this function is an $s-DF$ and its weight is at most $\gamma(G)+s$ and so $\gamma_s(G)\leq\gamma(G)+s$.

(ii) Suppose that $v_1,\cdots,v_{d+1}$ is a path of G with $d(v_1,v_{d+1})=d$. We assign 1 to each of $v_1,v_4,\ldots,v_{t+1},\ldots,v_{3(s-1)+1}$ and assign 0 to other vertices of $\{v_1,v_2,\ldots,v_{3(s-1)+2}\}$ and 1 to the other vertices of G. Then obviously f is an $s-DF$ of weight $n-2s+1$ and so $\gamma_s(G)\leq n-2s+1$. □

Theorem 2.18. *Suppose that G is a graph and $g=girth(G)$, and $g>3(s-1)$ with $s>1$. Then*
(i) $\gamma_{s-1}(G)\leq\gamma(G)+s$,
(ii) $\gamma_s(G)\leq n-2s+2$.

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Proof. (i) Suppose that $v_1, \ldots, v_g$ are the vertices of a cycle C of length g, and let f be a γ-function on G . We assign 1 to $v_1, v_4, \ldots, v_{3t+1}, \ldots, v_{3(s-1)+1}$ and assign to the other vertices of G the same value of f and call the resulting function h. We claim that h is an $(s-1)-DF$ on G. Clearly each vertex v in G with $h(v) = 0$ is adjacent with a vertex w with $h(w) = 1$ in G. Also each vertex v with $h(v) = 0$ on C is not adjacent with at least $s-2$ vertices z with $h(z) = 1$ (these $s-2$ vertices belong to $\{v_1, v_4, \ldots, v_{3t+1}, \ldots, v_{3(s-1)+1}\}$). Now if a vertex v with $h(v) = 0$ in $V(G) \setminus C$ is adjacent with two of these vertices with value 1, (i.e $v_1, v_4, \ldots, v_{3t+1}, \ldots, v_{3(s-1)+1}$) say u, w, then we have u, w are not adjacent. (Otherwise we have a cycle of length 3 but $g > 3$) . These two vertices partition the cycle to two parts. If the length of the smaller part is more than two then we may replace it with uvw, resulting in a shorter cycle than C which is impossible. Since u, w are not adjacent, the length of the smaller part is exactly two. So the length of the cycle is $3t+2$ for some positive integer, t while uvw with the smaller part of C form a cycle of length 4 which is impossible. So h is an $(s-1)-DF$ on G of weight at most $\gamma(G) + s$ and so $\gamma_{s-1}(G) \leq \gamma(G) + s$.

(ii) Suppose that $v_1, \cdots, v_g$ are the vertices of a cycle of length g. We assign 1 to $v_1, v_4, \cdots, v_{3(s-1)+1}$ and assign 0 to other vertices in $\{v_2, v_3, \cdots, v_{3(s-1)}\}$, and assign 1 to the other vertices of G. Obviously this function is an $s-DF$ of G of weight $n-(2s-2)$ and so $\gamma_s(G) \leq n-2s+2$. □

The proof of the next result is essentially similar to the proof of [1, Lemma 2].

Theorem 2.19. *If T is a tree and f is a γ-function such that for some vertices $v_1, \cdots, v_s$, $f(v_i) = 1$ and $d(v_i, v_j) \neq 2$ then $\gamma_s(T) = \gamma(T)$.*

Proof. Since $d(v_i, v_j) \neq 2$ and T is tree we have $N_T(v_i) \cap N_T(v_j) = \emptyset$ and so

$$N_{\overline{T}}[v_i] \cup N_{\overline{T}}[v_j] = V(T).$$

This together with the fact that $f(v_i) = 1$ and f is a γ-function implies that f is an $s-DF$ and the weight of f is $\gamma(T)$. Thus $\gamma_s(T) \leq \gamma(T)$. The converse is obvious. □

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